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ON THE UNIFICATION OF KUMMER AND ARTIN-SCHREIER-WITT THEORIES

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1. MOTIVATION

Our aim of this report is to give an explanation of the final version of our theory which unifies the Kummer theory and Artin-Schreier-Witt theory. The details of this report can be seen in the Bordeaux preprint [15].

First, we review the Kummer theory.

Let n be an integer with $n \geq 2$, and K be a field of charcteristic q with $q \not| n$ and $K \supset \mu_n = \{\zeta \mid \zeta^n = 1\}$.

Theorem 1.1 (Kummer Theory).

L/K: n-cyclic Galois extension

$\Longleftrightarrow\ \exists a \in K^*$ s.t. $L = K(\sqrt[n]{a})$

$$\Longleftrightarrow\ \exists a \in K^* \text{ s.t. } \begin{array}{ccccc} L = K \otimes_{K[X,X^{-1}]} K[X,X^{-1}] & \leftarrow & K[X,X^{-1}] & X^n \\ \uparrow & & \uparrow & \uparrow \\ K & \leftarrow & K[X,X^{-1}] & X \\ a & \leftarrow\!\dashv & X & \end{array}$$

$$\Longleftrightarrow\ \exists f : \operatorname{Spec} K \to \mathbb{G}_{m,K} \text{ s.t. } \begin{array}{ccc} \operatorname{Spec} L & \rightarrow & \mathbb{G}_{m,K} \\ \downarrow & \square & \downarrow \theta_n \\ \operatorname{Spec} K & \underset{f}{\rightarrow} & \mathbb{G}_{m,K} \end{array},$$

where $\theta_n : \mathbb{G}_{m,K} \longrightarrow \mathbb{G}_{m.K}\, ;\, x \mapsto x^n$.

Namely, the Kummer theory implies that the following exact sequence (so-called the **Kummer exact sequence**)of sheaves on the fppf (or étale) site on $\operatorname{Spec} K$ is essential in the world of cyclic coverings of K:

$$\begin{array}{cccccc} 1 \to \boldsymbol{\mu}_{n,K} \to & \mathbb{G}_{m,K} & \xrightarrow{\theta_n} & \mathbb{G}_{m,K} & \to 1 \\ & t & \longmapsto & t^n & \end{array}$$

In fact, from the exact sequence, for any K-scheme X we can deduce the exact sequence:

$$\mathbb{G}_{m,K}(X) \xrightarrow{\theta_n} \mathbb{G}_{m,K}(X) \xrightarrow{\partial} \mathrm{H}^1(X, \boldsymbol{\mu}_{n,K}) \to \mathrm{H}^1(X, \mathbb{G}_{m,K}) \to \mathrm{H}^1(X, \mathbb{G}_{m,K}).$$

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Here

$$\mathrm{H}^1(X, \boldsymbol{\mu}_{n,K}) = \text{the set of isomorphism classes of unramified } \boldsymbol{\mu}_n \text{ coverings of } X$$
$$\mathrm{H}^1(X, \mathbb{G}_{m,K}) = 0 \text{ for suitable } X\text{'s by Hilbert Theorem 90}$$

Next we review the Artin-Schreier-Witt theory.
Let k be a field of positive characteristic p.
$W_{n,k}$: the group scheme of Witt vectors of length n

$$\wp : W_{n,k} \longrightarrow W_{n,k};\ x \mapsto x^{(p)} - x$$

Theorem 1.2 (Artin-Schreier-Witt Theory).

K/k: p^n-cyclic Galois extension

$\iff \exists a \in W_n(k)$ s.t. $K = k(\wp^{-1}(a))$

$$\iff \exists a = (a_0, a_1, \dots, a_{n-1}) \in W_n(k) \text{ s.t. } \begin{array}{ccc} K = k \otimes_{k[\mathbb{X}]} k[\mathbb{X}] & \longleftarrow & k[\mathbb{X}] \\ \uparrow & & \uparrow \wp^* \\ k & \longleftarrow & k[\mathbb{X}] \\ a_i & \longleftarrow\!\dashv & X_i \end{array}$$

$$\iff \exists f : \operatorname{Spec} k \longrightarrow W_{n,k} \text{ s.t. } \begin{array}{ccc} \operatorname{Spec} K & \longrightarrow & W_{n,k} \\ \downarrow & \square & \downarrow \wp \\ \operatorname{Spec} k & \xrightarrow[f]{} & W_{n,k} \end{array},$$

where $\mathbb{X} = (X_0, X_1, \dots, X_{n-1})$.

Namely, the Artin-Schreier-Witt theory implies that the following exact sequence (so-called the **Artin-Schreier-Witt exact sequence**) of sheaves on the fppf (or étale) site on the $\operatorname{Spec} k$ is essential:

$$\begin{array}{cccccc} 0 \to \mathbb{Z}/p^n\mathbb{Z} \to & W_{n,k} & \xrightarrow{\wp} & W_{n,k} & \to 0. \\ & x & \longmapsto & x^{(p)} - x & \end{array}$$

In fact, from the exact sequence, for any k-scheme X we can deduce the exact sequence:

$$W_{n,k}(X) \xrightarrow{\wp} W_{n,k}(X) \xrightarrow{\partial} \mathrm{H}^1(X, \mathbb{Z}/p^n) \to \mathrm{H}^1(X, W_{n,k}) \to \mathrm{H}^1(X, W_{n,k}).$$

