数理解析研究所講究録 1202 巻 2001 年 33-37

WHAT IF λ IS A STRONG LIMIT SINGULAR CARDINAL ?

名古屋大学・人間情報学研究科 松原 洋 (Yo Matsubara)

Nagoya University

1. Background

Let κ denote a regular uncountable cardinal and λ a cardinal $\geq \kappa$. Let $\mathcal{P}_\kappa\lambda$ denote the set $\{x \subset \lambda | |x| < \kappa\}$. We refer the reader to Kanamori [6, Section 25] for basic facts about the combinatorics of $\mathcal{P}_\kappa\lambda$.

Suppose I is an ideal over $\mathcal{P}_\kappa\lambda$. Let $I^+ = \{X \subseteq \mathcal{P}_\kappa\lambda \mid X \notin I\}$. Let $\mathbb{P}_I$ denote the p.o. of members of I^+ ordered by $X \leq_{\mathbb{P}_I} Y \iff X \subseteq Y$.

Definition 1.1.
We say that an ideal I is precipitous if $\Vdash_{\mathbb{P}_I}$ "Ult$(V; G)$ is wellfounded".

Let $NS_{\kappa\lambda} = \{X \subseteq \mathcal{P}_\kappa\lambda \mid X$ is the non-stationary$\}$. $NS_{\kappa\lambda}$ is known as the non-stationary ideal over $\mathcal{P}_\kappa\lambda$. For a stationary $X \subseteq \mathcal{P}_\kappa\lambda$, let $NS_{\kappa\lambda} \mid X$ denote the ideal over $\mathcal{P}_\kappa\lambda$ defined by $Y \in NS_{\kappa\lambda} \mid X \iff Y \cap X \in NS_{\kappa\lambda}$.

Can $NS_{\kappa\lambda}$ or $NS_{\kappa\lambda} \mid X$ be precipitous ?

Answer. : *Yes (sometimes assuming ...).*

Note The existence of a precipitous ideal has the strength of some large cardinal because it provides us with a "*generic*" elementary embedding of $\mathbf{V}$.

Theorem 1.2 (Foreman, Magidor, Shelah, Goldring) [3][6].
If λ is regular and δ is a Woodin cardinal $>\lambda$, then $\Vdash_{Coll(\lambda,<\delta)}$ "$NS_{\kappa\lambda}$ is precipitous". ($Coll(\lambda, < \delta)$ is the Levy collapse of δ to λ^+.)

Question. *What if λ is singular ?*

Burke and Matsubara [1] conjectured that $NS_{\kappa\lambda}$ cannot be precipitous if λ is singular.

Definition 1.3. Let δ be a cardinal. We say that an ideal I is *δ-saturated* if $\mathbb{P}_I$ satisfies the δ chain condition .

Fact. If I is a λ^+-saturated κ-complete normal ideal over $\mathcal{P}_\kappa\lambda$, then I is precipitous.

Note. $NS_{\kappa\lambda}$ is the minimal κ-complete normal ideal over $\mathcal{P}_\kappa\lambda$.

Theorem 1.4 (Foreman-Magidor) [2].
Unless $\kappa = \lambda = \aleph_1$, $NS_{\kappa\lambda}$ cannot be λ^+-saturated.

What about $NS_{\kappa\lambda} \mid X$?

名古屋大学・人間情報学研究科 松原 洋 (YO MATSUBARA)

Menas' Conjecture. *Every stationary subset of $\mathcal{P}_\kappa\lambda$ can be partitioned into $\lambda^{<\kappa}$ disjoint stationary sets.*

It turned out that Menas' Conjecture is independent of **ZFC**.

Theorem 1.5. $L \models$ *"Menas' Conjecture holds".*

Theorem 1.6(Gitik) [5]. *Suppose that κ is supercompact and $\lambda > \kappa$. Then $\exists$ p.o. $\mathbb{P}$ that preserves cardinals $\geq \kappa$ such that $\Vdash_{\mathbb{P}}$ "κ is inaccessible and $\exists$ stationary $X \subseteq \mathcal{P}_\kappa\lambda$ such that X cannot be partitioned into κ^+ disjoint stationary sets".*

2. Main results

Theorem 2.1 (Matsubara-Shelah)[9]. *If λ is a strong limit singular cardinal then $NS_{\kappa\lambda}$ is nowhere precipitous (i.e. $NS_{\kappa\lambda} \mid X$ is not precipitous for every stationary $X \subseteq \mathcal{P}_\kappa\lambda$).*

Theorem 2.2 [9]. *If λ is a strong limit singular cardinal then every stationary subset of $\mathcal{P}_\kappa\lambda$ can be partitioned into $\lambda^{<\kappa}$ disjoint stationary sets.*

One of the ingredients of the proof is the following lemma.

