数理解析研究所講究録 1201 巻 2001 年 26-48

Absence of eigenvalues of time harmonic Maxwell equations

京都大学大学院理学研究科 大鍛治 隆司 (Takashi Ōkaji)

1 Introduction

It is well known that the eigenvalue problem for the Lapalce operator

$$-\Delta u=ku, \quad k>0 \tag{1.1}$$

in an exterior domain U of $\mathbf{R}^{d}$ has no positive eigenvalue. Indeed,

Theorem 1.1 (Rellich (1943)) *Let u be a solution to (1.1) belonging to $L^{2}(U)$. If $k>0$, then u is identically zero.*

T. Kato (1959) extended this result to the Schrödinger equation

$$-\Delta u+q(x)u=ku, x\in U, \tag{1.2}$$

where $k>0$ and

$$q(x)=o(|x|^{-1}), \qquad |x|\rightarrow\infty.$$

In addition, his result is generalized to a class of second order elliptic equations (Agmon, Simon, Jäger, Ikebe-Uchiyama).

On the other hand, an analogue to Rellich's theorem holds for symmetric elliptic systems. This result was shown by P.D.Lax and R.S.Phillips when d is odd and by N. Iwasaki when d is even. It is natural to ask whether an analogue to Kato's result holds for such systems or not. As for Dirac operators, many works are devoted to the study of this problem ([8], [21], [18] and [9]).

In this paper, we focus our attention to optical systems in general inhomogeneous media. We do not use the usual second order approach found in the works of [4], [13] and [17]. The second order approach is to convert such system into a system of second order, so that it requires that the coefficients belongs to the C^{2} class. Contrary to this, the first order approach we shall take requires only C^{1} regularity for the coefficients. Our strategy for proving absence of eigenvalues is similar to Vogelsang's one. Namely, we shall use weighted L^{2} estimates to prove absence of eigenvalues while T. Kato used differential inequalities of surface integrals of solutions to show the nonexistence of positive eigenvalues. As a result, we can greatly improve the known result ([4]).

We would like to mention that our problem is local one around infinity because it bears no relation to boundary conditions. In fact, as Kato has pointed out, if we transform the variables by inversion wih respect to the unit sphere according to

$$y = x/|x|^2, \quad v(y) = |x|^{n-2}u(x),$$

(1.2) is transformed into

$$-\Delta_y v + |y|^{-4}\left\{q(y/|y|^2) - k\right\} u = 0.$$

The potential of the above equation has stronger singularity than the usual one appeared in the strong unique continuation theory.

Finaly, as an important consequence of results on absence of eigenvalues, we can show local decay property of nonstatic solutions $U(t) = e^{-itA}u_0$ to the corresponding time evolution equation ([12]).

2 Maxwell operators

Let ε and μ be 3×3 real symmetric matrices defined in an exterior domain U of $\mathbf{R}^3$. They are supposed to be uniformly positive definite in U: There exists a positive constant δ_0 such that

(2.1) $$(\varepsilon(x)\zeta, \zeta) \geq \delta_0|\zeta|^2, \quad (\mu(x)\zeta, \zeta) \geq \delta_0|\zeta|^2, \quad \forall\zeta \in \mathbf{C}^3, \ \forall x \in U.$$

Let us define two 6×6 matrices as follows:

$$A = \begin{pmatrix} 0 & \text{curl} \\ -\text{curl} & 0 \end{pmatrix} \quad \text{and} \quad \Gamma = \begin{pmatrix} \varepsilon(x) & 0 \\ 0 & \mu(x) \end{pmatrix}.$$

The Maxwell equations are written as

$$\partial_t \Gamma u = Au, \quad u =^t (E, H),$$

where E and H are $\mathbf{C}^3$-valued unknown functions. We are concerned with existence of their particular solutions of the form

$$u(x,t) = e^{i\lambda t}u(x), \quad \lambda \in \mathbf{R}\backslash\{0\}.$$

The new unknown function $u(x)$ should satisfy the time harmonic Maxwell equation:

$$Au = i\lambda\Gamma u. \tag{2.2}$$

We define new unknown functions $\tilde{u}$ as

$$\tilde{u} = {}^t(\varepsilon^{1/2}E, \mu^{1/2}H)$$

and set

$$\tilde{A} = \begin{pmatrix} 0 & \varepsilon^{-1/2}\mathrm{curl}\mu^{-1/2} \\ -\mu^{-1/2}\mathrm{curl}\varepsilon^{-1/2} & 0 \end{pmatrix}.$$

Then, it is easily verified that (2.2) is equivalent to the standard form of eigenvalue problems:

$$\tilde{A}\tilde{u} = i\lambda\tilde{u}.$$

To describe our conditions, we introduce the function space $\mathcal{M}(U)$ as the set of all real positive symmetric matrices of third order whose components are continuously differentiable functions in U satisfying that there exist a symmetrix matrix $F_\infty(x) \in C^1(U)^{3\times 3}$ and a positive constant F_0 such that as $|x| \to \infty$

$$F(x) - F_\infty(x) = o(|x|^{-1}), \;\; F_\infty(x) - F_0 I = o(|x|^{-1/2}) \tag{2.3}$$

and

$$\nabla F(x) = o(|x|^{-1}). \tag{2.4}$$

Theorem 2.1 *Suppose that ε and μ belong to $\mathcal{M}(U)$ and there exists a positive constant κ such that*

$$\varepsilon_\infty(x) = \kappa\mu_\infty(x),$$

for all x in a neighborhood of infinity. If $u \in H^1_{\mathrm{loc}}(U) \cap L^2(U)$ is a solution to (2.2), then u has a compact support.

Corollary 2.2 *In addition to the assumptions of Theorem 2.1, we assume that there exists a scalar function $\kappa \in C^1(U)$ such that $\varepsilon(x) = \kappa(x)\mu(x)$. If $u \in H^1_{\mathrm{loc}}(U) \cap L^2(U)$ is a solution to (2.2), then u is identically zero in U.*

Remark 2.1 *If $u \in L^2(U)$ is a solution to (2.2), then $u \in H^1_{\mathrm{loc}}(U)$.*

For the isotropic case, we can show a sharper result. To state it, we prepare some notations. Let I_a be an interval $[a,\infty)$ for $a \geq 0$. We denote the positive part and the negative part of a real-valued function f defined in I_a by $[f]_+$ and $[f]_-$, respectively:

$$[f]_+ = \max(0, f(r)), \quad [f]_- = \max(0, -f(r)).$$

In what follows, f' denotes the derivative of $f(r)$. Define the subset $m(I_a)$ of $C^1(I_a)$ as

$$(2.5) \quad m(I_a) = \{q(r) \in C^1(I_a; \mathbf{R});\ \lim_{r\to\infty} q(r) = q_\infty > 0, \\ q'(r) = o(r^{-1/2}),\ [q']_- = o(r^{-1}),\ \}.$$

For $a > 0$, define $D_a = \{x \in \mathbf{R}^3;\ |x| > a\}$. Henceforth, we always choose a so large that $D_a \subset U$. We shall use the polar coordinates, $r = |x|$, $\omega = x/|x|$. For $q \in m(I_a)$ with $a > 0$, we say that $f(x) \in C^1(U)^{3\times 3}$ belongs to the class $S(q)$ if

$$(2.6) \qquad \partial_r^j(f(x) - q(r)) = o((r^{-(j+1)/2}), \qquad j = 0, 1.$$

Theorem 2.3 *Suppose that $\varepsilon(x)$ and $\mu(x)$ are positive scalar functions such that*

$$(2.7) \qquad \varepsilon \in S(q_1),\ \mu \in S(q_2),\quad q_j \in m(I_a),\ q_j' = o(r^{-1}),\ j = 1, 2.$$

If $u \in H^1_{\mathrm{loc}}(U) \cap L^2(U)$ is a solution to (2.2), then u is identically zero in U.

