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A remark on the derivative of the one-dimensional Hardy-Littlewood maximal function

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Abstract

For $p>1$ and $d\geq 1$ J. Kinnunen proved that if f is a function on the Sobolev space $W^{1,p}(\mathbf{R}^{d})$, then the Hardy-Littlewood maximal function $\mathcal{M}f$ has the first order weak partial derivatives which belong to $L^{p}(\mathbf{R}^{d})$ and whose L^{p}-norm are controlled by those of f. We improve Kinnunen's result to $p=1$ and $d=1$ by making an expression of the maximal function by the one-sided maximal functions. We also study some properties of the one-sided maximal functions.

1 Introduction

For f a locally integrable function on $\mathbf{R}^{d}$, $d\geq 1$, define the Hardy-Littlewood maximal function $\mathcal{M}f$ as

$$(\mathcal{M}f)(x)=\sup_{Q}\frac{1}{|Q|}\int_{Q}|f|\,dy,$$

where the supremum is taken over all cubes Q containing $x\in \mathbf{R}^{d}$. Here, $|Q|$ denotes the volume of the cube Q. The maximal function is a basic tool in real analysis. The well-known theorem of Hardy, Littlewood and Wiener asserts that if $f\in L^{p}(\mathbf{R}^{d})$, with $1<p\leq\infty$, then $\mathcal{M}f\in L^{p}(\mathbf{R}^{d})$ and

$$\|\mathcal{M}f\|_{p}\leq A_{p}\|f\|_{p}, \tag{1}$$

where the constant A_{p} depends only on p and the dimension d. Moreover, one knows that the constant A_{p} satisfies

$$A_{p}=O(1/(p-1)),\ \text{as } p\rightarrow 1. \tag{2}$$

(See [St].) Recall that the Sobolev space $W^{1,p}(\mathbf{R}^{d})$, $1\leq p\leq\infty$, consists of functions f in $L^{p}(\mathbf{R}^{d})$, whose first order weak partial derivatives $D_{i}f$, $i=1,2,\cdots,d$, belong to $L^{p}(\mathbf{R}^{d})$.

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In [K] [1] J. Kinnunen showed that if $f \in W^{1,p}(\mathbf{R}^d)$, with $1 < p < \infty$ and $d \geq 1$, then $\mathcal{M}f \in W^{1,p}(\mathbf{R}^d)$ and

$$|(D_i\mathcal{M}f)(x)| \leq (\mathcal{M}D_if)(x), \quad i = 1, 2, \cdots, d, \tag{3}$$

almost everywhere $x \in \mathbf{R}^d$. This implies by (1) that

$$\|D_i\mathcal{M}f\|_p \leq A_p\|D_if\|_p \quad i = 1, 2, \cdots, d. \tag{4}$$

Kinnunen's methods used to prove (3) mainly depend on the L^p-boundedness of $\mathcal{M}$ and the fact that the first order Sobolev space is a lattice (see [GT]). So, one cannot directly extend the result of (4) to the case $p = 1$ because then we do not have the Hardy-Littlewood-Wiener theorem available. But, according to the embedding theorem of Sobolev, $W^{1,1}(\mathbf{R}^d)$ can be continuously embedded in $L^{d/(d-1)}(\mathbf{R}^d)$. (See [St].) Therefore, by the Hardy-Littlewood-Wiener theorem we see that $\mathcal{M}f \in L^{d/(d-1)}(\mathbf{R}^d)$, if $f \in W^{1,1}(\mathbf{R}^d)$. In particular, $\mathcal{M}f$ becomes a locally integrable function and hence is differentiable in distribution sense.

Now, we have the following problems.

(I) If $f \in W^{1,1}(\mathbf{R}^d)$, are the derivatives of $\mathcal{M}f$ function or not?
(II) If the derivatives of $\mathcal{M}f$ are functions, is it possible to show some norm estimates?

The purpose of this note is to investigate the above problems for the one-dimensional case. As yet we have not been able to prove the corresponding results in the higher dimensions. Our result is the following.

THEOREM 1 *If $f \in W^{1,1}(\mathbf{R})$, then the derivative of $\mathcal{M}f$ becomes an integrable function and*

$$\|(\mathcal{M}f)'\|_1 \leq 2\|f'\|_1. \tag{5}$$

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2 The one-sided maximal functions

A crucial point in our argument is to consider one-sided maximal functions. In this section we shall discuss our problem for one-sided maximal functions.