Lemma 2.3. *If $2^{<\kappa} < \lambda^{<\kappa} = 2^\lambda$, then*

(i) *every stationary subset of $\mathcal{P}_\kappa\lambda$ can be partitioned into $\lambda^{<\kappa}$ disjoint stationary sets and*
(ii) *$NS_{\kappa\lambda}$ is nowhere precipitous.(Matsubara-Shioya).*

Remark.

(1) The hypothesis of Lemma 2.3 is satisfied if λ is a strong limit cardinal with $\mathrm{cf}(\lambda) < \kappa$.
(2) Under the hypothesis of Lemma 2.3, if $X \subseteq \mathcal{P}_\kappa\lambda$ has size $< 2^\lambda$ then X is bounded and therefore non-stationary.

For the proof of (i) see page 345 of Kanamori [8].

proof of (ii).
Consider the following game G_ω between two players, **Nonempty** and **Empty**.

Nonempty	X_1		X_2	...	X_n	...
Empty		Y_1		Y_2 ...	Y_n	...

Nonempty and **Empty** alternately choose stationary sets $X_n, Y_n \subseteq \mathcal{P}_\kappa\lambda$ respectively so that $X_n \supseteq Y_n \supseteq X_n$ for n=1,2,3,....

After ω moves, Empty wins G_ω if $\bigcap_{n=1}^{\infty} X_n = \emptyset$

Fact. $NS_{\kappa\lambda}$ is nowhere precipitous iff **Empty** has a winning strategy in G_ω.

For the proof of this fact, see [4]. Let $\langle f_\alpha \mid \alpha < 2^\lambda \rangle$ enumerate functions from $\lambda^{<\omega}$ into $\mathcal{P}_\kappa\lambda$.
For a function $f : \lambda^{<\omega} \to \mathcal{P}_\kappa\lambda$, let

$$\underbrace{C(f)}_{\text{club set generated by} f} = \{s \in \mathcal{P}_\kappa\lambda \mid \bigcup f\text{"}s^{<\omega} \subseteq s\}$$

Fact. $X \subseteq \mathcal{P}_\kappa\lambda$ is stationary iff $\forall\alpha < 2^\lambda$ $C(f_\alpha) \cap X \neq \emptyset$.

We now describe **Empty**'s strategy. Suppose **Nonempty** plays X_1. Choose a sequence $\langle s^1_\alpha \mid \alpha < 2^\lambda\rangle$ from X_1 by induction on α as follow: Pick an element from $X_1 \cap C(f_0)$ and call it s^1_0.
Given $\langle s^1_\alpha \mid \alpha < \beta\rangle$ for some $\beta < 2^\lambda$, pick $s^1_\beta \in X_1 \cap C(f_\beta) \setminus \underbrace{\{s^1_\alpha \mid \alpha < \beta\}}_{\text{non-stationary}}$.

Let **Empty** play $Y_1 = \{s^1_\alpha \mid \alpha < 2^\lambda\}$. Now suppose **Nonempty** plays X_n immediately following **Empty**'s move $Y_{n-1} = \{s^{n-1}_\alpha \mid \alpha < 2^\lambda\}$.
Choose $\langle s^n_\alpha \mid \alpha < 2^\lambda\rangle$ a sequence from X_n as follows:
Pick $s^n_0 \in (X \cap C(f_\beta)) \setminus \underbrace{(\{s^{n-1}_\alpha \mid \alpha \leq \beta\} \cup \{s^n_\alpha \mid \alpha < \beta\})}_{\text{non-stationary}}$.

Let **Empty** play $Y_n = \{s^n_\alpha \mid \alpha < 2^\lambda\}$.

Claim. *This is a winning strategy for* **Empty**

proof: We want to show that $\bigcap_{n=1}^{\infty} Y_n = \emptyset$.