When q_1 is equal to q_2, we can improve the previous result.

Theorem 2.4 *Suppose $q \in C^2(I_a)$ satisfies*

$$(2.8) \quad \inf_{I_a} q(r) > 0,\ [q'(r)]_- = o(r^{-1}q),\ (\frac{d}{dr})^j q(r) = o(r^{-j/2}q^{1+j/2}),\ j = 1, 2.$$

If $\varepsilon(x)$ and $\mu(x)$ are positive scalar functions belonging to $C^1(D_a)$ such that

$$|\partial_r^j(\varepsilon(x) - q(r))| + |\partial_r^j(\mu(x) - \beta q(r))| = o(\left(r^{-1/2}q^{1/2}\right)^{j+1}),\ \forall x \in D_a,\ j = 0, 1$$

for some positive number β, then the conclusion of Theorem 2.3 is still true.

Remark 2.2 *D. Eidus has studied the same problem by the second order approach. He has obtained an analogous result (Theorem 4.4 of [4]) for $U = \mathbf{R}^3$ under the assumption that ε and μ belong to $C^2(\mathbf{R}^3)$ and they satisfy a faster asymptotic property*

$$|\varepsilon - \varepsilon_0| + |\mu - \mu_0| + |\nabla\varepsilon| + |\nabla\mu| = o(|x|^{-1}).$$

Remark 2.3 *A similar result for Dirac operators with the potential growing at infinity has been obtained ([9]).*

We remark that each hypothesis of Theorems 2.1, 2.3 and 2.4 implies that if a is taken to be so large, there exists a positive number κ such that

$$(rV)' > \kappa, \quad \forall x \in D_a. \tag{2.9}$$

If $U = \mathbf{R}^3$ and there exists a positive constant β such that the virial condition

$$\partial_r(r\Gamma)(x) > \beta I, \tag{2.10}$$

holds for all $x \in \mathbf{R}^3$, we can easily show the absence of nonzero eigenvalues. Let $\mathcal{B}^1(U)$ be the subset of $C^1(U)$ consisting of all functions f satisfying

$$|f| + |\nabla f| \in L^\infty(U).$$

Theorem 2.5 *Let $U = \mathbf{R}^3$ and ε, $\mu \in \mathcal{B}^1(\mathbf{R}^3)^{3\times 3}$ satify (2.1). Suppose (2.10). If $u \in L^2(\mathbf{R}^3)$ satisfies (2.2), then $u = 0$ in $\mathbf{R}^3$.*

Remark 2.4 *Theorem 2.5 also improves Theorem 4.4 of [4].*

3 The Polar coordinates

Let $r = |x|$ and $\omega = x/|x|$. It holds

$$\partial_{x_j} = \omega_j \partial_r + r^{-1}\Omega_j,$$

where Ω is a vector field on $\mathbf{S}^2$. Define respectively two important matrices J_ω and J_Ω as $J_\omega u = \omega \wedge u$ and $J_\Omega u = \Omega \wedge u$: It is easily seen that

$$J_\omega = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}, \quad J_\Omega = \begin{pmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{pmatrix}.$$

Lemma 3.1

$$\mathrm{curl} = J_\omega \partial_r + r^{-1} J_\Omega$$

and

$$J_\omega \mathrm{curl} u = -\partial_r u + r^{-1} G u + (\mathrm{div} u)\omega,$$

where G is a selfadjoint operator in $L^2(\mathbf{S}^{d-1})$.

Remark 3.1 *G is given explicitly as*

$$G = \begin{pmatrix} 0 & -L_3 & L_2 \\ L_3 & 0 & -L_1 \\ -L_2 & L_1 & 0 \end{pmatrix},$$

where

$$L_1 = x_2\partial_3 - x_3\partial_2, \quad L_2 = x_3\partial_1 - x_1\partial_3, \quad L_3 = x_1\partial_2 - x_2\partial_1.$$

Let

$$\alpha = \begin{pmatrix} 0 & iI \\ -iI & 0 \end{pmatrix}, \ \mathcal{J}_\omega = \begin{pmatrix} J_\omega & 0 \\ 0 & J_\omega \end{pmatrix}.$$

Define

$$\hat{J}_\Omega = J_\Omega - J_\omega, \ \mathcal{J}_\Omega = \begin{pmatrix} \hat{J}_\Omega & 0 \\ 0 & \hat{J}_\Omega \end{pmatrix}, \ \mathcal{G} = \begin{pmatrix} G+1 & 0 \\ 0 & G+1 \end{pmatrix}.$$

Then we can show the following lemmata.

Lemma 3.2 *If $\tilde{u} = ru$, then it satisfies*

$$\{-\mathcal{J}_\omega\partial_r - r^{-1}\mathcal{J}_\Omega\}\alpha\tilde{u} = \lambda\Gamma\tilde{u}.$$

Proof: The equation (2.2) is equivalent to

$$\begin{pmatrix} \mathrm{curl} & 0 \\ 0 & \mathrm{curl} \end{pmatrix} \alpha u = -\lambda\Gamma u.$$

□

Lemma 3.3 *Suppose that the hypothesis of Theorem 2.3 is fulfilled. Let $v = ru$. It holds that*

$$\{-\mathcal{J}_\omega\partial_r - r^{-1}\mathcal{J}_\Omega\}\alpha v = \lambda\Gamma v \tag{3.1}$$

and

$$\{\partial_r - r^{-1}\mathcal{G} - Q\}\alpha v = \lambda\mathcal{J}_\omega\Gamma v, \tag{3.2}$$

where Q satisfies that

$$Q \in C^0(D_a;\ \mathbf{R})^{6\times 6},\ Q = o(r^{-1/2}). \tag{3.3}$$

Proof: We see that

$$A = \begin{pmatrix} 0 & J_\omega \\ -J_\omega & 0 \end{pmatrix} \partial_r + r^{-1} \begin{pmatrix} 0 & J_\Omega \\ -J_\Omega & 0 \end{pmatrix} \tag{3.4}$$

and

$$\begin{pmatrix} J_\omega & 0 \\ 0 & -J_\omega \end{pmatrix} A = -\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial_r + r^{-1} \begin{pmatrix} 0 & G \\ G & 0 \end{pmatrix} + \begin{pmatrix} 0 & \omega\mathrm{div} \\ \omega\mathrm{div} & 0 \end{pmatrix}. \tag{3.5}$$

