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Models of real-valued measurability

北見工業大学 情報システム工学科 渕野 昌 (Sakaé Fuchino)

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by

Sakaé Fuchino and Saharon Shelah

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Abstract

Starting from a model of ZFC with a measurable cardinal κ, we construct a generic extension in which κ is real-valued measurable, $2^{\aleph_0} > \kappa$ and club principle for κ holds. This gives a positive answer to a question of D.H. Fremlin asking the existence of models of real-valued measurability with some combinatorial behavior different from that of Solovay's model of real-valued measurablility. Some other models related to this question will be given in the forthcomming [4].

1 Introduction

In his celebrated paper [9], Solovay proved that, if κ is a measurable cardinal, then, by forcing with measure algebra $\mathcal{B}_\lambda$ of Maharam type $\lambda \geq \kappa$ for appropriate

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λ, we obtain a model in which κ is real-valued measurable. Recall that κ is said to be real-valued measurable if there is a κ additive atomless measure $\mu : \mathcal{P}(\kappa) \to [0,1]$. Existence of such a cardinal is equivalent to the extendability of Lebesque measure to a σ-additive measure on the whole $\mathcal{P}(\mathbb{R})$ — which, of course, cannot be translation invariant under the axiom of choice due to Vitali's theorem.

Some properties of Solovay's model follow simply from the fact that the model is obtained by adding random reals; for example we have the equations $cov(null) = \lambda$ and $non(null) = \aleph_1$ in the model while $\mathbf{b}$ and $\mathbf{d}$ remain as in the ground model.

On the other hand, it is also known that the existence of a real-valued measurable cardinal alone implies a lot of combinatorial consequences. For example:

Theorem 1.1 *Suppose that $\kappa \leq 2^{\aleph_0}$ is real-valued measurable. Then:*

(1) (see [2]) *$non(null) = \aleph_1$, $cov(null) \geq \kappa$, $\mathbf{b} \neq \kappa$, $\mathbf{d} \neq \kappa$.*

(2) *κ has the tree property.*

(3) (Kunen) *If $\kappa = 2^{\aleph_0}$ then $\diamondsuit_\kappa$ holds (actually $\diamondsuit_\kappa(S)$ holds for a lot of stationary $S \subseteq \kappa$* (Ketonen)*).* ❑

For more complete list of such implications see e.g. [2].

Against this back-ground, D.H. Fremlin asked if there is a model of real-valued measurability which is intrinsically different from Solovay's models. As one of the possible answers to this question, we present here a new model in which we have a real-valued measurable κ strictly less than the continuum while club principle for κ holds.

2 Preliminaries

For a set u let $B_{(u)} = Borel({}^u2)$, the set of all Borel sets in the generalized Cantor discontinuum ${}^u2$. $B_{(u)}$ is also seen as the Boolean algebra with usual set operations. In particular, for $a \in B_{(u)}$ we denote with $-a$ the complement of a, i.e. $-a = {}^u2 \setminus a$. Strictly speaking $-a$ depends on u and hence this notation is rather ambiguous. This ie because we often identify a with the corresponding element of $Borel({}^{u'}2)$ for some $u' \supseteq u$ (see below): in this case $-a$ should denote the complement of the Borel subset of ${}^{u'}2$ which corresponds to a. Nevertheless it should be always clear from the context what is meant with this notation.

For $r \in \mathbb{R}$, $r > 0$, a mapping $\mu : B_{(u)} \to [0,r]$ is said to be a $[0,r]$-measure if

(a) $\mu(\emptyset) = 0$; $\mu({}^u2) = r$;

(b) for any pairwise disjoint $s_i \in B_{(u)}$, $i < \omega$,

$$\mu(\bigcup_{i<\omega} s_i) = \sum_{i<\omega} \mu(s_i).$$

Kolmogoroff's extension theorem can be formulated as follows:

Theorem 2.1 (Kolmogoroff) *Suppose that $\langle S_i, \mathcal{A}_i \rangle$ is a measurable space for $i \in I$ and μ_E is a $[0,1]$-measure over the product spapce of $\langle S_i, \mathcal{A}_i \rangle$, $i \in E$ (seen as the subspace of the product space $\langle \prod_{i \in I} S_i, \otimes_{i \in I} \mathcal{A}_i \rangle$ by the canonical embedding) for each $E \in [I]^{<\aleph_0}$ such that μ_E, $E \in [I]^{<\aleph_0}$ are pairwise compatible (as mappings). Suppose that for each $i \in I$, there is $\mathcal{C}_i \subseteq \mathcal{A}_i$ such that*

