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L^p estimates for some Schrödinger type operators

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Abstract

We consider the Schrödinger operator $L=-\Delta+V$ with non-negative potentials V on $\mathbf{R}^n$, $n\geq 3$. We assume that the potential V belongs to the reverse Hölder class which includes non-negative polynomials. We show the L^p estimates for the operators V^kL^{-k} and $V^{k-1/2}\nabla L^{-k}$, where k is a positive integer.

1 Introduction

In this paper we consider the Schrödinger operator $L=-\Delta+V$ on $\mathbf{R}^n$, $V\geq 0$, $n\geq 3$. When V is a non-negative polynomial, Zhong ([Zh]) proved that the operators V^kL^{-k} and $V^{k-1/2}\nabla L^{-k}$, $k\in \mathbf{N}$, are bounded on L^p, $1<p\leq\infty$. For the potential V which belongs to the reverse Hölder class, which includes non-negative polynomials, Shen ([Sh]) generalized Zhong's results. Actually, he proved that the operators VL^{-1} and $V^{1/2}\nabla L^{-1}$ are bounded on $L^p(\mathbf{R}^n)$, $1\leq p\leq\infty$.

For the operator L with potentials V which belong to the reverse Hölder class, Kurata and the author generalized Shen's results as follows. In [KS1], we replace Δ by the second order uniformly elliptic operator $L_0=-\sum_{i,j=1}^n(\partial/\partial x_i)\{a_{ij}(x)(\partial/\partial x_j)\}$ and assume certain assumptions for a_{ij}. Then we showed that the operators $V(L_0+V)^{-1}$ and $V^{1/2}\nabla(L_0+V)^{-1}$ are bounded on weighted L^p space $(1<p<\infty)$ and Morrey spaces. Moreover, in [Su], the auther showed weighted L^p-L^q estimates of the operators $V^\alpha L^{-\beta}$ and $V^\alpha\nabla L^{-\beta}$ (α, $\beta\in(0,1]$) and their boundedness on Morrey spaces.

The purpose of this paper is to show the L^p boudedness of the operators V^kL^{-k} and $V^{k-1/2}\nabla L^{-k}$, $k\in\mathbf{N}$, where V belongs to the reverse Hölder class.

We shall repeat the definitions of the reverse Hölder class (e.g.[Sh]). Throughout this paper we denote by $B_r(x)$ the ball centered at x with radius r, and the letter C stands for a constant not necessarily the same at each occurrence.

Definition 1 (Reverse Hölder class) *Let* $V \geq 0$.

(1) *For* $1 < p < \infty$ *we say* $V \in (RH)_p$, *if* $V \in L^p_{loc}(\mathbf{R}^n)$ *and there exists a constant* C *such that*

$$\left(\frac{1}{|B_r(x)|}\int_{B_r(x)} V(y)^p dy\right)^{1/p} \leq \frac{C}{|B_r(x)|}\int_{B_r(x)} V(y)dy \tag{1}$$

holds for every $x \in \mathbf{R}^n$ *and* $0 < r < \infty$.

(2) *We say* $V \in (RH)_\infty$, *if* $V \in L^\infty_{loc}(\mathbf{R}^n)$ *and there exists a constant* C *such that*

$$\|V\|_{L^\infty(B_r(x))} \leq \frac{C}{|B_r(x)|}\int_{B_r(x)} V(y)dy \tag{2}$$

holds for every $x \in \mathbf{R}^n$ *and* $0 < r < \infty$.

Remark 1 If $P(x)$ is a polynomial and $\alpha > 0$, then $V(x) = |P(x)|^\alpha$ belongs to $(RH)_\infty$ ([Fe]). For $1 < p < \infty$, it is easy to see $(RH)_\infty \subset (RH)_p$.

In [Zh], Zhong proved the L^p estimates of the operators $V^k L^{-k}$ and $V^{k-1/2}\nabla L^{-k}$ with non-negative polynomials V by using the k times composition of the Hardy-Littlewood maximal operator M. In [KS1] we considered the uniformly elliptic operators L_0 and proved a pointwise bound $|Tf(x)| \leq CM(|f|)(x)$ where Mf is Hardy-Littlewood maximal function and T is either $V(L_0+V)^{-1}$ or $V^{1/2}\nabla(L_0+V)^{-1}$. Pointwise estimates are also used by Zhong in the polynomial case. Once we have these pointwise estimates the boundedness of these operators in any spaces on which the Hardy-Littlewood maximal operator is known to be bounded. Examples are weighted L^p space and Morrey spaces.

In this paper we establish pointwise estimates (see Lemma 3) which generalize Zhong's estimates we mentioned above. By using them we show the L^p boundedness of these operators (see Theorem 1).