Define $Q = Q_1 + Q_2$ with

$$Q_1 \begin{pmatrix} v_+ \\ v_- \end{pmatrix} = \begin{pmatrix} q_1^{-1}(\nabla q_1, v_+)\omega \\ q_2^{-1}(\nabla q_2, v_-)\omega \end{pmatrix},$$

$$Q_2 \begin{pmatrix} v_+ \\ v_- \end{pmatrix} = \begin{pmatrix} \omega\{\varepsilon^{-1}(\nabla \varepsilon, v_+) - q_1^{-1}(\nabla q_1, v_+)\} \\ \omega\{\mu^{-1}(\nabla \mu, v_-) - q_2^{-1}(\nabla q_2, v_-)\} \end{pmatrix}.$$

Then, it follows that $Q_1^* = Q_1$, $Q_2 = o(r^{-1/2})$ and $\partial_r Q_2 = o(r^{-1})$. □

In what follows, we denote the inner product and the norm of $L^2(\mathbf{S}^2)^6$ by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Then, we note that

$$\langle \hat{\mathcal{J}}_\Omega v, v \rangle = \langle v, \hat{\mathcal{J}}_\Omega v \rangle$$

and

$$\int \langle \partial_r v, v \rangle r^2 dr = \int \langle (\partial_r + r^{-1}) v, v \rangle r^2 dr = \int \langle \partial_r v, v \rangle dr.$$

4 The virial theorem

Note that

$$(\alpha)^* = \alpha, \quad \alpha^2 = I.$$

Define

$$F_v(r) = -\lambda r \mathrm{Re} \langle \mathcal{J}_\omega \partial_r \alpha v, v \rangle.$$

First of all, we need the following property on regularity of solutions.

Lemma 4.1 *Suppose that $F \in \mathcal{M}(\mathbf{R}^3)$. There exists a positive constant $C_F > 0$ such that*

$$\int |\nabla v|^2 dx \leq C_F \int \{|\mathrm{curl} v|^2 + |\mathrm{div} F v|^2 + |v|^2\} dx \tag{4.1}$$

for all $v \in C_0^1(\mathbf{R}^3)^3$.

Proof: Let $\{\sigma_j(x)\}_{j=1}^3$ be the set of all eigenvalues of $F(x)$. Define a diagonal matrix S as

$$S_{x_0} = \mathrm{diag}[\sigma_1(x_0), \sigma_2(x_0), \sigma_3(x_0)].$$

For every $x_0 \in U$, one can find an orthogonal transformation T_{x_0} such that

$$S_{x_0}^{-1/2} T_{x_0} F(x_0) T_{x_0}^{-1} S_{x_0}^{-1/2} = I.$$

Define

$$\tilde{F}(z; x_0) = S_{x_0}^{-1/2} T_{x_0} F\left(x_0 + T^{-1} S_{x_0}^{1/2} z\right) T_{x_0}^{-1} S_{x_0}^{-1/2}.$$

Then, making a change of variables

$$x = x_0 + T^{-1} S^{1/2} z, \quad \tilde{u}(x) = S^{1/2} T u, \tag{4.2}$$

we see that

$$\mathrm{div}_x(F(x)u) = \mathrm{div}_z(\tilde{F}(z; x_0)\tilde{u}) \tag{4.3}$$

and

$$\mathrm{curl}_x u = \frac{1}{\sqrt{\sigma_1(x_0)\sigma_2(x_0)\sigma_3(x_0)}} S_{x_0}^{1/2} \mathrm{curl}_z \tilde{u}. \tag{4.4}$$

We note that

$$\int |\nabla \tilde{u}|^2 dz = \int |\mathrm{curl}\tilde{u}|^2 dz + \int |\mathrm{div}\tilde{u}|^2 dz \tag{4.5}$$

for all $\tilde{u} \in C_0^\infty(\mathbf{R}^3)$. Combining (4.5) with (4.3) and (4.4) and using

$$\tilde{F}(z; x_0) - I = \mathcal{O}(|z|), \quad \text{as} \quad |z| \to 0,$$

one can find a small neighborhood U_{x_0} of x_0 such that

$$\int |\nabla u|^2 dx \le C\{\int |\mathrm{curl} u|^2 dx + \int |\mathrm{div} F(x) u|^2 dx + \int |u|^2 dx\} \tag{4.6}$$

for all $u \in C_0^1(U_{x_0})$. Here the positive constant C can be chosen independent of x_0. By use of a partition of unity, the inequality (4.1) follows from (4.6). □

The next is a kind of the virial theorem.

Lemma 4.2 *Let $v = ru$. Then,*

$$\lambda^2 \int_s^t \langle \partial_r [rV] v, v \rangle dr = F_v(t) - F_v(s).$$

Proof: From Lemma 4.1, it follows that the solution $u \in L^2(\mathbf{R}^3)^6$ to (2.2) belongs to $H^1(\mathbf{R}^3)$, Hence,

$$\int_0^\infty \|\nabla v\|^2 dr < \infty.$$

We approximate v by $\{v_n\}_{n=1}^\infty$ such that

$$\sum_{|\beta|\le 2} \int_0^\infty \|\partial_x^\beta v_n\|^2 dr < \infty$$

and

$$\lim_{n\to\infty} \int_0^\infty \{\|\nabla v_n - v\|^2 + \|v_n - v\|^2\} dr = 0.$$

Let $\Sigma_r = \{x \in \mathbf{R}^3;\ |x| = r\}$. Since the trace operator on the sphere is continuous from $H^{1/2}(\mathbf{R}^3)$ to $L^2(\Sigma_r)$, we see that for every $r \in (0, \infty)$,

$$\lim_{n\to\infty} \langle r^{-1}\mathcal{J}_\Omega \alpha v_n, v_n\rangle(r) = \langle r^{-1}\mathcal{J}_\Omega \alpha v, v\rangle(r).$$

Indeed,

$$\begin{aligned}
&|\langle r^{-1}\mathcal{J}_\Omega \alpha v_n, v_n\rangle(r) - \langle r^{-1}\mathcal{J}_\Omega \alpha v, v\rangle(r)| \\
&\qquad\le |\langle r^{-1}\mathcal{J}_\Omega \alpha v_n, v_n - v\rangle(r)| + |\langle r^{-1}\mathcal{J}_\Omega \alpha (v_n - v), v\rangle(r)| \\
&\qquad\le C\int_0^\infty \{\|\nabla v_n\|^2 + \|v_n\|^2\}\{\|\nabla(v_n - v)\|^2 + \|v_n - v\|^2\} dr \\
&\qquad\qquad + C\int_0^\infty \{\|\nabla(v_n - v)\|^2 + \|v_n - v\|^2\}\{\|\nabla v\|^2 + \|v\|^2\} dr.
\end{aligned}$$