For f a locally integrable function on the line define the one-sided maximal functions $\mathcal{M}_lf$ and $\mathcal{M}_rf$ as

$$(\mathcal{M}_lf)(x) = \sup_{s>0} \frac{1}{s} \int_{x-s}^{x} |f|\, dy,$$

[1] Roughly speaking, the Hardy-Littlewood maximal function may be distinguished into two types. The first is defined as the supremum taken over all balls centered at x, and the second is defined as the supremum taken over all cubes containing x. Kinnunen proved his results for the first type maximal function. But, one sees that the corresponding results hold for the second one by the slight modification of the argument.

and

$$(\mathcal{M}_r f)(x) = \sup_{t>0} \frac{1}{t} \int_x^{x+t} |f| \, dy.$$

We will refer $\mathcal{M}_l f$ and $\mathcal{M}_r f$ to the left maximal function and the right maximal function respectively.

The first lemma represents an expression of the one-dimensional Hardy-Littlewood maximal function by the one-sided maximal functions.

LEMMA 2 *Let f be a locally integrable function on the line. Then we can express $\mathcal{M}f$ by $\mathcal{M}_l f$ and $\mathcal{M}_r f$ as*

$$(\mathcal{M}f)(x) = \max\{(\mathcal{M}_l f)(x), \ (\mathcal{M}_r f)(x)\}.$$

Proof. It follows from the definitions that

$$\max\{(\mathcal{M}_l f)(x), \ (\mathcal{M}_r f)(x))\} \leq (\mathcal{M}f)(x). \tag{6}$$

To prove the reverse inequality we see that

$$\begin{aligned} \frac{1}{s+t} \int_{x-s}^{x+t} |f| \, dy &= \frac{s}{s+t} \frac{1}{s} \int_{x-s}^{x} |f| \, dy + \frac{t}{s+t} \frac{1}{t} \int_x^{x+t} |f| \, dy \\ &\leq \frac{s}{s+t} (\mathcal{M}_l f)(x) + \frac{t}{s+t} (\mathcal{M}_r f)(x) \leq \max\{(M_l f)(x), \ (\mathcal{M}_r f)(x)\}. \end{aligned} \tag{7}$$

Taking the supremum on the left hand side of (7) over $s, t > 0$, we obtain

$$(\mathcal{M}f)(x) \leq \max\{(\mathcal{M}_l f)(x), \ (\mathcal{M}_r f)(x)\}. \tag{8}$$

(6) and (8) imply the desired relation. ∎

Next, we shall prove some elementary propositions. We will state only the case $\mathcal{M}_l$, but the corresponding results hold for the case $\mathcal{M}_r$ as well. In the following we assume that the function f is continuous. With this assumption, we note that

$$\lim_{s \to 0} \frac{1}{s} \int_{x-s}^{x} |f| \, dy = |f(x)|, \tag{9}$$

and

$$(\mathcal{M}_l f)(x) \geq |f(x)|$$

for every $x \in \mathbf{R}$.

PROPOSITION 3 *Let $f \in C(\mathbf{R})$. Assume that there exists a point $x_0 \in \mathbf{R}$ such that*

$$(\mathcal{M}_l f)(x_0) = \infty.$$

Then we have

$$\mathcal{M}_l f \equiv \infty.$$

Proof. By the assumption there exists a sequence $\{s_n\}$ of positive numbers, which converges to ∞, such that

$$\frac{1}{s_n}\int_{x_0-s_n}^{x_0} |f|\,dy > n. \tag{10}$$

Fix a point x on the line and take s_n sufficiently large, then by (10) we see that

$$\begin{aligned} n &< \frac{1}{s_n}\int_{x_0-s_n}^{x_0} |f|\,dy = \frac{s_n + x - x_0}{s_n}\frac{1}{s_n + x - x_0}\int_{x_0-s_n}^{x} |f|\,dy - \frac{1}{s_n}\int_{x_0}^{x} |f|\,dy \\ &\leq (1 + \frac{x - x_0}{s_n})(\mathcal{M}_l f)(x) - \frac{\int_{x_0}^{x} |f|\,dy}{s_n}. \end{aligned}$$

Letting n tend to ∞, we obtain

$$(\mathcal{M}_l f)(x) = \infty. \quad \blacksquare$$

Corollary 4 *Let $f \in C(\mathbf{R})$. Assume that there exists a point $x_0 \in \mathbf{R}$ such that*

$$(\mathcal{M}_l f)(x_0) < \infty.$$

Then we have

$$(\mathcal{M}_l f)(x) < \infty$$

for every $x \in \mathbf{R}$.