Here

$$\mathrm{H}^1(X, \mathbb{Z}/p^n) = \text{ the set of isomorphism classes of unramified } \mathbb{Z}/p^n\mathbb{Z} \text{ coverings of } X$$
$$\mathrm{H}^1(X, W_{n,k}) = 0 \quad \text{for affine schemes } X$$

Therefore, the **Kummer theory** implies that in the world of unramified p^n-cyclic coverings in characteristic 0, the **Kummer exact sequence** is the **Buddha**, and any such coverings is deduced from the sequence. On the other hand, the **Artin-Schreier-Witt theory** imples that in the world of unramified p^n-cyclic coverings in characteristic p, the

Artin-Schreier-Witt exact sequence is the **Buddha**, and any such covering is deduced from the sequence. But our religion asserts that every Buddha should be deduced from the unique eseential **Buddha** (**Mahāvairocanaḥ**). Hence, behind the two Buddhas, there should exist a more essential Buddha unifying them.

So we arrive at the following problems:

- Search for the Buddha unifying the Kummer and ASW sequences.
- Construct the deformations of the group schemes of Witt vectors of finite length to tori.
- Such deformations should keep the filtarations of the group schemes of Witt vectors.

2. 1 DIMENSIONAL CASE

Let $(A, \mathfrak{m})$ be a DVR with f.f.$A = K$ and $A/\mathfrak{m} = k$, and $\lambda \in \mathfrak{m} \setminus \{0\}$. Now we look at the plane curve over A:

$$\mathcal{C}: \quad Y^2Z - \lambda XYZ - X^3 = 0 \subset \mathbb{P}^2,$$

whose generic fibre is a nodal curve and the special fibre is a cuspidal curve. Therefore the Picard scheme of the curve gives a deformation of an additive group scheme to a torus:

$$\mathrm{Pic}^0(\mathcal{C}/A) \cong \mathrm{Spec}\, A[X, 1/(1 + \lambda X)],$$

with group law $x \cdot y = \lambda xy + x + y$. Hereafter we denote this group scheme by $\mathcal{G}^{(\lambda)}$:

$$\mathcal{G}^{(\lambda)} := \mathrm{Spec}\, A[X, 1/(1 + \lambda X)].$$

The important fact is that any deformations of $\mathbb{G}_a$ to $\mathbb{G}_m$ over A are only the type of $\mathcal{G}^{(\lambda)}$'s. In fact, we have the following.

Theorem 2.1 ([17, Th. 2.5]). *Let $\mathcal{G}$ be a flat group scheme over* $\mathrm{Spec}\, A$ *with gereric fibre $\mathbb{G}_m$ and special fibre $\mathbb{G}_a$. Then there exists a non-zero element λ of $\mathfrak{m}$, uniquely up to unit factors, such that*

$$\mathcal{G} \cong \mathcal{G}^{(\lambda)}.$$

3. HIGHER DIMENSIONAL CASE

If we obtain a deformation $\mathcal{W}_{n-1}$ of W_{n-1} to $\mathbb{G}_{m,K}^{n-1}$, then since the Witt vectors has the filtaration

$$\begin{array}{ccccccccc} 0 \to & \mathbb{Z}/p & \to & \mathbb{Z}/p^n & \to & \mathbb{Z}/p^{n-1} & \to 0 \\ & \cap & & \cap & & \cap & \\ 0 \to & \mathbb{G}_{a,k} & \to & W_{n,k} & \to & W_{n-1,k} & \to 0, \end{array}$$

we can expect the next one fits into an extension

$$0 \to \mathcal{G}^{(\lambda)} \to \mathcal{W}_{n+1} \to \mathcal{W}_n \to 0 \in \mathrm{Ext}^1(\mathcal{W}_n, \mathcal{G}^{(\lambda)}).$$

Definition 3.1. *Let $(A, \mathfrak{m})$ be a DVR, and $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathfrak{m} \setminus \{0\}$. If $\mathcal{W}_n$ is given by the extensions*

$$0 \to \mathcal{G}^{(\lambda_2)} \to \mathcal{W}_2 \to \mathcal{G}^{(\lambda_1)} \to 0 \in \mathrm{Ext}^1(\mathcal{G}^{(\lambda_1)}, \mathcal{G}^{(\lambda_2)})$$
$$0 \to \mathcal{G}^{(\lambda_3)} \to \mathcal{W}_3 \to \mathcal{W}_2 \to 0 \in \mathrm{Ext}^1(\mathcal{W}_2, \mathcal{G}^{(\lambda_3)})$$
$$\dots\dots$$
$$0 \to \mathcal{G}^{(\lambda_n)} \to \mathcal{W}_n \to \mathcal{W}_{n-1} \to 0 \in \mathrm{Ext}^1(\mathcal{W}_{n-1}, \mathcal{G}^{(\lambda_n)}),$$

we call it a **group scheme of type $(\lambda_1, \lambda_2, \dots, \lambda_n)$**.

To compute the group $\mathrm{Ext}^1(\mathcal{W}_\ell, \mathcal{G}^{(\lambda_{\ell+1})})$ for a group scheme $\mathcal{W}_\ell$ of type $(\lambda_1, \lambda_2, \dots, \lambda_\ell)$, the following exact sequence of sheaves on each the small Zariski, fppf or étale site on $\mathrm{Spec}\, A$ is essential:

$$\begin{array}{cccccccc} 0 \to & \mathcal{G}^{(\lambda)} & \xrightarrow{\alpha^{(\lambda)}} & \mathbb{G}_{m,A} & \xrightarrow{\rho^{(\lambda)}} & \iota_* \mathbb{G}_{m,A_\lambda} & \to 0. \\ & x & \mapsto & 1 + \lambda x & & & \\ & & & t & \mapsto & t \bmod \lambda & \end{array}$$

where $\iota : \mathrm{Spec}(A/\lambda) \hookrightarrow \mathrm{Spec}\, A$ is the canonical inclusion.

