Normal ultrafilters without the partition property

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1 Introduction

Let $\kappa$ be a measurable cardinal and $\kappa \leq \lambda$. Concerning the partition property of a normal ultrafilter on $\mathcal{P}_\kappa \lambda$, Solovay (see Menas [6]) proved the existence of a normal ultrafilter without the partition property under the assumption of that the existence of a certain large cardinal greater than $\kappa$. After Solovay established this result, Kunen (see Kunen-Pelletier [3]) improved his results, and proved that the existence of a normal ultrafilter without the partition property implies the existence of a certain large cardinal above $\kappa$. On the other hand, Menas [6] proved that there exist $2^{2^{\lambda^<\kappa}}$ normal ultrafilters with the partition property, if $\kappa$ is $2^{\lambda^<\kappa}$ supercompact. In the talk, we prove

Theorem 1 Suppose that $U$ is a normal ultrafilter on $\mathcal{P}_\kappa \lambda$ without the partition property. Define $\theta$ by

$$\text{Ult}_U(V) \models "\theta \text{ is the first Mahlo cardinal greater than } \lambda."$$

Then, it holds that

$$\text{Ult}_U(V) \models "\kappa \text{ is } \gamma\text{-supercompact for all } \gamma < \theta."$$

As a corollary, we have the following which has been proved in [1].

Corollary 2 If $\kappa$ is $\lambda$-supercompact, then there exists a normal ultrafilter on $\mathcal{P}_\kappa \lambda$ with the partition property.

2 Notations and definitions

We use standard $\mathcal{P}_\kappa \lambda$-combinatorial terminologies (e.g., see [2]). Throughout this paper, $\kappa$ denotes a regular uncountable cardinal. Let $A$ be a set such
that \( \kappa \subset A \). \( P_{\kappa}A \) denotes the set \( \{ x \subset A \mid |x| < \kappa \} \).

Let \( Y \subset P_{\kappa}A \). \([Y]^2\) denotes the set \( \{ (x, y) \in Y \times Y \mid x \subset y \text{ and } x \neq y \} \).

For any function \( f : [Y]^2 \to 2 \), a subset \( H \) of \( Y \) is said to be homogeneous for \( f \), if \( |f[H]^2| = 1 \).

For each \( x \in P_{\kappa}A \), \( \hat{x} \) denotes the set \( \{ y \in P_{\kappa}A \mid x \subset y \text{ and } x \neq y \} \).

Let \( U \) be a \( \kappa \)-complete ultrafilter on \( P_{\kappa}A \). The ultrapower of the universe \( V \) modular \( U \) is denoted by \( \text{Ult}_U(V) \). We say that \( U \) is fine, if \( \hat{x} \in u \) for all \( x \in P_{\kappa}A \). A fine ultrafilter \( U \) is said to be normal, if it is closed under the diagonal intersection. \( U \) has the partition property, if for any \( X \in U \) and any \( f : [X]^2 \to 2 \), there exists \( Y \in U \) such that \( Y \subset X \) and \( Y \) is homogeneous for \( f \).

3 Preparations for a proof of Theorem 1

In this section, we prove a lemma which will be used to prove the theorem.

Define \( X_0 \subset P_{\kappa}\lambda \) by:

\( x \in X_0 \) if and only if \( x \in P_{\kappa}\lambda \) and the following (1) and (2) hold.

(1) \( x \cap \kappa \) is a Mahlo cardinal.

(2) \( \xi \) is inaccessible iff \( \text{ot}(x \cap \xi) \) is inaccessible, for all \( \xi \in x \cup \{ \lambda \} \).

Since \( \langle \text{ot}(x \cap \xi) \mid x \in P_{\kappa}\lambda \rangle \) represents \( \xi \) in \( \text{Ult}_U(V) \) for every \( \xi \leq \lambda \), \( X_0 \in U \) for every normal ultrafilter \( U \) on \( P_{\kappa}\lambda \). Now we can prove the lemma.

Lemma 3 Let \( U \) be a normal ultrafilter on \( P_{\kappa}\lambda \) and \( \kappa \leq \gamma \leq \lambda \). Suppose that

\( \forall X \in U \exists (x, y) \in [X]^2 \ ( x \cap \gamma = y \cap \gamma ) . \)

Let \( \sigma \) be the least ordinal \( \delta \leq \lambda \) which satisfies

\( \forall X \in U \exists (x, y) \in [X]^2 \ ( x \cap \gamma = y \cap \gamma \text{ and } x \cap \delta \neq y \cap \delta ) . \)

Then, \( \sigma \) is a Mahlo cardinal.

Proof For each \( \xi \in [\gamma, \sigma) \), take a \( Y_\xi \in U \) such that

\( \forall (x, y) \in [Y_\xi]^2 \ ( \text{ if } x \cap \gamma = y \cap \gamma \text{ then } x \cap \xi = y \cap \xi ) . \)

Set \( X_1 = X_0 \cap \Delta_{\gamma \leq \xi < \sigma} Y_\xi \). Note that, for any \( (x, y) \in [X_1]^2 \), if \( x \cap \gamma = y \cap \gamma \)

and \(x \cap \sigma \neq y \cap \sigma\) then \(y \cap \sigma\) is an end extension of \(x \cap \sigma\).