Next, if $f \in C(\mathbf{R})$ and $\mathcal{M}_l f < \infty$, then the lower semi-continuity of $\mathcal{M}_l f$ implies that the set E:

$$E = \{x \in \mathbf{R} \mid (\mathcal{M}_l f)(x) > |f(x)|\}$$

is open. Hence, E can be written as $E = \sum_j I_j$, where I_j denotes an open interval.

Proposition 5 *Let $f \in C(\mathbf{R})$, $\mathcal{M}_l f < \infty$ and $E = \sum_j I_j$. Then $(\mathcal{M}_l f)(x)$ becomes a non-increasing function of x on each I_j.*

Proof. Fix $x \in I_j$ and set $\epsilon = (\mathcal{M}_l f(x) - |f(x)|)/2 > 0$. By the continuity of $|f|$ there exists a $\delta > 0$ such that $|x - y| \leq \delta$ implies

$$|f(y)| < |f(x)| + \epsilon. \tag{11}$$

(11) yields

$$(\mathcal{M}_l f)(x) = \sup_{s>\delta} \frac{1}{s}\int_{x-s}^{x} |f|\,dy. \tag{12}$$

For $x - h \in I_j \cap (x - \delta,\ x)$, and $s > \delta$ it follows from (11) that

$$\begin{aligned} \frac{1}{s}\int_{x-s}^{x} |f|\,dy &= \frac{s-h}{s}\cdot\frac{1}{s-h}\int_{x-s}^{x-h} |f|\,dy + \frac{h}{s}\cdot\frac{1}{h}\int_{x-h}^{x} |f|\,dy \\ &\leq \max\{(\mathcal{M}_l f(x-h),\ |f(x)| + \epsilon\}. \end{aligned} \tag{13}$$

Taking the supremum on the left hand side of (13) over $s > \delta$, we have

$$(\mathcal{M}_l f)(x) \leq \max\{(\mathcal{M}_l f(x-h),\ |f(x)| + \epsilon\}$$

by (12). Since, the relation: $(\mathcal{M}_l f)(x) \le |f(x)| + \epsilon$ and the choice of ϵ give

$$(\mathcal{M}_l f)(x) \le |f(x)|,$$

which contradicts $x \in I_j$, we obtain

$$(\mathcal{M}_l f)(x) \le (\mathcal{M}_l f)(x-h). \quad \blacksquare$$

PROPOSITION 6 *Let the assumptions and notation be the same as those of Proposition 5. Then $(\mathcal{M}_l f)(x)$ becomes a locally Lipschitz function of x on each I_j. In particular, $\mathcal{M}_l f$ becomes an absolutely continuous function on each I_j.*

Proof. Take $K = [\alpha, \beta] \subset I_j$. By the lower semi-continuity of $\mathcal{M}_l f$ and the continuity of $|f|$ there exists an $\epsilon > 0$ such that $x \in K$ implies

$$(\mathcal{M}_l f)(x) > |f(x)| + \epsilon. \tag{14}$$

By the uniform continuity of $|f|$ there then exists a $\delta > 0$ such that $x \in K$ and $|y - x| \le \delta$ imply

$$|f(y)| < |f(x)| + \frac{\epsilon}{2}. \tag{15}$$

(14) and (15) yield that $x \in K$ implies

$$(\mathcal{M}_l f)(x) = \sup_{s>\delta} \frac{1}{s} \int_{x-s}^{x} |f|\, dy. \tag{16}$$