By an explicit computation of cocycles, we have

Proposition 3.1.

$$\mathrm{Ext}^1(\mathcal{G}^{(\lambda)}, \mathbb{G}_{m,A}) = 0.$$

Therefore inductively we have

$$\mathrm{Ext}^1(\mathcal{W}_\ell, \mathbb{G}_{m,A}) = 0,$$

for any group scheme of type $(\lambda_1, \lambda_2, \dots, \lambda_\ell)$.

Hence, by using the above exact sequence we obtain the following.

Theorem 3.2. *Let $\mathcal{W}_n$ be a group scheme of type $(\lambda_1, \lambda_2, \dots, \lambda_n)$, and $\lambda \in \mathfrak{m} \setminus \{0\}$. Then we have*

$$\mathrm{Ext}^1(\mathcal{E}, \mathcal{G}^{(\lambda)}) \cong \mathrm{Hom}(\mathcal{E}, \iota_* \mathbb{G}_{m,A/\lambda}) \Big/ (\rho^{(\lambda)})_* \left(\mathrm{Hom}(\mathcal{E}, \mathbb{G}_{m,A})\right).$$

From this theorem, we can deduce the following.

Theorem 3.3. *Let $\mathcal{W}_n$ be a group scheme of type $(\lambda_1, \lambda_2, \dots, \lambda_n)$. Then there exists a homomorphism*

$$D_\ell : \mathcal{W}_\ell \to \iota_* \mathbb{G}_{m,A/\lambda_{\ell+1}}$$

for each ℓ $(2 \le \ell \le n-1)$, and each $\mathcal{W}_\ell$ is given by

$$\mathcal{W}_\ell \cong \mathrm{Spec}\, A[X_0, \dots, X_{\ell-1}, \frac{1}{1 + \lambda_1 X_0}, \frac{1}{D_1(X_0) + \lambda_2 X_1}, \dots, \frac{1}{D_{\ell-1}(X_0, \dots, X_{\ell-2}) + \lambda_\ell X_{\ell-1}}].$$

Moreover, the group law of $\mathcal{W}_\ell$ *is the one which makes the morphism*

$$\begin{array}{rcl} \alpha^{(\ell)} : \mathcal{W}_\ell & \to & (\mathbb{G}_{m,A})^\ell \\ (X_0, \dots, X_{\ell-1}) & \mapsto & \begin{array}{l}(1+\lambda_1 X_0, D_1(X_0) + \lambda_2 X_1, \\ \quad \dots, D_{\ell-1}(X_0, \dots, X_{\ell-2}) + \lambda_\ell X_{\ell-1})\end{array} \end{array}$$

a group-schematic homomorphism.

Definition 3.2. *Suppose that* A *dominates* $\mathbb{Z}_{(p)}[\mu_{p^n}]$, *and put* $\lambda = \lambda_{(1)}$. *We call a group scheme* $\mathcal{W}_1 = \mathcal{G}^{(\lambda)}, \mathcal{W}_2, \dots, \mathcal{W}_n$ *over* A *of type* $(\lambda)^n = (\overbrace{\lambda, \lambda, \dots, \lambda}^{n})$ *a* **KASW group scheme** *over* A, *if there exists an inclusion* $i_\ell : \mathbb{Z}/p^\ell \hookrightarrow \mathcal{W}_\ell$ *for each* ℓ *satisfying a commtative diagram*

$$\begin{array}{ccccccccc} 0 \to & (\mathbb{Z}/p)_A & \to & (\mathbb{Z}/p^\ell)_A & \to & (\mathbb{Z}/p^{\ell-1})_A & \to 0 \\ & \downarrow i_1 & & \downarrow i_\ell & & \downarrow i_{\ell-1} & \\ 0 \to & \mathcal{G}^{(\lambda)} & \to & \mathcal{W}_\ell & \xrightarrow{r_\ell} & \mathcal{W}_{\ell-1} & \to 0. \end{array}$$

Once we obtain a KASW group scheme, then it embodies the unified Kummer-Artin-Schreier-Witt theory.

Theorem 3.4 (KASW theory). *Let* $\mathcal{W}_n$ *be a KASW group scheme over* A. *Let* B *and* C *are local flat* A*-algebras such that* C *is an unramified* p^n*-cyclic covering over* B. *Then there exists an* A*-morphism* $f : \operatorname{Spec} B \to \mathcal{W}_n/(\mathbb{Z}/p^n)$, *and the covering* $\operatorname{Spec} C \to \operatorname{Spec} B$ *is given by the fibre product*

$$\begin{array}{ccc} \operatorname{Spec} C & \longrightarrow & \mathcal{W}_n \\ \downarrow & \square & \downarrow \\ \operatorname{Spec} B & \xrightarrow{f} & \mathcal{W}_n/(\mathbb{Z}/p^n). \end{array}$$

By these argument, our work is concentrated upon the calculation of $\operatorname{Ext}^1(\mathcal{W}_n, \mathcal{G}^{(\lambda)})$, namely of $\operatorname{Hom}(\mathcal{W}_n, \mathbb{G}_{m,A/\lambda})$.