On the other hand, an integration by parts implies

$$\int_s^t \mathrm{Re}\langle \lambda\Gamma v, 2\lambda r\partial_r v\rangle dr = -\lambda^2 \int_s^t \mathrm{Re}\langle (r\Gamma)' v, v\rangle dr + \lambda^2 \left[\langle r\Gamma v, v\rangle\right]_s^t, \tag{4.7}$$

$$2\mathrm{Re} \int_s^t \langle r^{-1}\mathcal{J}_\Omega \alpha v_n, \lambda r (v_n)_r\rangle dr = \lambda \mathrm{Re}\left[\langle \mathcal{J}_\Omega \alpha v_n, v_n\rangle\right]_s^t \tag{4.8}$$

and

$$\lambda \mathrm{Re}\langle i\mathcal{J}_\omega D_r \alpha v, 2r v_r\rangle = 0. \tag{4.9}$$

Letting $n \to \infty$ in (4.8), we obtain

$$2\mathrm{Re} \int_s^t \langle r^{-1}\mathcal{J}_\Omega \alpha v, \lambda r v_r\rangle dr = \lambda \mathrm{Re}\left[\langle \mathcal{J}_\Omega \alpha v, v\rangle\right]_s^t. \tag{4.10}$$

$$\lambda\Gamma v + r^{-1}\mathcal{J}_\Omega \alpha v + i\mathcal{J}_\omega D_r \alpha v = 0,$$

we see that

$$0 = -\lambda^2 \int_s^t \langle \partial_r[r\Gamma]v, v\rangle dr + \lambda^2 \left[\langle r\Gamma v, v\rangle\right]_s^t + \lambda \mathrm{Re}\left[\langle \mathcal{J}_\Omega \alpha v, v\rangle\right]_s^t. \tag{4.11}$$

From (3.1), it follows that

$$\lambda^2\langle r\Gamma u, u\rangle(r) + \lambda\mathrm{Re}\langle \mathcal{J}_\Omega \alpha v, v\rangle(r) = -\lambda\mathrm{Re}\langle ri\mathcal{J}_\omega D_r \alpha v, v\rangle(r). \tag{4.12}$$

In view of (4.11) and (4.12), we arrive at the desired identity. □

5 Proof of Theorem 2.5

Theorem 2.5 follows from the virial theorem. Since $u \in H^1(\mathbf{R}^3)$, we see that

$$\int_0^\infty r^{-1}|F_v|dr < \infty.$$

Thus, it holds that

$$\liminf_{r\to 0} |F_v|(r) = 0, \quad \liminf_{r\to\infty} |F_v(r)| = 0.$$

Performing $s = s_j \to 0$ and $t = t_j \to \infty$ in (4.2), we obtain

$$\lambda^2 \int_0^\infty \langle \partial_r[r\Gamma]v, v\rangle dr \le 0,$$

which implies $v = 0$ since $\partial_r[r\Gamma] > 0$. □

Remark 5.1 *From Lemma 4.2 and the fact that*

$$\liminf_{r\to\infty} |F_v(r)| = 0,$$

it follows that $F_v(r) \le 0$ for every sufficient large r.

The essential difficulty arises when the virial condition (2.9) is valid only in a neighborhood of infinity.

6 Isotropic cases

In this section we shall consider the isotropic case.

Define

$$q_0(r) = \sqrt{q_1 q_2},\ \Gamma_\infty(r) = \begin{pmatrix} q_1 I & 0 \\ 0 & q_2 I \end{pmatrix}$$

and

$$Q_3 = -\frac{1}{2}\begin{pmatrix} q_2^{-1}q_2' I & 0 \\ 0 & q_1^{-1}q_1' I \end{pmatrix}.$$

Lemma 6.1 *Let* $v = \Gamma_\infty^{1/2} r u$. *Then,*

$$\{-\mathcal{J}_\omega \partial_r - r^{-1}\mathcal{J}_\Omega - \mathcal{J}_\omega Q_3\}\alpha v = \lambda V v \tag{6.1}$$

and

$$\{\partial_r - r^{-1}\mathcal{G} - \mathcal{Q} - \mathcal{J}_\omega^2 Q_3\}\alpha v = \lambda \mathcal{J}_\omega V v, \tag{6.2}$$

where $V \in C^1(D_a)$ *satisfies that*

$$V^* = V,\ V = q_0(1+\tilde{V}),\ \partial_r^j \tilde{V} = o(r^{-(j+1)/2}),\ j = 1, 2. \tag{6.3}$$

Proof: Let

$$V_2 = \Gamma_\infty^{-1/2}(\Gamma - \Gamma_\infty)\Gamma_\infty^{-1/2}.$$

Multiplying (3.4) and (3.5) by $\Gamma_\infty^{-1/2}$ from the left and by $\Gamma_\infty^{-1/2}$ from the right, we observe that if u is a solution to (2.2), $\tilde{u} = \Gamma_\infty^{1/2} u$ satisfies

$$\begin{aligned} &q_0^{-1}\begin{pmatrix} 0 & J_\omega \\ -J_\omega & 0 \end{pmatrix}\partial_r \tilde{u} + r^{-1} q_0^{-1}\begin{pmatrix} 0 & J_\Omega \\ -J_\Omega & 0 \end{pmatrix}\tilde{u} \\ &\quad + \begin{pmatrix} 0 & q_1^{-1/2} J_\omega (q_2^{-1/2})' \\ -q_2^{-1/2} J_\omega (q_1^{-1/2})' & 0 \end{pmatrix}\tilde{u} = i\lambda\{\tilde{u} + V_2 \tilde{u}\}. \end{aligned} \tag{6.4}$$

Multiplying the last identity $\mathcal{J}_\omega$, we obtain

$$\begin{aligned} &-q_0^{-1}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\partial_r \tilde{u} + r^{-1} q_0^{-1}\begin{pmatrix} 0 & G \\ G & 0 \end{pmatrix}\tilde{u} + q_0^{-1}\begin{pmatrix} 0 & \omega\mathrm{div} \\ \omega\mathrm{div} & 0 \end{pmatrix}\tilde{u} \\ &\quad + \mathcal{J}_\omega^2 \begin{pmatrix} 0 & q_1^{-1/2}(q_2^{-1/2})' \\ q_2^{-1/2}(q_1^{-1/2})' & 0 \end{pmatrix}\tilde{u} = i\lambda\begin{pmatrix} J_\omega & 0 \\ 0 & -J_\omega \end{pmatrix}\{\tilde{u} + V_2\tilde{u}\}. \end{aligned} \tag{6.5}$$

If $V = q_0(1+V_2)$, then

$$\{-\mathcal{J}_\omega(\partial_r + r^{-1}) - r^{-1}\mathcal{J}_\Omega - \mathcal{J}_\omega Q_3\}\alpha\tilde{u} = \lambda V\tilde{u} \tag{6.6}$$

and

$$\{\partial_r - r^{-1}\mathcal{G} - Q - \mathcal{J}_\omega^2 Q_3\}\alpha\tilde{u} = \lambda\mathcal{J}_\omega V\tilde{u}. \tag{6.7}$$

Since $v = r\tilde{u}$ satisfies

$$\partial_r v = r(\partial_r + r^{-1})\tilde{u},$$

we arrive at the conclusion. □

Let δ be a small nonnegative integer which will be chosen later. Define

$$G_v(r) = -\lambda r \mathrm{Re}\langle \mathcal{J}_\omega \partial_r \alpha v, v\rangle + \delta q_0^{-1}\langle v, r^{-1}\mathcal{G}v\rangle.$$