Suppose otherwise, say $t \in \bigcap_{n=1}^{\infty} Y_n$. For each $n < \omega$, $\exists\,!\ \alpha_n < 2^\lambda$ such that $t = s^n_{\alpha_n}$.
It is easy to see that $\alpha_n > \alpha_{n+1}$ for each n. ($s^n_\beta \notin \{s^n_\alpha \mid \alpha \leq \beta\}$ etc ...)

We now prove Theorem 2.2 assuming Theorem 2.1 and Lemma 2.3 (i).

proof of Theorem 2.2. : Let λ be a strong limit singular cardinal . If $\mathrm{cf}(\lambda) < \kappa$ then by Lemma 2.3 (i) we are done.

Assume $\mathrm{cf}(\lambda) \geq \kappa$. In this case $\lambda^{<\kappa} = \lambda$. So it is enough to show that $NS_{\kappa\lambda} \mid X$ is not λ-saturated for every stationary $X \subseteq \mathcal{P}_\kappa\lambda$.

But this is a consequence of $NS_{\kappa\lambda}$ being nowhere precipitous . In fact we know that $NS_{\kappa\lambda} \mid X$ cannot be λ^+-saturated for every stationary $X \subseteq \mathcal{P}_\kappa\lambda$.

proof of Theorem 2.1. : We now tamper with the definition of $\mathcal{P}_\kappa\lambda$.
From now on we let $\mathcal{P}_\kappa\lambda = \{s \subseteq \lambda \mid |s| < \kappa, s \cap \kappa \in \kappa\}$. This set is club in $\{s \subseteq \lambda \mid |s| < \kappa\}$. The following is the advantage of this change:
$X \subseteq \mathcal{P}_\kappa\lambda$ is stationary iff $\forall f : \lambda^{<\omega} \to \lambda \quad C[f] \cap X \neq \emptyset$
where $C[f] = \{s \in \mathcal{P}_\kappa\lambda \mid s \text{ is closed under} f\}$.

Let λ be a strong limit singular cardinal. By Lemma 2.3 (ii) we may assume that $\mathrm{cf}(\lambda) \geq \kappa$. Let $\langle\lambda_i \mid i < \mathrm{cf}(\lambda)\rangle$ be a continuous increasing sequence of strong limit singular cardinals converging to λ. Let $T = \{i < \mathrm{cf}(\lambda) \mid \mathrm{cf}(i) < \kappa\}$.
For each $i \in T$, let $E_i = \{s \in \mathcal{P}_\kappa\lambda \mid \sup(s) = \lambda_i, \lambda_i \notin s\}$

Note.

(i) $|E_i| = 2^{\lambda_i}$
(ii) $\bigcup_{i\in T} E_i$ is club in $\mathcal{P}_\kappa\lambda$.

For each $i \in T$, let $\langle f^i_\epsilon \mid \epsilon < 2^{\lambda_i}\rangle$ enumerate all of the functions whose domain $\subseteq \lambda_i^{<\omega}$ and range $\subseteq \lambda_i$.

Definition 2.4. $C^i[f^i_\epsilon] = \{s \in E_i \mid s^{<\omega} \subseteq dom(f^i_\epsilon)$ and s is close

To show $NS_{\kappa\lambda}$ is nowhere precipitous we will present a win **Empty** in G_ω.
Suppose W_1 is **Nonempty**'s first move in G_ω. For each $i \in T$, we a "*local game*" where each player altenately chooses subsets of E_i **Nonempty**'s first move is $W_1 \cap E_i$.
Local game $G(i)$

For each $i \in T$, define a game $G(i)$ as follows:
Nonempty and **Empty** alternately choose $X_n, Y_n \subseteq E_i$ resp $1, 2, \ldots,$ so that $X_n \supseteq Y_n \supseteq X_{n+1}$ and $\forall \epsilon < 2^{\lambda_i}$ ($|C^i[f^i_\epsilon] \cap$ $C^i[f^i_\epsilon] \cap Y_n \neq \emptyset$).
Empty wins $G(i)$ iff $\bigcap_{n=1}^{\infty} X_n = \emptyset$.
Just as in the proof of Lemma 2.3 (ii) we can show that **Empt** strategy, say τ_i in G_i.

$$
\begin{array}{llcccc}
G_\omega & \textbf{Nonempty} & W_1 & & W_2 & \\
 & \textbf{Empty} & \downarrow & \bigcup_{i\in T} \tau_i(\langle W \cup E_i \rangle) & \downarrow & \bigcup_{i \in} \\
G(i) & & W_1 \cap E_i & \uparrow & W_2 \cap E_i & \\
(i \in T) & & & \tau_i(\langle W \cap E_i \rangle) & &
\end{array}
$$

The following lemma tells us that we can combine τ_i's for $i \in T$ for G_ω.