4. Determination of $\operatorname{Ext}^1(\mathcal{W}_n, \mathcal{G}^{(\lambda)})$

We provide some notations.

$(A, \mathfrak{m})$: DVR dominating $\mathbb{Z}_{(p)}$, $\lambda \in \mathfrak{m} \setminus \{0\}$
$\Phi_n(\mathbb{T}) = T_0^{p^n} + pT_1^{p^{n-1}} + \dots + p^n T_n$: Witt polynomial
$\tilde{a} := (a, 0, 0, \dots) \in W(A)$ for $A \in A$
$[p] : W_A \to W_A$; $[p]\boldsymbol{b} := (0, b_0^p, b_1^p, \dots)$ for $\boldsymbol{b} = (b_0, b_1, \dots)$
$V : W_A \to W_A$: Verschiebung endomorphism
$F : W_A \to W_A$: the generalized Frobenius endomorphism
$F^{(\lambda)} := F - (\lambda^{p-1})^\sim$
For $\boldsymbol{a} \in W(A)$, we define $T_{\boldsymbol{a}} : W(A) \to W(A)$ by

$$\Phi_n(T_{\boldsymbol{a}}\boldsymbol{x}) = a_0^{p^n}\Phi_n(\boldsymbol{x}) + pa_1^{p^{n-1}}\Phi_{n-1}(\boldsymbol{x}) + \dots + p^n a_n \Phi_0(\boldsymbol{x}) \quad (n \geq 0)$$

for $x \in W(A)$. Then we have $T_a = \sum_{k\geq 0} V^k \cdot \tilde{a_k}$.
If A is a ring (not necessarily a $\mathbb{Z}_{(p)}$-algebra),

$$\widehat{W}_n(A) = \left\{ (a_0, a_1, \dots, a_{n-1}) \in W_n(A) \; ; \; a_i \text{ is nilpotent for all } i \right\}$$

and

$$\widehat{W}(A) = \left\{ (a_0, a_1, a_2, \dots) \in W_n(A) \; ; \; \begin{array}{l} a_i \text{ is nilpotent for all } i \text{ and} \\ a_i = 0 \text{ for all but a finite number of } i \end{array} \right\}.$$

Moreover we need to deform the Artin-Hasse exponential series

$$\begin{aligned} E_p(X) &:= \exp\left(X + \frac{X^p}{p} + \frac{X^{p^2}}{p^2} + \cdots\right) \\ &= e^X e^{\frac{X^p}{p}} e^{\frac{X^{p^2}}{p^2}} \cdots \in \mathbb{Z}_{(p)}[[X]]. \end{aligned}$$

The well-known formula $\lim_{\lambda\to 0}(1+\lambda x)^{\alpha/\lambda} = e^{\alpha x}$ can be seen that $(1+\lambda x)^{\alpha/\lambda}$ is a deformation of $e^{\alpha x}$. From this point of view, we obtain the deformations of Artin-Hasse exponential series:

$$\begin{aligned} E_p(U, \Lambda; X) &:= (1+\Lambda X)^{\frac{U}{\Lambda}} \prod_{k=1}^{\infty} \left(1 + \Lambda^{p^k} X^{p^k}\right)^{\frac{1}{p^k}\left(\left(\frac{U}{\Lambda}\right)^{p^k} - \left(\frac{U}{\Lambda}\right)^{p^{k-1}}\right)} \\ &\in \mathbb{Z}_{(p)}[U, \Lambda][[X]]. \end{aligned}$$

Moreover for a Witt vector $\boldsymbol{a} \in W(A)$, we define a formal power series as follows:

$$\begin{aligned} E_p(\boldsymbol{a}, \lambda; X) &:= \prod_{k=0}^{\infty} E_p(a_k, \lambda^{p^k}; X^{p^k}) \\ &= (1+\lambda X)^{\frac{a_0}{\lambda}} \prod_{k=1}^{\infty} \left(1 + \lambda^{p^k} X^{p^k}\right)^{\frac{1}{p^k \lambda^{p^k}} \Phi_{k-1}(F^{(\lambda)}\boldsymbol{a})}. \end{aligned}$$

The boundary of this power series $E_p(\boldsymbol{a}, \lambda; X)$ is given by the following.

$$\begin{aligned} \left(\partial E_p(\boldsymbol{a}, \lambda; \cdot)\right)(X, Y) &= \frac{E_p(\boldsymbol{a}, \lambda; X) E_p(\boldsymbol{a}, \lambda; Y)}{E_p(\boldsymbol{a}, \lambda; X + Y + \lambda XY)} \\ &= \prod_{k=1}^{\infty} \left(\frac{(1+\lambda^{p^k} X^{p^k})(1+\lambda^{p^k} Y^{p^k})}{1 + \lambda^{p^k}(X+Y+\lambda XY)^{p^k}}\right)^{\frac{1}{p^k \lambda^{p^k}} \Phi_{k-1}(F^{(\lambda)}\boldsymbol{a})}. \end{aligned}$$

Now replacing $F^{(\lambda)}\boldsymbol{a}$ with a Witt vector $\boldsymbol{b} = (b_0, b_1, \dots)$ in the right hand side of this equation, we define a cocyle as follows.

$$\begin{aligned} F_p(\boldsymbol{b}, \lambda; X, Y) &:= \prod_{k=1}^{\infty} \left(\frac{(1+\lambda^{p^k} X^{p^k})(1+\lambda^{p^k} Y^{p^k})}{1 + \lambda^{p^k}(X+Y+\lambda XY)^{p^k}}\right)^{\frac{1}{p^k \lambda^{p^k}} \Phi_{k-1}(\boldsymbol{b})} \\ &\in \mathbb{Z}_{(p)}[\boldsymbol{b}, \lambda][[X, Y]]. \end{aligned}$$

Using these deformed Artin-Hasse exponential series, we can obtain the following.