For $x, x+h \in K$, $h > 0$, and $s > \delta$ it follows from the previous proposition that

$$\begin{aligned} &\frac{1}{s}\int_{x-s}^{x} |f|\, dy - \frac{1}{s+h}\int_{x-s}^{x+h} |f|\, dy \le \frac{1}{s}\int_{x-s}^{x} |f|\, dy - \frac{1}{s+h}\int_{x-s}^{x} |f|\, dy \\ &= \frac{1}{s+h}\cdot\frac{1}{s}\int_{x-s}^{x} |f|\, dy \cdot h \le \frac{(\mathcal{M}_l f)(x)}{\delta}\cdot h \le \frac{(\mathcal{M}_l f)(\alpha)}{\delta}\cdot h. \end{aligned} \tag{17}$$

Moving $s > \delta$ on the left hand side of (17) freely, we obtain

$$0 \le (\mathcal{M}_l f)(x) - (\mathcal{M}_l f)(x+h) \le Ch. \quad \blacksquare$$

PROPOSITION 7 *Let the assumptions and notation be the same as those of Proposition 5. Then $(\mathcal{M}_l f)(x)$ is continuous at the boundary of E.*

Proof. Fix a point a at the boundary of E. Then we have

$$(\mathcal{M}_l f)(a) = |f(a)|. \tag{18}$$

Clearly, we see

$$\liminf_{x\to a} (\mathcal{M}_l f)(x) \ge |f(a)| \tag{19}$$

by the lower semi-continuity of $\mathcal{M}_l f$.

Now, we assume that

$$\limsup_{x\to a} (\mathcal{M}_l f)(x) - |f(a)| = \epsilon > 0. \tag{20}$$

With this assumption, by the continuity of $|f|$ there exists a $\delta > 0$ such that $|y - a| \leq 2\delta$ implies

$$|f(y)| < |f(a)| + \frac{\epsilon}{2},$$

and hence we can select by (20) the sequences $\{x_n\}$ and $\{s_n\}$, where $\{x_n\}$ converges to a and $s_n > \delta$, such that

$$\frac{1}{s_n}\int_{x_n - s_n}^{x_n} |f|\, dy > |f(a)| + \frac{\epsilon}{2}. \tag{21}$$

Taking η_n so that

$$(1 + \eta_n)s_n = a - x_n + s_n, \tag{22}$$

we have

$$|\eta_n| \leq \frac{|x_n - a|}{\delta} \tag{23}$$

by the fact that $s_n > \delta$. It follows from (21) and (22) that

$$\begin{aligned} |f(a)| + \frac{\epsilon}{2} &< \frac{1}{s_n}\int_{x_n - s_n}^{x_n} |f|\, dy \\ &= \frac{1 + \eta_n}{a - x_n + s_n}\left(\int_{x_n - s_n}^{a} |f|\, dy - \int_{x_n}^{a} |f|\, dy\right) \\ &\leq (1 - \eta_n)(\mathcal{M}_l f)(a) + \frac{1 + \eta_n}{a - x_n + s_n}\left|\int_{x_n}^{a} |f|\, dy\right|. \end{aligned}$$

Letting n tend to ∞, we obtain

$$|f(a)| + \frac{\epsilon}{2} \leq (\mathcal{M}_l f)(a) \tag{24}$$

by (23) and the fact that $s_n > \delta$. (24) contradicts (18) and hence we obtain

$$\lim_{x \to a}(\mathcal{M}_l f)(x) = |f(a)|$$

by (20) and (19). ∎

The next theorem is the key of our argument, and will be proved by using the above propositions.

Theorem 8 *If $f \in W^{1,1}(\mathbf{R})$, then the distribution derivatives of $\mathcal{M}_l f$ and $\mathcal{M}_r f$ become integrable functions and*

$$\|(\mathcal{M}_l f)'\|_1 \leq \|f'\|_1, \quad \|(\mathcal{M}_r f)'\|_1 \leq \|f'\|_1. \tag{25}$$

Proof. It suffices to prove only the case $\mathcal{M}_l f$. We note that if $f \in W^{1,1}(\mathbf{R})$, then $|f| \in W^{1,1}(\mathbf{R})$ and

$$\||f|'\|_1 = \|f'\|_1. \tag{26}$$

(See [GT].) By the fact that $|f| \in W^{1,1}(\mathbf{R})$ we note further that $|f|$ becomes a continuous function and hence the assumptions of Propositions 5–7 are satisfied.