We first show that \(\sigma\) is a strong limit cardinal. To get a contradiction, assume that there is a \(\delta < \sigma\) such that \(\sigma \leq 2^\delta\). Put
\[
Y_0 = \{ x \in X_1 \mid \delta \in x \text{ and } \text{ot}(x \cap \sigma) \leq 2^{\text{ot}(x \cap \delta)} \}.
\]
Since \(\sigma \leq 2^\delta\) also holds in \(\text{Ult}_U(V)\) and \(\langle \text{ot}(x \cap \delta) \mid x \in P_\kappa \lambda \rangle\) represents \(\delta\), we have that \(Y_0 \in U\). For each \(\alpha < \kappa\), take an injection \(f_\alpha : 2^\alpha + 1 \to P(\alpha)\). For each \(x \in Y_0\), let \(\pi_x : \text{ot}(x \cap \delta) \to x \cap \delta\) be the order isomorphism, and put \(a_x = \pi_x''f_{\text{ot}(x \cap \delta)}(\text{ot}(x \cap \sigma))\). Since \(a_x \subset x \cap \delta\) for all \(x \in Y_0\), there is an \(A \subset \delta\) such that
\[
Y_1 = \{ x \in Y_0 \mid a_x = A \cap x \} \in U.
\]
Take a pair \((x, y) \in [Y_1]^2\) such that \(x \cap \gamma = y \cap \gamma\) and \(x \cap \sigma \neq y \cap \sigma\). Since \(\delta \in x \subset y\), it holds that \(x \cap \delta = y \cap \delta\). By this, we have \(\pi_x = \pi_y\) and \(a_x = A \cap x \cap \delta = A \cap y \cap \delta = a_y\). So, \(\text{ot}(x \cap \sigma) = \text{ot}(y \cap \sigma)\). This contradicts that \(y \cap \sigma\) is an end extension of \(x \cap \sigma\).

Next, we show that \(\sigma\) is a regular cardinal. To get a contradiction, assume that \(\delta = \text{cof}(\sigma) < \sigma\). Take a normal cofinal function \(f : \delta \to \sigma\). Put
\[
Y_2 = \{ x \in X_1 \mid \delta \in x \text{ and } x \text{ is } f, f^{-1}\text{-closed and } f''x \cap \delta \text{ is cofinal in } x \cap \sigma \}.
\]
It is easy to check that \(Y_2 \in U\). So, there is a pair \((x, y) \in [Y_2]^2\) such that \(x \cap \gamma = y \cap \gamma\) and \(x \cap \sigma \neq y \cap \sigma\). Since \(\delta \in x \subset y\), it holds that \(x \cap \delta = y \cap \delta\). So, we have that \(\text{sup}(x \cap \sigma) = \text{sup}f''x \cap \delta = \text{sup}f''y \cap \delta = \text{sup}(y \cap \sigma)\). This contradicts that \(y \cap \sigma\) is an end extension of \(x \cap \sigma\).

Finally we show that \(\sigma\) is a Mahlo cardinal. Note that \(\text{ot}(x \cap \sigma)\) is inaccessible for all \(x \in X_1\), since \(X_0 \subset X_1\) and \(\sigma\) is inaccessible. Put \(S = \{ \alpha < \sigma \mid \alpha \text{ is inaccessible} \}\). To get a contradiction, assume that \(S\) is non-stationary. Take a closed unbounded subset \(C\) of \(\sigma\) such that \(\text{min}C > \gamma\) and \(S \cap C = \emptyset\). For each \(x \in P_\kappa \lambda\), let \(\rho_x : \text{ot}(x \cap \sigma) \to x \cap \sigma\) be an order isomorphism and put \(C_x = \rho^{-1}(x \cap C)\). Since \(\langle C_x \mid x \in P_\kappa \lambda \rangle\) represents \(C\) in \(\text{Ult}_U(V)\), it holds that
\[
Y_3 = \{ x \in X_1 \mid C_x \text{ is club in } \text{ot}(x \cap \sigma) \} \in U.
\]
Take a pair \((x, y) \in [Y_3]^2\) such that \(x \cap \gamma = y \cap \gamma\) and \(x \cap \sigma \neq y \cap \sigma\). Let \(\eta\) be
the least element of $y \cap \sigma \setminus x \cap \sigma$ and $\bar{\eta} = \rho^{-1}(\eta)$. Since $\text{ot}(x \cap \sigma) = \text{ot}(y \cap \eta)$, we have that $\rho_x = \rho_y \upharpoonright \bar{\eta}$ and $\text{ot}(y \cap \eta)$ is inaccessible. So, $\bar{\eta} \in C_y$ and $\eta$ is inaccessible. Hence $\eta \in C \cap S$. This is a contradiction. \hfill \square

4 Proofs of Theorem 1 and Corollary 2

In order to prove the theorem, we need the notion of $\omega$-Jonsson functions and some known results. Let $S$ be an infinite set. We denote by $\omega S$ the set of functions from $\omega$ to $S$. A function $F$ from $\omega S$ to $S$ is called an $\omega$-Jonsson function for $S$ if $F^\omega T = S$ for any $T \subset S$ with $|T| = |S|$. Concerning $\omega$-Jonsson functions, Erdős-Hajnal (e.g., see [2, Theorem 23.13]) proved:

Lemma 4 (Erdős-Hajnal) For any infinite set $S$, there exists an $\omega$-Jonsson function for $S$.