Lemma 6.2 *Suppose that (6.3) and (2.9). Then, it holds that*

$$\lambda^2 \int_s^t \|q_0^{1/2}v\|^2 dr \le G_v(t) - G_v(s), \quad t > s \gg 1.$$

Proof: In the same manner as in the proof of Lemma 4.2, we see that

$$\lambda^2 \int_s^t \langle \partial_r[rV]v, v\rangle dr - 2\lambda \mathrm{Re}\int_s^t \langle r\mathcal{J}_\omega Q_3 \alpha v, \partial_r v\rangle = F_v(t) - F_v(s). \tag{6.8}$$

Since $Q_3 = o(r^{-1})$, it holds that

$$\begin{aligned} 2|\lambda \mathrm{Re}\int_s^t \langle r\mathcal{J}_\omega Q_3\alpha v, \partial_r v\rangle dr| &\le |\lambda| \int_s^t \|o(1)q_0^{1/2}v\|\|q_0^{-1/2}\partial_r \alpha v\| dr \\ &\le \int_s^t o(1)\lambda^2 q_0 \|v\|^2 dr + \int_s^t o(1)\|q_0^{-1/2}\partial_r\alpha v\|^2 dr. \end{aligned} \tag{6.9}$$

Let

$$X = q_0^{-1/2}\partial_r \alpha v, \; Y = q_0^{-1/2} r^{-1}\mathcal{G}\alpha v.$$

Then, in view of

$$\int_s^t \{\|X\|^2 + \|Y\|^2\} dr = \int_s^t \|f\|^2 dr - 2\mathrm{Re}\int_s^t \langle X, Y\rangle dr, \tag{6.10}$$

where

$$f = q_0^{-1/2}\{\mathcal{J}_\omega \lambda V v + (Q + \mathcal{J}_\omega^2 Q_3)\alpha v\}.$$

An integration by parts implies

$$(6.11)\quad 2\mathrm{Re}\int_s^t \langle X,Y\rangle dr = \int_s^t \langle (r^{-1}q_0^{-1})'\alpha v, \mathcal{G}\alpha v\rangle dr + [\langle q_0^{-1}\alpha v, r^{-1}\mathcal{G}\alpha v\rangle]_s^t$$
$$\leq [\langle q_0^{-1}\alpha v, r^{-1}\mathcal{G}\alpha v\rangle]_s^t + \int_s^t r^{-1}o(1)q_0\|v\|^2 dr + \frac{1}{2}\int_s^t \|Y\|^2 dr.$$

On the other hand, from $Q = o(r^{-1/2})$, it follows that

$$(6.12)\qquad \int_s^t \|f\|^2 dr \leq \int_s^t (1+o(1))\lambda^2 q_0\|v\|^2 dr.$$

As a result, from (6.10),(6.11) and (6.12), we obtain

$$(6.13)\qquad \delta\int_s^t \|X\|^2 dr \leq C\delta\int_s^t \lambda^2 q_0\|v\|^2 dr + \delta[\langle q_0^{-1}\alpha v, r^{-1}\mathcal{G}\alpha v\rangle]_s^t.$$

If $\delta > 0$ is chosen small enough, (6.8), (6.9) and (6.13) imply the conclusion. □

As the first step, from the virial theorem we shall derive a weighted L^2 inequality. Let $\varphi \in C^2(I_a;\mathbf{R})$ be a nonnegative function such that $\varphi' \geq 0$.

Lemma 6.3 *Suppose $G_v(r) \leq 0$ for all $r \gg 1$. There exists a positive constant C such that if $t \geq s \geq a$, then*

$$\lambda^2\int_s^t e^{2\varphi}\|q_0^{1/2}v\|^2 dr \leq Ce^{2\varphi(s)}\int_s^t \|q_0^{1/2}v\|^2 dr - \int_s^t 2\varphi' e^{2\varphi}G_v(r)dr.$$

Proof:

$$\int_s^t (e^{2\varphi})'(\tau)\int_\tau^t \|q_0^{1/2}v\|^2 dr d\tau = \left[e^{2\varphi(\tau)}\int_\tau^t \|q_0^{1/2}v\|^2 dr\right]_{\tau=s}^t + \int_s^t e^{2\varphi}\|q_0^{1/2}v\|^2 d\tau.$$

From Lemma 4.2, we arrive at the conclusion. □

Let $\chi \in C_0^\infty(\mathbf{R})$ be a nonnegative cut-off function supported in $[s-1, t+1]$ such that

$$\chi(r) = 1, \quad r \in [s,t].$$

Define

$$w = \chi e^{\varphi} q_0^{-1/2} v.$$

Let

$$\tilde{Q} = Q + \mathcal{J}_\omega^2 Q_3.$$

Lemma 6.4 *Under the same assumption as in Lemma 6.3, it holds*

$$(6.14)\quad -2\chi^2\varphi' e^{2\varphi}G_v \leq -\mathrm{Re}\langle 2r\varphi'(i\lambda V\mathcal{J}_\omega + i\tilde{Q}\alpha)^*(-i\partial_r)\alpha w, w\rangle$$
$$+ C\delta\{\varphi' r\|\partial_r w\|^2 + o(1)\left\{(\varphi')^2 + \varphi' + 1\right\}\|w\|^2 + o(1)\varphi'|\chi'|\|e^{\varphi}v\|^2\}.$$

Proof: Since $\mathcal{J}_\omega^* = -\mathcal{J}_\omega$, it holds

$$(6.15)\quad -2\chi^2\varphi' e^{2\varphi} G_v = 2\lambda r\varphi' \mathrm{Re}\langle q_0 \mathcal{J}_\omega \partial_r \alpha w, w\rangle + 2\delta\chi^2\varphi' e^{2\varphi} q_0^{-1}\langle \alpha v, r^{-1}\mathcal{G}\alpha v\rangle.$$

Note that

$$\lambda q_0 \mathcal{J}_\omega = -(\lambda V \mathcal{J}_\omega + \tilde{Q}\alpha)^* + o(r^{-1/2})$$

and

$$r^{-1}\mathcal{G}\alpha v = (\partial_r - \tilde{Q})\alpha v - \lambda \mathcal{J}_\omega V v.$$

Since $\tilde{Q} = o(r^{-1/2})$, we arrive at the conclusion. □

Thus, w satisfies

$$\{-\partial_r + r^{-1}\mathcal{G} + \varphi' + \tilde{Q}\}\alpha w + \lambda \mathcal{J}_\omega V w = -\chi' e^{\varphi}\alpha v.$$

Let $f_\chi = -\chi' e^{\varphi}\alpha v$. We shall consider the integral

$$(6.16)\qquad -2\mathrm{Re}\int_{s-1}^{t+1} r\varphi' \langle \partial_r \alpha w, \lambda \mathcal{J}_\omega V w + \tilde{Q}\alpha w\rangle + \mathrm{Re}\int_{s-1}^{t+1} r\varphi'' \langle f_\chi, \alpha w\rangle.$$