We denote by $\Gamma(x,y)$ the fundamental solution for L. The operator L^{-1} is the integral operator with $\Gamma(x,y)$ as its kernel. Let $f \in C_0^\infty(\mathbf{R}^n)$. Then we have $L^{-1}f \in L^p(\mathbf{R}^n)$ for $1 \leq p \leq \infty$. For any integer $k \geq 2$, we define L^{-k} as follows.

$$L^{-k}f(x) = \int_{\mathbf{R}^n} \Gamma(x,y)L^{-(k-1)}f(y)dy.$$

Now we state our theorem.

Theorem 1 *Suppose* $V \in (RH)_\infty$. *Then there exist constants* C, C' *such that*

$$\|V^k L^{-k} f\|_{L^p(\mathbf{R}^n)} \leq C\|f\|_{L^p(\mathbf{R}^n)} \quad \text{for} \quad f \in C_0^\infty(\mathbf{R}^n), \tag{3}$$

$$\|V^{k-1/2}\nabla L^{-k} f\|_{L^p(\mathbf{R}^n)} \leq C'\|f\|_{L^p(\mathbf{R}^n)} \quad \text{for} \quad f \in C_0^\infty(\mathbf{R}^n), \tag{4}$$

where $1 < p \leq \infty$ *and* $k \in \mathbf{N}$.

Remark 2 In Theorem 1 the case $k = 1$ was shown in [Sh, Remark 2.9, Theorem 4.13].

The plan of this paper is as follows. In section 2, we recall Shen's lemmas which we use to prove Theorem 1. In section 3, we prove Theorem 1.

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2 Preliminaries

In [Sh], Shen defined the auxiliary function $m(x,V)$ and established the estimates of the fundamental solution of L (see Lemma 1). By using the estimates he proved L^p boundedness of the operators VL^{-1} and $V^{1/2}\nabla L^{-1}$. We also need them to prove our theorem.

We recall the definition of the function $m(x,V)$.

Definition 2 ([Sh, Definition 1.3]) *Let* $V \in (RH)_{n/2}$ *and* $V \not\equiv 0$. *Then it is well-known that there exists* $\epsilon > 0$ *such that* $V \in (RH)_{n/2+\epsilon}$ ([Ge]). *Then the function* $m(x,V)$ *is well-defined by*

$$\frac{1}{m(x,V)} = \sup\left\{r > 0 : \frac{r^2}{|B_r(x)|}\int_{B_r(x)} V(y)dy \leq 1\right\}$$

and satisfies $0 < m(x,V) < \infty$ *for every* $x \in \mathbf{R}^n$.

Remark 3 If $V \in (RH)_\infty$ then there exists a constant C such that $V(x) \leq Cm(x,V)^2$ ([Sh, Remark 2.9]).

We recall the estimates of the fundamental solution for L.

Lemma 1 ([Sh])

(1) *Suppose* $V \in (RH)_{n/2}$. *Then for any positive integer* N *there exists a constant* C_N *such that*

$$(0 \leq)\Gamma(x,y) \leq \frac{C_N}{\{1 + m(x,V)|x-y|\}^N} \cdot \frac{1}{|x-y|^{n-2}}.$$

(2) *Suppose* $V \in (RH)_n$. *Then for any positive integer* N *there exists a constant* C_N *such that*

$$|\nabla_x \Gamma(x,y)| \leq \frac{C_N}{\{1+m(x,V)|x-y|\}^N} \cdot \frac{1}{|x-y|^{n-1}}.$$

The following Lemma is also needed to prove our theorem.

Lemma 2 ([Sh, Lemma 1.4(c)]) *Suppose* $V \in (RH)_{n/2}$. *Then there exist positive constants* C, k_0 *such that*

$$m(y,V) \geq \frac{Cm(x,V)}{\{1+m(x,V)|x-y|\}^{k_0/(k_0+1)}}.$$

3 Proof

Theorem 1 is easily proved by the following pointwise estimates. These estimates generalize the results in [Zh, Lemma 3.2] to the Schrödinger operators with reverse Hölder class potentials.

Lemma 3 *Let* k *be a positeve integer. The opeator* M^k *stands for the* k *times composition of the Hardy-Littlewood maximal operator* M.

(1) *Suppose* $V \in (RH)_{n/2}$. *Then there exist a constant* C *such that*

$$|m(x,V)^{2k} L^{-k} f(x)| \leq CM^k(|f|)(x) \quad \textit{for} \quad f \in C_0^\infty(\mathbf{R}^n). \tag{5}$$

(2) *Suppose* $V \in (RH)_n$. *Then there exist a constant* C *such that*

$$|m(x,V)^{2k-1} \nabla L^{-k} f(x)| \leq CM^k(|f|)(x) \quad \textit{for} \quad f \in C_0^\infty(\mathbf{R}^n). \tag{6}$$

Remark 4 In Lemma 3 the case $k=1$ was shown in [KS, Theorem 1.3].