Theorem 4.1 (Explicit Formula in 1 Dimensional Case).

$$\begin{array}{rcl}
\xi_0^1 : \mathrm{Ker}\left(\widehat{W}(A/\lambda_2) \xrightarrow{F^{(\lambda_1)}} \widehat{W}(A/\lambda_2)\right) & \xrightarrow{\sim} & \mathrm{Hom}(\mathcal{G}^{(\lambda_1)}, \iota_*\mathbb{G}_{m,A/\lambda_2}), \\
\boldsymbol{a} & \mapsto & E_p(\boldsymbol{a}, \lambda_1; X) \\
\xi_1^0 : \mathrm{Coker}\left(\widehat{W}(A/\lambda_2) \xrightarrow{F^{(\lambda_1)}} \widehat{W}(A/\lambda_2)\right) & \xrightarrow{\sim} & \mathrm{H}_0^2(\mathcal{G}^{(\lambda_1)}, \iota_*\mathbb{G}_{m,A/\lambda_2}). \\
\boldsymbol{b} & \mapsto & F_p(\boldsymbol{b}, \lambda_1; X, Y)
\end{array}$$

Therefore

$$\xi_0^1 : \frac{\mathrm{Ker}\left(\widehat{W}(A/\lambda_2) \xrightarrow{F^{(\lambda_1)}} \widehat{W}(A/\lambda_2)\right)}{< \widetilde{\lambda_1} >} \xrightarrow{\sim} \frac{\mathrm{Hom}(\mathcal{G}^{(\lambda_1)}, \iota_*\mathbb{G}_{m,A/\lambda_2})}{\mathrm{Hom}(\mathcal{G}^{(\lambda_1)}, \mathbb{G}_{m,A})} \xrightarrow{\sim} \mathrm{Ext}^1(\mathcal{G}^{(\lambda_1)}, \mathcal{G}^{(\lambda_2)}).$$

In higher dimentional case, we need more notations. For a vector $\mathbb{U} = (U_0, U_1, \dots)$, we define

$$[p]E_p(\mathbb{U}, \Lambda; X) := E_p([p]\mathbb{U}, \Lambda; X),$$
$$[p]F_p(\mathbb{U}, \Lambda; X, Y) := F_p([p]\mathbb{U}, \Lambda; X, Y).$$

Moreover

$$H(X,Y) := \frac{1}{\Lambda_2}\{F_p(\mathbb{U}, \Lambda_1; X, Y) - 1\},$$

$$G_p(\mathbb{A}, \Lambda_2; E) := \prod_{\ell \geq 1}\left(\frac{1 + (E-1)^{p^\ell}}{[p]^\ell E}\right)^{\frac{1}{p^\ell \Lambda_2^{p^\ell}}\Phi_{\ell-1}(\mathbb{A})},$$

$$G_p(\mathbb{A}, \Lambda_2; F) := \prod_{\ell \geq 1}\left(\frac{1 + (F-1)^{p^\ell}}{[p]^\ell F}\right)^{\frac{1}{p^\ell \Lambda_2^{p^\ell}}\Phi_{\ell-1}(\mathbb{A})}$$
$$\in \mathbb{Z}_{(p)}[\mathbb{A}, \frac{\mathbb{U}}{\Lambda_2}, \Lambda_1, \Lambda_2][[X, Y]]$$

For a series of variables $\Lambda_1, \Lambda_2, \dots$, and a series of vectors $\mathbb{A}_j^i = (A_{j0}^i, A_{j1}^i, \dots)$ $(1 \leq i;\ 1 \leq j \leq i)$, we denote

$$\mathbb{A}^i = (\mathbb{A}_\ell^i)_{1 \leq \ell \leq i} := \begin{pmatrix} \mathbb{A}_1^i \\ \mathbb{A}_2^i \\ \vdots \\ \mathbb{A}_i^i \end{pmatrix} \quad \text{and} \quad (\Lambda_\ell)_{1 \leq \ell \leq i} := \begin{pmatrix} \Lambda_1 \\ \Lambda_2 \\ \vdots \\ \Lambda_i \end{pmatrix}.$$

We define vectors $\mathbb{B}_j^i$ $(1 \leq j < i)$ inductively by

$$\mathbb{B}_1^2 := \frac{1}{\Lambda_2} F^{(\Lambda_1)} \mathbb{A}_1^1,$$

and for $k \geq 2$,

$$\begin{cases} \mathbb{B}_j^{k+1} := \frac{1}{\Lambda_{k+1}}\left(F^{(\Lambda_j)}\mathbb{A}_j^k - \sum_{\ell=j+1}^k T_{\mathbb{B}_j^\ell}\mathbb{A}_\ell^k\right) & 1 \leq j \leq k-1 \\ \mathbb{B}_k^{k+1} := \frac{1}{\Lambda_{k+1}} F^{(\Lambda_k)} \mathbb{A}_k^k. \end{cases}$$