To estimate the first integral of (6.16) we use the expression

$$\begin{aligned}(6.17)\quad & -\mathrm{Re}\langle 2r\varphi'\partial_r \alpha w, \lambda \mathcal{J}_\omega V w + \tilde{Q}\alpha w\rangle \\ &= -r\varphi'\{\|\partial_r \alpha w\|^2 + \|\partial_r \alpha w - f_\chi\|^2 - \|f_\chi\|^2\} + 2\mathrm{Re}\langle \partial_r \alpha w, \varphi'(\mathcal{G} + r\varphi')\alpha w\rangle \\ &= -r\varphi'\{\|\partial_r \alpha w\|^2 + \|\partial_r \alpha w - f_\chi\|^2 - \|f_\chi\|^2\} - \mathrm{Re}\langle \alpha w, \{\varphi''\mathcal{G} + \big(r(\varphi')^2\big)'\}\alpha w\rangle.\end{aligned}$$

As a result, we obtain

Proposition 6.5 *Suppose that (6.3) and (2.9) hold and $G_v(r) \le 0$ for all $r \ge a$. It holds that*

$$\begin{aligned}(6.18)\quad & \lambda^2(1 - o(1))\int_s^t \left\{\|q_0^{1/2} e^{\varphi} v\|^2 + \frac{1}{2} r\varphi' \|\partial_r(e^{\varphi} v)\|^2\right\} dr + \int_{s-1}^{t+1} \chi^2 k_\varphi \|e^{\varphi} v\|^2 dr \\ &\le C\left\{e^{2\varphi(s)}\int_s^t \|q_0^{1/2} v\|^2 dr + \int_{s-1}^{t+1} r(\varphi' + |\varphi''|)|\chi'|\|e^{\varphi} v\|^2 dr\right\}.\end{aligned}$$

Here,

$$k_\varphi = r\varphi'\{(\varphi'' + (r^{-1} - o(r^{-1}))\varphi'\} - \frac{1}{2}(r\varphi'')' - o(1)\varphi' - o(q^{1/2})\varphi'.$$

Lemma 6.6 *Let $u \in L^2(U)^6$ be a solution to (2.2). Then, there exists a positive number a such that*

$$G_v(r) \le 0, \quad \forall r \ge a.$$

Now we are going to show

$$(\log r)^n v, \quad r^n v, \quad \exp\{nr^\rho\}v \in L^2(D_a), \quad \forall n \in \mathbf{N}, \ \forall \rho \in (0,1).$$

Choosing respectively $q(r) = \log^{1/2} r$, $r^{b/2}$ and finally $e^{r^b(\log r)^2}$ as the weight function of (6.18), we obtain three kind of weighted inequalities. The first one is as follows.

$$(6.19) \quad \int_s^t (\log r)^n \|u\|^2 dr \le C\{\int_{s-1}^{t+1} o(1)(1+n^2(\log r)^{-2})(\log r)^n \|u\|^2 dr \\ + (\log s)^n \int_s^t \|u\|^2 dr + \left\{\int_t^{t+1} + \int_{s-1}^s\right\} n(\log r)^{n-1}\|u\|^2 dr.$$

We shall use

$$\liminf_{N\to\infty} N \int_N^{N+1} \|u\|^2 dr = 0.$$

By letting $t \to \infty$ in (6.19), an induction procedure implies that if $v \in L^2(D_a)^6$,

$$(\log r)^{n/2} v \in L^2(I_a)^6, \quad \forall n = 0,1,2,\dots .$$

In view of

$$r^m = \exp\{m \log r\} = \sum_{n=0}^{\infty} (m \log r)^n / n!,$$

we can conclude that $r^m v \in L^2(I_a)^6$. In the same manner, we see that

$$(6.20) \quad \int_s^\infty \sum_{n=2}^N \frac{1}{n!}(mr^b)^n \|u\|^2 dr \\ \le C \int_{s-1}^\infty r^{-2(1-b)} m^2 \sum_{n=2}^N \frac{1}{(n-2)!}(mr^b)^{n-2}\|u\|^2 dr + C_m(u)$$

for all $N = 2,3,\dots$. Finally, we arrive at

$$e^{nr^b} v \in L^2(I_a)^6, \quad \forall n = 1,2,\dots .$$

for any $b \in (0,1)$.

Applying the weighted inequality with $e^{2\varphi} = e^{nr^b(\log r)^2}$, we can conclude that

Lemma 6.7 *For every $n \in \mathbf{N}$ and every $s \ge a+1$,*

$$(6.21) \qquad \int_s^\infty e^{nr^b(\log r)^2}\|v\|^2 dr \le C e^{n(a+1)^b(\log(a+1))^2} \int_{a+1}^\infty \|v\|^2 dr.$$

Proof: To prove this, we have to show that $k_\chi > 0$. Indeed, if $e^\varphi = \{r^b(\log r)^2\}^n$, it holds that

$$\varphi'/n = (r^b(\log r)^2)' = br^{b-1}(\log r)^2 + 2r^{b-1}\log r,$$

$$\varphi''/n = b(b-1)r^{b-2}(\log r)^2 + 2br^{b-2}(\log r) + 2(b-1)r^{b-2}\log r + 2r^{b-2}.$$

Therefore,

$$r\varphi'(\varphi'' + r^{-1}\varphi') = n^2b^2r^{b-2}(\log r)^2 br^b(\log r)^2(1+o(1)) = n^2b^3r^{2b-2}(\log r)^4(1+o(1))$$

and

$$(r\varphi'')' + \varphi' o(1) = nb(b-1)^2 r^{b-2}(\log r)^2 + no(r^{b-1}(\log r)^2).$$

Let $X = nr^{b-1}(\log r)^2$. Then, there exists a positive number σ_0 such that

$$\lambda q_0 + b^3X^2 - o(X) - o(X^2) \geq \sigma_0(1+X^2), \quad \forall X \geq 0.$$

□

Now, we are in the final step for proving Theorem 2.3.

Let $\phi = r^b(\log r)^2$. From (6.21), it follows that

$$\int_{2a+1}^{\infty} \|v\|^2 dr \leq C\exp\{2n(\phi(a+1) - \phi(2a+1))\}\int_{a+1}^{\infty} \|v\|^2 dr.$$

Since $\phi(r)$ is monotone increasing, we see

$$0 < e^{\varphi(a+1)-\varphi(2a+1)} < 1.$$

Letting $n \to \infty$, we conclude that $v = 0$ in D_{2a+1}. On account of unique continuation theorem for the time harmonic Maxwell equations, we see that $v = 0$ in U. □

7 Potentials growing at infinity

In this section we shall prove Theorem 2.4. Suppose that $q \in C^2(I_a)$ satisfies

(7.1) $\inf q(r) > 0,\ [q'(r)]_- = o(r^{-1}q),\ (\frac{d}{dr})^j q(r) = o(r^{-j/2}q^{1+j/2}),\ j = 1, 2.$

We say that $f(x) \in C^1(U)$ belongs to the class $\tilde{S}(q)$ if

$$\partial_r^j(f(x) - q(r)) = o(\left(r^{-1/2}q^{1/2}\right)^{j+1}), \quad \forall x \in D_a, \quad j = 0, 1.$$

$$h(r) = q(q' + \frac{1}{2}r^{-1}q)^{-1/2}.$$

and

$$G_v(r) = -\lambda r \mathrm{Re}\langle \mathcal{J}_\omega \partial_r \alpha v, v\rangle + \frac{1}{2} r h^{-2}(r)\langle v, r^{-1}\mathcal{G} v\rangle.$$

