数理解析研究所講究録
1198 巻 2001 年 117-121



# A Gradient Flow Approach to a Free Boundary Problem with Volume Constraint

G. S. Weiss
Graduate School of Mathematical Sciences
University of Tokyo
3-8-1 Komaba, Meguro, 153-8914 Tokyo

Consider the following free boundary problems with volume constraint, of which the first one has been applied to the construction of optimal heat conductors while the second one has been introduced as a model for the equilibrium state of two immiscible fluids described by an order parameter:

1) $u\geq 0$ , $\Delta u=0$ in $\Omega\cap\{u>0\}$ , $|\nabla u|$ is constant on $\Omega\cap\partial\{u>0\}$ ,

$$\mathcal{L}^{n}(\Omega\cap\{u>0\})=c\ .$$

2) $0\leq v\leq 1$, $\Delta v=0$ in $\Omega\cap\{0<v<1\}$, $|\nabla v|$ is constant on $\Omega\cap\partial\{v>0\}$, $|\nabla v|$ is constant on $\Omega\cap\partial\{v<1\}, \mathcal{L}^{n}(\Omega\cap\{v=0\})=c_{1}, \mathcal{L}^{n}(\Omega\cap\{v=1\})=c_{2}$ .

Here we suggest a *gradient flow approach* that avoids numerical difficulties arising from a minimization process.

Problem 1) has been derived from the minimization problem

$$\int_{\Omega}|\nabla u|^{2}\rightarrow\min$$

with the constraint $\mathcal{L}^n(\Omega \cap \{u > 0\}) = \omega$ (see [3]). Existence of a minimizer has been shown in [1] via a penalization method. Aguilera, Alt and Caffarelli prove that for small $\epsilon > 0$ each minimizer of the penalized functional

$$J_\epsilon(w) := \int_\Omega (|\nabla w|^2 + f_\epsilon(\mathcal{L}^n(\Omega \cap \{w > 0\})))$$

where

$$f_\epsilon(s) = \begin{cases} \frac{1}{\epsilon}(s - \omega), s \geq \omega \\ \epsilon(s - \omega), s \leq \omega \end{cases}$$

is automatically a solution of the problem

$$\int_\Omega |\nabla u|^2 \to \min \ , \ \mathcal{L}^n(\Omega \cap \{u > 0\}) = \omega \ .$$

In two dimensions it follows then ([1]) that $u$ is a classical solution of the problem, i.e. $u \in C^2(D \cap \overline{\{u > 0\}})$ for each $D \subset\subset \Omega$ and $\Omega \cap \partial\{u > 0\}$ is an analytic surface. In higher dimensions $u$ is still a "weak solution", and the results in [6] can be applied to infer that the singularities of the free boundary are in the case of three dimensions isolated in $\Omega$ and the singular set $\Sigma$ satisfies $\dim(\Sigma) \leq n - 3$ in general.
Let us also mention the result [3], where the asymptotic behavior as $\mathcal{L}^n(\Omega \cap \{u = 0\}) \to 0$ is studied.

Problem 2) has been derived as a model for the equilibrium state of two immiscible fluids described by an order parameter ([4]).
In [2] existence has been proved and the asymptotic behavior as the interfacial volume tends to $0$, i.e. $\mathcal{L}^n(\Omega \cap \{u = 0\}) \to \gamma$ and $\mathcal{L}^n(\Omega \cap \{u = 1\}) \to \mathcal{L}^n(\Omega) - \gamma$, has been studied. It turns out that the limits of $\Omega \cap \{u = 0\}$ minimize the perimeter and satisfy the volume constraint.
Tilli proves in [5] continuity of minimizers which makes it possible to apply the regularity theory in [1] and [6] to obtain as before in two dimensions the existence of a classical solution as well as that the singularities are in the case of three dimensions isolated in $\Omega$ and form in general a set of dimension

$\leq n-3$.

We suggest here a *(surface) gradient flow approach* to the construction of solutions of problem 1) and 2) which makes it possible to dispense with the penalization of [1]. This is important for the numerical analysis as penalization would mean introduction of an additional parameter (in addition to the regularization parameter and the parameter of the Galerkin approximation), in which case we would expect the solution to depend sensitively on the relation of the parameters. Another point in favor of our approach is that we can explicitly use in computations the quantity (called below $\lambda_\epsilon(t)$) converging to our Lagrange multiplier as time goes to infinity.
We describe the approach for problem 1). Problem 2) can be treated in the same way. Let $\epsilon \in (0,1)$, $\beta_\epsilon(z) = \frac{1}{\epsilon}\beta(\frac{z}{\epsilon})$, $\beta \in C_0^1([0,1])$, $\beta > 0$ in $(0,1)$ and $\int \beta = \frac{1}{2}$. The serves as regularization of the nondifferentiable nonlinearity in the problem ($\epsilon$ is a regularization parameter which is not related to the penalization parameter $\epsilon$ above). In addition we need the primitive $B_\epsilon(z) := \int_0^z \beta_\epsilon(s)\,ds$. As $2B_\epsilon(z) \to \chi_{\{z>0\}}$ as $\epsilon \to 0$, $\int_\Omega 2B_\epsilon(u_\epsilon)$ will serve as a regularization of the volume term $\mathcal{L}^n(\Omega \cap \{u>0\})$.
For initial data $u_\epsilon^0 \geq 0$ satisfying the constraint $\int_\Omega 2B_\epsilon(u_\epsilon^0) = \omega$ our approach consists now in solving the evolution equation