Using these symboles, we define triangle matrices U^n's by

$$U^n := \begin{pmatrix} F^{(\Lambda_1)} & -T_{\mathbf{B}_1^2} & -T_{\mathbf{B}_1^3} & \cdots & -T_{\mathbf{B}_1^n} \\ 0 & F^{(\Lambda_2)} & -T_{\mathbf{B}_2^3} & \cdots & -T_{\mathbf{B}_2^n} \\ 0 & 0 & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & -T_{\mathbf{B}_{n-1}^n} \\ 0 & 0 & \cdots & 0 & F^{(\Lambda_n)} \end{pmatrix}.$$

We define inductively a series of formal power series $D_k(X_0, X_1, \dots, X_{k-1})$'s by

$$D_0 = 1,$$
$$D_1(X_0) = E_p(\mathbb{A}_1^1, \Lambda_1; X_0),$$

and for $k \geq 1$,

$$\begin{aligned} D_{k+1}(X_0, X_1, \dots, X_k) &= E_p(\mathbb{A}^{k+1}, (\Lambda_\ell)_{1\leq \ell\leq k+1}; X_0, X_1, \dots, X_k) \\ &:= \prod_{i=1}^{k+1} E_p(\mathbb{A}_i^{k+1}, \Lambda_i; \frac{X_{i-1}}{D_{i-1}(X_0, \dots, X_{i-1})}). \end{aligned}$$

Hereafter, we put $\mathbb{X} = (X_0, X_1, \dots)$, $\mathbb{Y} = (Y_0, Y_1, \dots)$ and $\mathbf{\Sigma} := \mathbb{X} \dot{+} \mathbb{Y} \in W$. We define

$$\begin{aligned} F^{(k)} &:= \partial(D_k(\mathbb{X})) = \frac{D_k(\mathbb{X})D_k(\mathbb{Y})}{D_k(\mathbf{\Sigma})} \\ H_k(\mathbb{X}, \mathbb{Y}) &:= \frac{1}{\Lambda_{k+1}}(F^{(k)} - 1) \\ F_p(\mathbb{V}_1, \Lambda_1; \mathbb{X}, \mathbb{Y}) &:= F_p(\mathbb{V}_1, \Lambda_1; X_0, Y_0) \\ F_p((\mathbb{V}_i)_{1\leq i\leq n}&, (\Lambda_i)_{1\leq i\leq n}; \mathbb{X}, \mathbb{Y}) \\ &= \prod_{i=1}^{n} F_p(\mathbb{V}_i, \Lambda_i; \frac{X_{i-1}}{D_{i-1}(\mathbb{X})}, \frac{Y_{i-1}}{D_{i-1}(\mathbb{Y})}) \\ &\times \prod_{i=2}^{n} F_p(\mathbb{V}_i, \Lambda_i; H_{i-1}, \frac{X_{i-1}}{D_{i-1}(\mathbb{X})} \dot{+} \frac{Y_{i-1}}{D_{i-1}(\mathbb{Y})}) \\ &\times \prod_{i=2}^{n} G_p(\mathbb{V}_i, \Lambda_i; F^{(i-1)})^{-1}. \end{aligned}$$

Then the important thing is the following result.

Theorem 4.2. *For each $n \geq 1$, we have*

$$F^{(n)} = \frac{D_n(\mathbb{X})D_n(\mathbb{Y})}{D_n(\mathbf{\Sigma})} = F_p(U^n\mathbb{A}^n, (\Lambda_i)_{1\leq i\leq n}; \mathbb{X}, \mathbb{Y}).$$

By using this theorem, we can obtain the explicit determination of $\mathrm{Ext}^1(\mathcal{W}_n, \mathcal{G}^{(\lambda)})$. in fact, let $(A, \mathfrak{m})$ be a DVR dominating $\mathbb{Z}_{(p)}$, and $\lambda, \lambda_1, \lambda_2, \dots$ be non-zero elements of $\mathfrak{m}$. We choose Witt vectors

$$\overline{\boldsymbol{a}}^i = (\overline{\boldsymbol{a}}_j^i)_{1\leq j\leq i} \in \mathrm{Ker}\left(U^i : \widehat{W}(A/\lambda_{i+1})^i \to \widehat{W}(A/\lambda_{i+1})^i\right)$$

inductively by the following recursive conditions:

$$U^1 = F^{(\lambda_1)},$$
$$\overline{\boldsymbol{a}}^1 = \overline{\boldsymbol{a}}_1^1 \in \operatorname{Ker}\left(U^1 : \widehat{W}(A/\lambda_2) \to \widehat{W}(A/\lambda_2)\right),$$
$$\boldsymbol{b}_1^2 = \frac{1}{\lambda_2}\boldsymbol{a}_1^1, \quad U^2 = \begin{pmatrix} F^{(\lambda_1)} & -T_{\boldsymbol{b}_1^2} \\ 0 & F^{(\lambda_2)} \end{pmatrix},$$

and for $k \geq 2$, we choose

$$\overline{\boldsymbol{a}}^k = (\overline{\boldsymbol{a}}_i^k)_{1 \leq i \leq k} \in \operatorname{Ker}\left(U^k : \widehat{W}(A/\lambda_{k+1})^k \to \widehat{W}(A/\lambda_{k+1})^k\right),$$