First, we shall prove that $\mathcal{M}_l f$ have a weak derivative. Recall that

$$E = \{x \in \mathbf{R} \mid (\mathcal{M}_l f)(x) > |f(x)|\}$$

$$E = \sum_j I_j = \sum_j (\alpha_j,\ \beta_j).$$

Set $F = \mathbf{R} \setminus E$. From Propositions 5, 6 $\mathcal{M}_l f$ has a weak derivative $v \leq 0$ on each I_j. For a test function $\phi \in \mathcal{D}(\mathbf{R})$ we see then that

$$\int_{I_j} \mathcal{M}_l f \phi' \, dy = [|f(\beta_j)|\phi(\beta_j) - |f(\alpha_j)|\phi(\alpha_j)] - \int_{I_j} v\phi \, dy \tag{27}$$

by Proposition 7. It follows from (27) that

$$\begin{aligned}
\int_{\mathbf{R}} \mathcal{M}_l f \phi' \, dy &= \int_{E+F} \mathcal{M}_l f \phi' \, dy \\
&= \sum_j [|f(\beta_j)|\phi(\beta_j) - |f(\alpha_j)|\phi(\alpha_j)] - \int_E v\phi \, dy + \int_F |f|\phi' \, dy \\
&= \int_E |f|\phi' \, dy + \int_E |f|'\phi \, dy - \int_E v\phi \, dy + \int_F |f|\phi' \, dy \\
&= \int_{\mathbf{R}} |f|\phi' \, dy + \int_E |f|'\phi \, dy - \int_E v\phi \, dy = -\int_{\mathbf{R}} (\chi_E v + \chi_F |f|')\phi \, dy.
\end{aligned}$$

Here, χ_E and χ_F denote the indicator functions of the sets E and F respectively. This relation implies that $\mathcal{M}_l f$ has a weak derivative $(\mathcal{M}_l f)' = \chi_E v + \chi_F |f|'$.

Lastly, we shall prove (25). For each finite interval I_j, by the fact that $v \leq 0$ and Proposition 7 we have

$$\begin{aligned}
\int_{I_j} |v| \, dy &= (\mathcal{M}_l f)(\alpha_j) - (\mathcal{M}_l f)(\beta_j) \\
&= |f(\alpha_j)| - |f(\beta_j)| = \int_{I_j} |f|' \, dy \leq \int_{I_j} ||f|'| \, dy.
\end{aligned} \tag{28}$$

If there exists an infinite interval I_{j_1} such that

$$I_{j_1} = (-\infty,\ \beta_{j_1}),$$

then from Proposition 5 and the definition of I_{j_1} we see that

$$(\mathcal{M}_l f)(x) \geq (\mathcal{M}_l f)(\beta_{j_1}) > 0, \quad \forall x \in I_{j_1}. \tag{29}$$

(29) contradicts the weak type (1, 1) inequality for $\mathcal{M}_l f$. (See [St].)
If there exists an infinite interval I_{j_2} such that

$$I_{j_2} = (\alpha_{j_2},\ \infty),$$

then we have

$$\begin{aligned}
\int_{\alpha_{j_2}}^{r} |v| \, dy &= (\mathcal{M}_l f)(\alpha_{j_2}) - (\mathcal{M}_l f)(r) \\
&\leq |f(\alpha_{j_2})| - |f(r)| = \int_{\alpha_{j_2}}^{r} |f|' \, dy \leq \int_{\alpha_{j_2}}^{r} ||f|'| \, dy
\end{aligned} \tag{30}$$

for $\alpha_{j_1} < r$. From (28) and (30) we obtain

$$\|(\mathcal{M}_l f)'\|_1 \leq \||f|'\|_1 = \|f'\|_1$$

by (26). Thus, we have proved the theorem. ∎

3 Proof of Theorem 1

The proof of Theorem 1 follows now easily from Theorem 8. We shall need only the following lemma.

LEMMA 9 *Let f and g be the (real valued) integrable functions on the line. Set*

$$F(x) = \int_{-\infty}^{x} f\,dy,$$

$$G(x) = \int_{-\infty}^{x} g\,dy,$$

and

$$H(x) = \max\{F(x),\ G(x)\}.$$

Then the weak derivative of H becomes an integrable function and

$$\|H'\|_1 \le \|f\|_1 + \|g\|_1.$$

This lemma can be proved easily. (cf. [GT], Lemma 7.6.)

Theorem 1 can be now proved by Lemmas 2, 9 and Theorem 8.

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