$$\partial_t u_\epsilon - \Delta u_\epsilon = -\left(\int_\Omega \beta_\epsilon(u_\epsilon)^2\right)^{-1}\left(\int_\Omega \beta_\epsilon(u_\epsilon)\Delta u_\epsilon\right)\beta_\epsilon(u_\epsilon)$$

in $(0,\infty)\times\Omega$. Note that $\lambda_\epsilon := \left(\int_\Omega \beta_\epsilon(u_\epsilon)^2\right)^{-1}\left(\int_\Omega \beta_\epsilon(u_\epsilon)\Delta u_\epsilon\right)$ depends only on the time variable $t$ and plays the role of the Lagrange multiplier as $t \to \infty$.
As

$$\partial_t \int_\Omega 2B_\epsilon(u_\epsilon(t)) = \int_\Omega 2\beta_\epsilon(u_\epsilon(t))\partial_t u_\epsilon(t)$$
$$= \int_\Omega 2\beta_\epsilon(u_\epsilon(t))\Delta u_\epsilon(t) - \int_\Omega 2\beta_\epsilon(u_\epsilon(t))\Delta u_\epsilon(t) = 0,$$

$u_\epsilon$ satisfies the volume constraint $\int_\Omega 2B_\epsilon(u_\epsilon(t)) = \omega$ for each $t \geq 0$.

Existence, time-asymptotic limits $(v_\epsilon(x), \lambda_\epsilon) \leftarrow (u_\epsilon(t,x), \lambda_\epsilon(t))$ (as $t \to +\infty$),

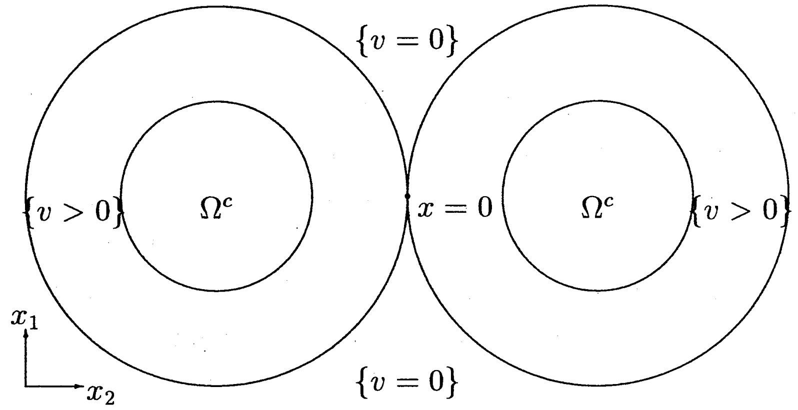


Figure 1: A singularity for the limit problem in two space dimensions

uniform bounds for $v_\epsilon$ and $\lambda_\epsilon$ are going to be studied in collaboration with I. Fonseca and M. Kowalcyk.
Concerning the $\epsilon$-limit the following can be said: there is a sequence $\epsilon_m \to 0$ such that $v_{\epsilon_m} \to v$ in $H^{1,2}(\Omega)$ , $\lambda_{\epsilon_m} \to \lambda$ as $m \to \infty$ and $v$ is a solution in the sense of domain variations, i.e. $v$ is smooth in $\Omega \cap \{v > 0\}$ and satisfies

$$\int_\Omega [|\nabla v|^2 \mathrm{div}\, \xi \; - \; 2\nabla v D\xi \nabla v] \; = \; -\lambda \int_R \xi \cdot \nu \, d\mathcal{H}^{n-1} \tag{1}$$

for every $\xi \in C_0^{0,1}(\Omega; \Omega)$ . Here

$$R := \{x \in \Omega \cap \partial\{v > 0\} \; : \; \text{there is } \nu(x) \in \partial B_1(0) \text{ such that } v_r(y) = \frac{v(x+ry)}{r} \to \max(-y \cdot \nu(x), 0) \text{ locally uniformly in } y \in \mathbf{R}^n \text{ as } r \to 0\}$$

is a countably $n-1$-rectifiable subset of the free boundary.
This is optimal in the sense that there are for each $\theta \in (0,1]$ solutions $v_\epsilon(x) \to \theta|x_n|$ as $\epsilon \to 0$ , that is, there are in general large singular sets.
The relative boundary of $R$ may also be non-empty as shown in Figure 1.

## References


[1] AGUILERA, N., ALT, H. W., CAFFARELLI, L. A., An optimization problem with volume constraint, *SIAM J. Control Optim.*, 24 (1986), 191-198.

[2] AMBROSIO, L., FONSECA, I., MARCELLINI, P., TARTAR, L., On a volume-constrained variational problem., *Arch. Ration. Mech. Anal.*, 149 (1999), 23-47.

[3] FLUCHER, M., An asymptotic formula for the minimal capacity among sets of equal area, *Calc. Var. Partial Differential Equations*, 1 (1993), 71-86.

[4] GURTIN, M., POLIGNONE, D., VINALS, J., Two-phase binary fluids and immiscible fluids described by an order parameter, *Math. Models Methods Appl. Sci.*, 6 (1996), 815-831.

[5] TILLI, P., On a constrained variational problem with an arbitrary number of free boundaries, *Interfaces Free Bound.*, 2 (2000), 201-212.

[6] WEISS, G.S., Partial regularity for weak solutions of an elliptic free boundary problem, *Commun. Partial Differ. Equations*, 23 (1998), 439-457.