Solovay proved:

Lemma 5 (Solovay [5]) Let $U$ be a normal ultrafilter on $P_\kappa \lambda$ and $F : \omega \lambda \to \lambda$ an $\omega$-Jonsson function. Then
\[ \{ x \in P_\kappa \lambda \mid F \upharpoonright \omega x \text{ is an } \omega\text{-Jonsson function for } x \} \in U. \]

The next lemma is due to Magidor.

Lemma 6 (Magidor [4]) If $\kappa$ is $\lambda$-supercompact and $\lambda$ is $\theta$-supercompact, then $\kappa$ is $\theta$-supercompact.

The next lemma is due to Menas.

Lemma 7 (Menas [6]) Let $U$ be a normal ultrafilter on $P_\kappa \lambda$. Then, the following (a) and (b) are equivalent.
(a) $U$ has the partition property.
(b) There exists an $X \in U$ such that $\forall (x, y) \in [X]^2 \ (|x| < |y \cap \kappa|)$.

Now we can prove the theorem.
Theorem 1. Suppose that $U$ is a normal ultrafilter on $\mathcal{P}_\kappa \lambda$ without the partition property. Define $\theta$ by
\[ \text{Ult}_U(V) \models "\theta \text{ is the first Mahlo cardinal greater than } \lambda". \]
Then, it holds that
\[ \text{Ult}_U(V) \models "\kappa \text{ is } \gamma\text{-supercompact for all } \gamma < \theta". \]

Proof. To get a contradiction, assume that
\[ \text{Ult}_U(V) \models "\kappa \text{ is not } \gamma\text{-supercompact for some } \gamma < \theta". \]
Define $f : \mathcal{P}_\kappa \lambda \to \kappa$ by
\[ f(x) = \text{the least Mahlo cardinal greater than } \text{ot}(x). \]
Since $f$ represents $\theta$ in $\text{Ult}_U(V)$,
\[ Y_0 = \{ x \in X_0 \mid x \cap \kappa \text{ is not } \xi\text{-supercompact for some } \xi < f(x) \} \in U. \]
Let $\gamma = \sup \{ \delta \leq \lambda \mid \delta \text{ is a Mahlo cardinal} \}$. Since $\gamma$ satisfies the same statement in $\text{Ult}_U(V)$, it holds that
\[ Y_0 = \{ x \in X_0 \mid \text{ot}(x \cap \gamma) = \sup \{ \delta \leq \text{ot}(x) \mid \delta \text{ is a Mahlo cardinal} \} \in U. \]
Furthermore, since
\[ \text{Ult}_U(V) \models "\kappa \text{ is } \xi\text{-supercompact for all } \xi < \gamma", \]
it holds that
\[ Y_1 = \{ x \in Y_0 \mid x \cap \kappa \text{ is } \xi\text{-supercompact for all } \xi < \text{ot}(x \cap \gamma) \} \in U. \]
By the previous lemma, we can take a $Z \in U$ such that $x \cap \gamma \neq y \cap \gamma$, for all $(x, y) \in Z$. Take an $\omega$-Jonsson function $F$ for $\gamma$ and put
\[ Y_3 = \{ x \in Y_2 \cap Z \mid F \upharpoonright \text{inf}(x \cap \gamma) \text{ is } \omega\text{-Jonsson function for } x \cap \gamma \} \in U. \]
Note that $|x \cap \gamma| < |y \cap \gamma|$ for all $(x, y) \in [Y_3]^2$. Since $U$ does not have the partition property, there is a pair $(x, y) \in [Y_3]^2$ such that $y \cap \kappa \leq |x|$. Since $x \in Y_1$ and $y \cap \kappa$ is Mahlo, it holds that $y \cap \kappa \leq \text{ot}(x \cap \gamma)$. So, $x \cap \kappa$ is $\xi$-supercompact for all $\xi < y \cap \kappa$. By this, since $y \in X_2$, it holds that
\[ x \cap \kappa \text{ is } \xi\text{-supercompact for all } \xi < \text{ot}(y \cap \gamma). \]
So, $f(x) \leq \text{ot}(y \cap \gamma)$. Hence, it holds that
\[ x \cap \kappa \text{ is not } \xi\text{-supercompact for some } \xi < \text{ot}(y \cap \gamma). \]
This is a desired contradiction. \qed
Corollary 2 directly follows from Theorem 1 and the following Menas's result.

**Lemma 8** (Menas [5]) *If κ is λ-supercompact, then there exists a normal ultrafilter U on \( P_\kappa \lambda \) such that
\[
\text{Ult}_U(V) \models \kappa \text{ is not } \lambda\text{-supercompact.}
\]

**References**