and we define

$$\begin{cases} \boldsymbol{b}_j^{k+1} := \frac{1}{\lambda_{k+1}}\left(F^{(\lambda_j)}\boldsymbol{a}_j^k - \sum_{\ell=j+1}^k T_{\boldsymbol{b}_j^\ell}\boldsymbol{a}_\ell^k\right) & 1 \leq j \leq k-1 \\ \boldsymbol{b}_k^{k+1} := \frac{1}{\lambda_{k+1}}F^{(\lambda_k)}\boldsymbol{a}_k^k, \end{cases}$$

$$U^{k+1} := \begin{pmatrix} F^{(\lambda_1)} & -T_{\boldsymbol{b}_1^2} & -T_{\boldsymbol{b}_1^3} & \cdots & -T_{\boldsymbol{b}_1^{k+1}} \\ 0 & F^{(\lambda_2)} & -T_{\boldsymbol{b}_2^3} & \cdots & -T_{\boldsymbol{b}_2^{k+1}} \\ 0 & 0 & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & -T_{\boldsymbol{b}_k^{k+1}} \\ 0 & 0 & \cdots & 0 & F^{(\lambda_{k+1})} \end{pmatrix},$$
$$\overline{\boldsymbol{a}}^{k+1} = (\overline{\boldsymbol{a}}_i^{k+1})_{1 \leq i \leq k+1} \in \operatorname{Ker}\left(U^{k+1} : \widehat{W}(A/\lambda_{k+2})^{k+1} \to \widehat{W}(A/\lambda_{k+2})^{k+1}\right).$$

We define formal power series $D_k(\mathbb{X}) = D_k(X_0, \dots, X_{k-1})$ $(k \geq 1)$ by

$$D_0 = 1,$$
$$D_1(X_0) = E_p(\boldsymbol{a}_1^1, \lambda_1; X_0),$$

and for $k \geq 1$,

$$D_{k+1}(X_0, X_1, \dots, X_k) = E_p(\boldsymbol{a}^{k+1}, (\lambda_\ell)_{1 \leq \ell \leq k+1}; X_0, X_1, \dots, X_k)$$
$$:= \prod_{i=1}^{k+1} E_p(\boldsymbol{a}_i^{k+1}, \lambda_i; \frac{X_{i-1}}{D_{i-1}(X_0, \dots, X_{i-1})}).$$

We put

$$\mathcal{W}_n := \operatorname{Spec} A[X_0, \dots, X_{n-1}, \frac{1}{1+\lambda_1 X_0}, \frac{1}{D_1(X_0) + \lambda_2 X_1}, \dots, \frac{1}{D_{n-1}(\mathbb{X}) + \lambda_n X_{n-1}}].$$

Theorem 4.3 (Explicit Formula in General Case). *Let $B = A/\lambda$. Then we have*

$$\xi_0^n : \operatorname{Ker}(\widehat{W}(B)^n \xrightarrow{U^n} \widehat{W}(B)^n) \xrightarrow{\sim} \operatorname{Hom}(\mathcal{W}_{n,B}, \mathbb{G}_{m,B});$$
$$\overline{\boldsymbol{v}}^n = (\overline{\boldsymbol{v}}_i^n)_{1 \leq i \leq n} \mapsto E_p(\overline{\boldsymbol{v}}^n, (\lambda_i)_{1 \leq i \leq n}; X_0, X_1, \dots, X_{n-1})$$
$$\xi_1^n : \operatorname{Coker}(\widehat{W}(B)^n \xrightarrow{U^n} \widehat{W}(B)^n) \xrightarrow{\sim} \operatorname{H}_0^2(\mathcal{W}_{n,B}, \mathbb{G}_{m,B}).$$
$$\overline{\boldsymbol{w}}^n = (\overline{\boldsymbol{w}}_i^n)_{1 \leq i \leq n} \mapsto F_p(\overline{\boldsymbol{w}}^n, (\lambda_i)_{1 \leq i \leq n}; \mathbb{X}, \mathbb{Y})$$

Theorem 4.4.

$$\overline{\xi}_0^n : \frac{\mathrm{Ker}(U^n : \widehat{W}(B)^n \to \widehat{W}(B/\lambda)^n)}{< c^0, c^1, \dots , c^{n-1} >} \xrightarrow{\sim} \mathrm{Ext}^1(\mathcal{W}_{n,B}, \mathcal{G}_B^{(\lambda)}),$$

where $< c^0, c^1, \dots , c^{n-1} >$ *is the subgroup generated by the vectors* $c^0 = (\widetilde{\lambda}_1, 0, \dots , 0), c^1 = (a^1, \widetilde{\lambda}_2, 0, \dots , 0), \dots , c^\ell = (a^\ell, \widetilde{\lambda}_{\ell+1}, 0, \dots , 0), \dots , c^{n-1} = (a^{n-1}, \widetilde{\lambda}_n)$.

5. Reductions of Extensions

The special fibres of the group schemes of type $(\lambda_1, \lambda_2, \dots , \lambda_n)$ can be decided as follows.