Lemma 7.1 *Suppose that ε and μ are scalar functions belonging to $\tilde{S}(q)$. Let v $q^{-1/2}ru$. Then, it holds that*

$$\lambda^2 \int_s^t \|q^{1/2}v\|^2 dr \le G_v(t) - G_v(s), \quad t > s \gg 1.$$

Proof: First of all, we see that v satisfies

$$\{\mathcal{J}_\omega \partial_r + r^{-1}\mathcal{J}_\Omega + \frac{1}{2}\mathcal{J}_\omega q^{-1}q'\}\alpha v = \lambda \Gamma v.$$

Thus, it holds that

$$\text{(7.2)} \qquad \lambda^2 \int_s^t \langle \partial_r[r\Gamma]v, v\rangle dr - 2\lambda \mathrm{Re}\int_s^t \langle r\mathcal{J}_\omega q^{-1}q'\alpha v, \partial_r v\rangle = F_v(t) - F_v(s).$$

Note that

$$(r\Gamma)' = q + rq' + o(1).$$

$$\text{(7.3)} \quad 2|\lambda \mathrm{Re}\int_s^t \langle r\mathcal{J}_\omega q^{-1}q'\alpha v, \partial_r v\rangle dr| \le |\lambda| \int_s^t \|q^{-1}q' r^{1/2} h v\| \|\mathcal{J}_\omega \partial_r r^{1/2} h^{-1}\alpha v\| dr$$
$$\le \frac{1}{2}\int_s^t \lambda^2 r[q']_+ \|v\|^2 dr + \frac{1}{2}\int_s^t \|\partial_r r^{1/2}h^{-1}\alpha v\|^2 dr.$$

Let

$$X = \partial_r r^{1/2} h^{-1}\alpha v, \ Y = r^{-1}\mathcal{G} r^{1/2} h^{-1}\alpha v.$$

Then, in view of

$$\text{(7.4)} \qquad \int_s^t \{\|X\|^2 + \|Y\|^2\} dr = \int_s^t \|f\|^2 dr - 2\mathrm{Re}\int_s^t \langle X, Y\rangle dr,$$

where

$$f = \mathcal{J}_\omega \lambda V r^{1/2} h^{-1} v + \left\{ Q r^{1/2} h^{-1} + \frac{d}{dr}[r^{1/2}h^{-1}] \right\} \alpha v.$$

$$\text{(7.5)} \quad 2\mathrm{Re}\int_s^t \langle X, Y\rangle dr = \int_s^t \langle r^{-1} r^{1/2} h^{-1}\alpha v, Y\rangle dr$$
$$\le [\langle r^{1/2}h^{-1}\alpha v, r^{-1}\mathcal{G} r^{1/2}h^{-1}\alpha v\rangle]_s^t + o_+(1)\int_s^t \|q^{1/2}v\|^2 dr + \frac{1}{2}\int_s^t \|Y\|^2 dr.$$

On the other hand, it is easily verified that

$$\int_s^t \|f\|^2 dr \le \int_s^t \lambda^2(\frac{1}{2}q + r[q']_+)\|v\|^2 dr + o(1)\int_s^t \|q^{1/2}v\|^2 dr. \tag{7.6}$$

As a result, from (7.4),(7.5) and (7.6), we obtain

$$\frac{1}{2}\int_s^t \|X\|^2 dr \le \frac{1}{2}\int_s^t \lambda^2(\frac{1}{2}q + r[q']_+)\|v\|^2 dr + o(1)\int_s^t \|q^{1/2}v\|^2 dr. \tag{7.7}$$

Combining (7.2) with (7.3) and (7.7), we arrive at the conclusion. □

8 Nonisotropic cases

To study non-isotropic tropic cases, we shall use a scalar operator which shall play as the radiation derivative ∂_r in the isotropic case. This operator was firstly introduced in [22]. For $F(x) \in \mathcal{M}(U)$, define the scalar operator $\mathcal{D}_F$ as

$$\mathcal{D}_F u = (\omega, F\omega)^{-1}(\omega, F\nabla u), \quad u \in C^1(U)$$

and

$$\mathcal{L}_F u = \mathrm{curl} u - J_\omega \mathcal{D}_F u, \quad u \in \{C^1(U)\}^3.$$

These operators have the following useful properties (cf. [22], Lemma 3.2 and Lemma 3.3).

Lemma 8.1 *Suppose that $F \in \mathcal{M}(U)$ and $F_0 = 1$. For any u, $v \in C_0^1(D_a)$, any $b(\omega) \in C^1(\mathbf{S}^2)$ and $f(r) \in C^1(I_a)$, it holds that*

$$\int_a^\infty \langle \tilde{D}_F u, v\rangle r^2 dr = -\int_a^\infty \langle u, \tilde{D}_F v\rangle r^2 dr - 2\int_a^\infty r^{-1}\langle u, v\rangle r^2 dr + \int_a^\infty o(r^{-1})\langle u, v\rangle r^2 dr,$$

$$\tilde{D}_F(b(\omega)u) = b(\omega)\tilde{D}_F u + o(r^{-1})u,$$

$$\tilde{D}_F f(r) = f'(r), \quad \tilde{L}_F(f(r)u) = f(r)\tilde{L}_F u,$$

$$\int_a^\infty \langle \mathcal{L}_F u, v\rangle r^2 dr = \int_a^\infty \langle u, \mathcal{L}_F v\rangle r^2 dr - \int \langle u, 2r^{-1}J_\omega v\rangle r^2 dr + \int_a^\infty o(r^{-1})\|u\|\|v\|r^2 dr$$

and

$$\tilde{D}_F \tilde{L}_F u = \tilde{L}_F \tilde{D}_F - r^{-1}\tilde{L}_F u + \sum_{j=1}^3 o(r^{-1})\partial_{x_j} u + o(r^{-2})u.$$

Proof: Note that

$$\partial_{x_j}\omega_k = \delta_{jk}r^{-1} - r^{-1}\omega_k\omega_j,$$

where δ_{jk} is equal to one if $j = k$ and 0 otherwise. Hence,

$$\tilde{D}_F\omega_k = (\omega, F\omega)^{-1}\sum_{i=1}^{3}\omega_i F_{ik}r^{-1} - r^{-1}\omega_k = o(r^{-1}).$$

Since $\nabla F = o(r^{-1})$ and $F - F_0 I = o(r^{-1/2})$, we have

$$(8.1)\quad \sum_{j=1}^{3}\partial_{x_j}\left(\sum_{k=1}^{3}\omega_k F_{kj}\right)$$
$$= 3r^{-1}F_0 - r^{-1}\sum_{k,j=1}^{3}\omega_k\omega_j F_{kj} + o(r^{-1}) = 2r^{-1}F_0 + o(r^{-1})$$

and

$$(8.2)\quad \partial_{x_j}(\omega, F\omega)^{-1} = -(\omega, F\omega)^{-2}\partial_{x_j}(\omega, F\omega)$$
$$= -(\omega, F\omega)^{-2}\partial_{x_j}\{F_0 + (\omega, (F - F_0)\omega)\} = o(r^{-1}).$$