Proof of Theorem 1. Since $V(x) \leq Cm(x,V)^2$, estimate (3) immediately follows from (5) and the fact that the Hardy-Littlewood maximal operator is bounded on $L^p(\mathbf{R}^n)$, $1 < p \leq \infty$. The proof of (4) can be done in the same way as above by using (6). □

Proof of Lemma 3. Let $f \in C_0^\infty(\mathbf{R}^n)$. We prove estimate (5) by induction on k. For the proof of the case $k = 1$, see [KS1, Theorem 1.3]. We assume it is true for $k = l$, that is, there exists a constant C such that

$$|m(x,V)^{2l}L^{-l}f(x)| \leq CM^l(|f|)(x) \tag{7}$$

and show the case $k = l + 1$. It follows from Lemma 1 (1) and Lemma 2 that

$$\begin{aligned}
& |m(x,V)^{2(l+1)}L^{-(l+1)}f(x)| \\
\leq\ & \left|Cm(x,V)^2 \int_{\mathbf{R}^n} \Gamma(x,y)m(x,V)^{2l}L^{-l}f(y)dy\right| \\
\leq\ & CC_N m(x,V)^2 \int_{\mathbf{R}^n} \frac{\{1+m(x,V)|x-y|\}^{2lk_0/(k_0+1)}|m(y,V)^{2l}L^{-l}f(y)|}{\{1+m(x,V)|x-y|\}^N|x-y|^{n-2}}dy.
\end{aligned}$$

Therefore we obtain the desired estimate in the same way as the case $k = 1$.

The proof of (6) can be done in the same way as the proof of (5) by using Lemma 1 (2). □

Remark 5 Let $s \in (0,\infty)$. We can obtain the estimate for the operator V^sL^{-s} as follows. Suppose $V \in (RH)_{n/2}$ and $\alpha \in (0,1]$. Then there exists a constant C such that

$$|m(x,V)^{2\alpha}L^{-\alpha}f(x)| \leq CM(|f|)(x) \quad \text{for} \quad f \in C_0^\infty(\mathbf{R}^n) \tag{8}$$

(see [Su Theorem 1]). Combining (8) and the argument in the proof of Lemma 3, we arrive at the following pointwise estimate:

$$|m(x,V)^{2s}L^{-s}f(x)| \leq CM^{s^*}(|f|)(x) \quad \text{for} \quad f \in C_0^\infty(\mathbf{R}^n), \tag{9}$$

where $s \in (0,\infty)$ and

$$s^* = \begin{cases} s, & \text{if } \ s \ \text{ is an integer,} \\ [s]+1, & \text{otherwise,} \end{cases}$$

where $[s]$ is the largest integer smaller than or equal to s. We should remark that, for the case V is a non-negative polynomial, Zhong proved the L^p boundedness (only for $1 < p < \infty$) of the operator V^sL^{-s}, $s \in (0,\infty)$ ([Zh, Corollary 1.5]).

Remark 6 Zhong also showed that the L^p estimate of the operator $V^{k-q/2}\Delta^{q/2}L^{-k}$ with non-negative polynomials V, where q and k are positive integers and $2 \leq q \leq 2k$ ([Zh, Theorem 1.3]). He proved this results by using the fact that the functions

$m(x,V)^{2k}L^{-k}f(x)$ and $m(x,V)^{2k-1}\nabla L^{-k}f(x)$ are bounded by the k times composition of the Hardy-Littlewood maximal function and there exists a constant C such that

$$|\Delta^{q/2}V(x)| \leq Cm(x,V)^{q+2} \tag{10}$$

which holds for non-negative polynomials V. Hence if we assume the inequality (10), we can obtein the L^p estimate of the operator $V^{k-q/2}\Delta^{q/2}L^{-k}$ with potentials V which belong to the reverse Hölder class in the same way as for polynomial potentials by using Lemma 3 and the assumption (10).

Remark 7 Shen proved that the operator $\nabla^2 L^{-1}$ is bounded on L^p, $1 < p < \infty$ ([Sh]). In [KS1] Kurata and the author extended this result to the uniformly elliptic operators. They also showed that the estimate for the kernel of the opeator $\nabla^2 L^{-1}$ ([KS2]). However, it is not known that whether the operator $V^{k-1}\nabla^2 L^{-k}$, $k \geq 2$ is bounded on L^p or not.

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