Theorem 5.1. *Let*

$$\mathcal{W}_n = \mathrm{Spec}\, A[X_0, X_1, \dots , X_{n-1}, \frac{1}{1 + \lambda_1 X_0}, \frac{1}{D_1(X_0) + \lambda_2 X_1}, \dots , \frac{1}{D_{n-1}(\mathbb{X}) + \lambda_n X_{n-1}}]$$

be the group scheme of type $(\lambda_1, \lambda_2, \dots , \lambda_{n-1})$ *defined by*

$$D_1(X_0) = E_p(a_1^1, \lambda_1; X_0)$$

and for $1 \le k \le n-2$,

$$D_{k+1}(\mathbb{X}) = E_p(a^{k+1}, (\lambda_\ell)_{1 \le \ell \le k+1}; X_0, X_1, \dots , X_k),$$

and

$$\overline{a}^k \in \mathrm{Ker}(U^k : \widehat{W}(A/\lambda_{k+1})^k \to \widehat{W}(A/\lambda_k)^k).$$

Here

$$b^i = {}^t(\, b_1^i, b_2^i, \dots , b_{i-1}^i) = \frac{1}{\lambda_i} U^{i-1} a^{i-1} \qquad (i = 2, \dots , n).$$

If $b_\ell^k \equiv 0 \pmod{\mathfrak{m}}$ *for* $3 \le k \le n$, $1 \le \ell \le k-2$, *and* $b_{k-1}^k \equiv (1, 0, , \dots) \pmod{m}$, *then we have*

$$\mathcal{W}_{n,k} = \mathcal{W}_n \otimes_A k = W_{n,k}.$$

6. Conditions for KRSW group schemes

Let

$$\mathcal{W}_n = \mathrm{Spec}\, A[X_0, \dots , X_{n-1}, \frac{1}{1 + \lambda_{(1)} X_0}, \frac{1}{D_1(X_0) + \lambda_{(1)} X_1}, \dots , \frac{1}{D_{n-1}(X_0, \dots , X_{n-2}) + \lambda_{(1)} X_{n-1}}]$$

be a KASW group scheme over a DVR $(A, \mathfrak{m})$, and λ be an element of $\mathfrak{m} \setminus \{0\}$. Here D_i's are given by

$$D_i(\mathbb{X}) = E_p(a^i, (\lambda_{(1)})^n; \mathbb{X})$$

with

$$\overline{a}^i \in \mathrm{Ker}(U^i : \widehat{W}(A/\lambda_{(1)}) \to \widehat{W}(A/\lambda_{(1)})).$$

We look at the exact sequence

$$0 \to (\mathbb{Z}/p^n)_A \xrightarrow{i_n} \mathcal{W}_n \xrightarrow{\psi_n} \mathcal{W}_n/(\mathbb{Z}/p^n)_A \to 0.$$

Then we have

$$\begin{array}{ccc} i_n^* : \mathrm{Ext}^1_A(\mathcal{W}_n, \mathcal{G}^{(\lambda)}) & \longrightarrow & \mathrm{Ext}^1_A((\mathbb{Z}/p^n)_A, \mathcal{G}^{(\lambda)}) \\ \| \wr & & \| \wr \\ \frac{\mathrm{Ker}(\widehat{W}(A/\lambda)^n \xrightarrow{U^n} \widehat{W}(A/\lambda))}{<c^0, c^1, \dots, c^{n-1}>} & \longrightarrow & (1+\lambda A) \big/ (1+\lambda A)^{p^n} \\ \boldsymbol{a}^n & \mapsto & \prod_{r>0} \left(E_p(a^n_{1,r}, \lambda^{p^r}_{(1)}; 1)^{p^n} \prod_{i=2}^n E_p(a^n_{i,r}, \lambda^{p^r}_{(1)}; \left(\frac{c_{i-1}}{D_{i-1}(i_n(1))}\right)^{p^r})^{p^n} \right) \end{array}$$

Under these notations, we have the following.

Theorem 6.1. *Let $\mathcal{W}_{n+1} \in \mathrm{Ext}^1(\mathcal{W}_n, \mathcal{G}^{(\lambda)})$ be the extension corresponding to a vector $\boldsymbol{a}^n = (a_i^n)_{1 \le i \le n}$ by the isomorphism*

$$\frac{\mathrm{Ker}(U^n : \widehat{W}(A/\lambda)^n \to \widehat{W}(A/\lambda)^n)}{(\boldsymbol{c}^1, \boldsymbol{c}^2, \dots, \boldsymbol{c}^{n-1})} \xrightarrow{\sim} \mathrm{Ext}^1(\mathcal{W}_n, \mathcal{G}^{(\lambda)}).$$

Then there exists an inclusion $(\mathbb{Z}/p^{n+1})_A \subset \mathcal{W}_{n+1}$ fitting into a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (\mathbb{Z}/p)_A & \longrightarrow & (\mathbb{Z}/p^{n+1})_A & \longrightarrow & (\mathbb{Z}/p^n)_A & \longrightarrow & 0 \\ & & \downarrow i_1 & & \downarrow i_{n+1} & & \downarrow i_n & & \\ 0 & \longrightarrow & \mathcal{G}^{(\lambda)} & \longrightarrow & \mathcal{W}_{n+1} & \longrightarrow & \mathcal{W}_n & \longrightarrow & 0, \end{array}$$

if and only if

$$E_p(\boldsymbol{a}^n, (\lambda, \dots, \lambda); i_n(1))^{p^n} = \zeta_1.$$

Using these results, we construct explicitly the KASW group schemes.

Theorem 6.2 (Main Theorem). *For each positive integer n, we construct explicitly a standard KASW group scheme $\mathcal{W}_n$ over $\mathbb{Z}_{(p)}[\mu_{p^n}]$.*

Finally, we remark that for a KASW group scheme $\mathcal{W}_n$, the quotient $\mathcal{V}_n := \mathcal{W}_n/(\mathbb{Z}/p^n)$ is a group scheme of type $(\lambda^p_{(1)})^n$, and which is given explicitly.

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