Thus, from (8.1) and (8.2), it follows that

$$\partial_{x_j}\left\{(\omega, F\omega)^{-1}\sum_{k=1}^{3}\omega_k F_{kj}\right\} = 2F_0 r^{-1} + o(r^{-1}).$$

□

Lemma 8.2 *Under the same assumption as in Lemma 8.1, it holds*

$$(8.3)\quad FJ_{F\omega}\text{curl}u = -\mathcal{D}_F(Fu) + \{FJ_{F\omega}\mathcal{L}_F + (FJ_{F\omega}\mathcal{L}_F)^*\}$$
$$+ (\text{div}Fu)F\omega - r^{-1}(\omega, Fu)F\omega - r^{-1}Fu + o(r^{-1/2})\mathcal{D}_F Fu + o(r^{-1})u.$$

The proof of Lemma 8.3 is given in [15].

Let

$$\Gamma_0 = \begin{pmatrix} \kappa & 0 \\ 0 & 1 \end{pmatrix}.$$

Making a change of coordinates $\tilde{u} = \Gamma_0^{1/2}u$, we may assume that $\varepsilon_\infty = \mu_\infty$. Define

$$\hat{D}_F = \mathcal{D}_F + r^{-1}, \quad \hat{L}_F = \mathcal{D}_F - r^{-1}J_\omega,$$

$$G_F = \{FJ_{F\omega}\mathcal{L}_F + (FJ_{F\omega}\mathcal{L}_F)^*\} - r^{-1}(\omega, Fu)F\omega$$

and

$$\mathcal{G} = r\begin{pmatrix} G_\varepsilon & 0 \\ 0 & G_\mu \end{pmatrix}, \quad \mathcal{D} = \begin{pmatrix} \hat{\mathcal{D}}_\varepsilon I & 0 \\ 0 & \hat{\mathcal{D}}_\mu I \end{pmatrix}.$$

Then, from Lemma 8.1, it follows that

$$[\mathcal{D}, \mathcal{G}] = \sum_{j=1}^{3} o(1)\partial_{x_j} u + o(r^{-1})u.$$

In view of

$$\mathcal{D}_{kF} = \mathcal{D}_F, \ \mathcal{L}_{kF} = \mathcal{L}_F, \quad \forall k > 0,$$

we may change the notations to denote $\varepsilon_0^{-1}\varepsilon$ and $\mu_0^{-1}\mu$ by the same letters ε and μ, respectively. Thus, we may assume that

$$\varepsilon(0) = \mu(0) = I.$$

In addition, we shall use the following notations.

$$\mathcal{D}_\infty = \begin{pmatrix} \hat{\mathcal{D}}_{\mu_\infty} I & 0 \\ 0 & \hat{\mathcal{D}}_{\mu_\infty} I \end{pmatrix}, \quad \alpha = \begin{pmatrix} 0 & iI \\ -iI & 0 \end{pmatrix}, \quad \mathcal{L}_\infty = \begin{pmatrix} \hat{L}_{\varepsilon_\infty} & 0 \\ 0 & \hat{L}_{\mu_\infty} \end{pmatrix},$$

$$\mathcal{J} = \begin{pmatrix} \varepsilon J_{\varepsilon\omega} & 0 \\ 0 & \mu J_{\mu\omega} \end{pmatrix}, \quad \Gamma_\infty = \begin{pmatrix} \mu_\infty & 0 \\ 0 & \mu_\infty \end{pmatrix},$$

$$V = \kappa\left\{\Gamma_\infty + \Gamma_0^{-1/2}\begin{pmatrix} \varepsilon - \varepsilon_\infty & 0 \\ 0 & \mu - \mu_\infty \end{pmatrix}\Gamma_0^{-1/2}\right\}.$$

In the same manner as in the isotropic case, we see that $v = \Gamma_0^{1/2} r u$ satisfies

$$Av = \{-\mathcal{J}_\omega \mathcal{D}_\infty - \mathcal{L}_\infty\}\alpha v = \lambda V v.$$

Since

$$\mathrm{div}(\varepsilon E) = \mathrm{div}(\mu H) = 0,$$

(8.3) implies that

$$\{\mathcal{D} - r^{-1}\mathcal{G} + \tilde{Q}\}\alpha v = \lambda \Gamma \mathcal{J} V v,$$

where

$$\tilde{Q}v = \begin{pmatrix} o(r^{-1/2})\mathcal{D}_\varepsilon \varepsilon & 0 \\ 0 & o(r^{-1/2})\mathcal{D}_\mu \mu \end{pmatrix} \alpha v + o(r^{-1})\alpha v.$$

Define

$$F_v(r) = -\lambda r \mathrm{Re}\langle \mathcal{J}_\omega \mathcal{D}_\infty \alpha v, v\rangle$$

and

$$G_v(r) = F_v(r) + \nu\langle \alpha v, r^{-1}\mathcal{G}\alpha v\rangle,$$

where ν is a sufficiently small positive number.

We consider

$$\mathrm{Re}\int_s^t \langle Av, 2r\mathcal{D}_\infty v\rangle dr = \mathrm{Re}\int_s^t \langle \lambda V v, 2r\mathcal{D}_\infty v\rangle dr.$$

Note that

$$\mathrm{Re}\int_s^t \langle \mathcal{J}_\omega \mathcal{D}_\infty \alpha v, 2r\mathcal{D}_\infty v\rangle dr = 0,$$

$$\int_s^t \langle \mathcal{D}_\infty f, g\rangle dr = -\int_s^t \langle f, \mathcal{D}_\infty g\rangle dr + [\langle f, g\rangle]_s^t + \int_s^t \langle o(r^{-1})f, g\rangle dr$$

and

$$\mathrm{Re}\int_s^t \langle \mathcal{L}_\infty \alpha v, 2r\mathcal{D}_\infty v\rangle dr = [\langle \mathcal{L}_\infty \alpha v, 2rv\rangle]_s^t + \mathrm{Re}\int_s^t \langle o(1)\nabla \alpha v, v\rangle dr.$$

We note that

$$\mathcal{J}_\omega \mathcal{D}_\infty = \mathcal{J}_\omega(\mathcal{D}_\infty - \mathcal{D}) + (\mathcal{J}_\omega - \mathcal{J})\mathcal{D} + \mathcal{J}\mathcal{D}.$$

From $\Gamma - \Gamma_\infty = o(r^{-1})$ and $\Gamma_\infty - I = o(r^{-1/2})$, it follows that

$$F_v = -\lambda r \mathrm{Re}\langle \mathcal{J}\mathcal{D}\alpha v, v\rangle + \mathrm{Re}\langle o(r^{1/2})\mathcal{D}\alpha v, v\rangle + \mathrm{Re}\langle o(1)\nabla \alpha v, v\rangle.$$

Using the same reasoning as in the isotropic case, we can arrive at the conclusion of Theorem 2.1. We omit the detail for saving pages.

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