Lemma 2.5. *Suppose $W \subseteq \mathcal{P}_\kappa\lambda$ is stationary. If $U \subseteq \mathcal{P}_\kappa\lambda$ satis condition (♯) then U is stationary.*
(♯) For each $i \in T$, $\forall \epsilon < 2^{\lambda_i}$($|C^i[f^i_\epsilon] \cap W| = 2^{\lambda_i} \longrightarrow C^i[f^i_\epsilon] \cap U \neq$

Now we describe **Empty**'s (combined) strategy σ in G_ω. Sup plays W_1.
Let **Empty** play $\bigcup_{i\in T} \tau_i(\langle W_1 \cap E_i \rangle) \overset{def}{=} \sigma(\langle W_1 \rangle)$.
Suppose

$$
\begin{array}{ccccc}
W_1 & & W_2 & & \ldots \quad W_n \\
 & \sigma(\langle W_1 \rangle) & & \sigma(\langle W_1, W_2 \rangle) & \ldots
\end{array}
$$

is the run of the game G_ω so far.
Let

$$\sigma(\langle W_1, W_2, \ldots, W_n \rangle) \overset{def}{=} \bigcup_{i\in T} \tau_i(\langle W_1 \cap E_i, W_2 \cap E_i, \ldots, W$$

Lemma 2.5 guarantees that σ provides **Empty** a legal move i.e. st of **Nonempty**'s last move. This σ is a winning strategy for **Emp** The proof of Lemma 2.5 depends upon the following lemma whos theory.

Lemma 2.6. *Suppose $U \subseteq \mathcal{P}_\kappa\lambda$. If $\forall i \in T$ $|U \cap E_i| < 2^{\lambda_i}$, then U is*

To prove the last lemma, we need the following fact from pcf theo

pcf Fact. *$\exists$ club $C \subseteq cf(\lambda)$ such that $pp(\lambda_i) = 2^{\lambda_i}$ for every $i \in C$.*

See Shelah "Cardinal Arithmetic" [12] Conclusion 5.13 page 414 and Hotz,Steffens,Weitz "Introduction to Cardinal Arithmetic" [7] Theorem 9.1.3 page 271.

References

1. D. Burke and Y. Matsubara, *The extent of strength of the club filters*, Israel Journal of Mathematics **114** (1999), 253–263.
2. M. Foreman, and M. Magidor, *Mutually stationary sequences of sets and the non-saturation of the non-stationary ideal on $\mathcal{P}_\kappa\lambda$*.
3. M. Foreman, M. Magidor and S. Shelah, *Martin's Maximum, saturated ideals, and non-regular ultrafilters. Part I*, Annals of Mathematics **127** (1988), 1–47.
4. F. Galvin, T. Jech and M. Magidor, *An ideal game*, Journal of Symbolic Logic **43** (1978), 284–292.
5. M. Gitik, *Nonsplitting subset of $\mathcal{P}_\kappa(\kappa^+)$*, Journal of Symbolic Logic **50** (1985), 881–894.
6. N. Goldring, *The entire NS ideal on $\mathcal{P}_\gamma(\mu)$ can be precipitous*, Journal of Symbolic Logic **62** (1997), 1162–1172.
7. M. Holz, K. Steffens and E. Weitz, *Introduction to Cardinal Arithmetic*, Birkhäuser, 1999.
8. A. Kanamori, *The Higner Infinite*, Springer-Verlag, 1994.
9. Y. Matsubara, and S. Shelah, *Nowhere precipitousness of the non-stationary ideal over $\mathcal{P}_\kappa\lambda$*.
10. Y. Matsubara and M. Shioya, *Nowhere precipitousness of some ideals*, Journal of Symbolic Logic **63** (1998), 1003–1006.
11. T. Menas, *On strong compactness and supercompactness*, Annals of Mathematical Logic **7** (1974), 327–359.
12. S. Shelah, *Cardinal Arithmetic*, Oxford Science Publications, 1994.