(i) *if $c_n \in \mathcal{C}_i$ for $n \in \omega$ and $\bigcap_{n \in \omega} c_n = \emptyset$, then there is $n_0 \in \omega$ such that $\bigcap_{n \in n_0} c_n = \emptyset$, and*

(ii) *$\mu_{\{i\}}(a) = \sup\{\mu_{\{i\}}(c) : c \in \mathcal{C}_i,\ c \subseteq a\}$ for all $a \in \mathcal{A}_i$.*

Then there is the unique $[0,1]$-measure over $\langle \prod_{i \in I} S_i, \otimes_{i \in I} \mathcal{A}_i \rangle$ extending all μ_E, $E \in [I]^{<\aleph_0}$.

In our context we may apply the theorem in the following form:

Corollary 2.2 *Suppose that $\mathcal{U}$ is a family of sets closed under union of finitely many members and μ_u is a $[0,1]$-measure on $B_{(u)}$ for $u \in \mathcal{U}$. If μ_u, $u \in \mathcal{U}$ are compatible to each other, then, letting $u^* = \bigcup \mathcal{U}$, there is the unique $[0,1]$-measure $\mu^* : B_{(u^*)} \to [0,1]$ extending all μ_u, $u \in \mathcal{U}$.*

Proof Let $I = u^*$. For $E \in [I]^{<\aleph_0}$, let $u \in \mathcal{U}$ be such that $E \subseteq u$ and $\mu_E = \mu_u \restriction B_{(E)}$. Applying Kolmogoroff's theorem to μ_E, $E \in [I]^{<\aleph_0}$, we obtain a $[0,1]$-measure μ^* on $B_{(u^*)} \cong \otimes_{i \in I} B_{(\{i\})}$ which is an extension of each μ_u, $u \in \mathcal{U}$ because of the uniqueness of $\mu^* \restriction B_{(u)}$. □ (Corollary 2.2)

For a $[0,1]$-measure μ on $B_{(u)}$, let $null(\mu) = \{a \in B_{(u)} : \mu(a) = 0\}$. For $a \in B_{(u)} \setminus null(\mu)$, $\mu || a$ is the $[0,1]$-measure on $B_{(u)}$ defined by

$$\mu || a\ (b) = \frac{\mu(b \cap a)}{\mu(a)}$$

for $b \in B_{(u)}$.

The following is easily seen:

Lemma 2.3 *For a $[0,1]$-measure μ on $B_{(u)}$ and $a, a' \in B_{(u)} \setminus null(\mu)$, $\mu || a = \mu || a'$ if and only if $a \triangle a' \in null(\mu)$.*

For u_1, u_2 with $u_1 \cap u_2 = \emptyset$ and $[0,1]$-measures $\mu_1 : B_{(u_1)} \to [0,1]$ and $\mu_2 : B_{(u_2)} \to [0,1]$, let $\mu_1 \otimes \mu_2 : B_{(u_1 \cup u_2)} \to [0,1]$ denote the product measure of μ_1 and μ_2. $\mu_1 \otimes \mu_2$ is characterized by:

$$\mu_1 \otimes \mu_2(a_1 \cap a_2) = \mu_1(a_1) \cdot \mu_2(a_2)$$

for all $a_1 \in B_{(u_1)}$ and $a_2 \in B_{(u_2)}$.

Lemma 2.4 (1) *For a $[0,1]$-measure μ on $B_{(u)}$ and $a, b \in B_{(u)} \setminus null(\mu)$ with $b \subseteq a$,*

$$\mu || b = (\mu || a) || b.$$

(2) *Suppose that $u_1 \cap u_2 = \emptyset$, $\mu_1 : B_{(u_1)} \to [0,1]$ and $\mu_2 : B_{(u_2)} \to [0,1]$ are $[0,1]$-measures, $a_1 \in B_{(u_1)} \setminus null(\mu_1)$ and $a_2 \in B_{(u_2)} \setminus null(\mu_2)$. Then*

$$(\mu_1 \otimes \mu_2) || (a_1 \cap a_2) = (\mu_1 || a_1) \otimes (\mu_2 || a_2).$$

Proof (1): Let $c \in B_{(u)}$. Then

$$\begin{aligned}((\mu||a)||b)(c) &= \frac{(\mu||a)(c \cap b)}{(\mu||a)(b)} \\ &= \frac{\mu(c \cap b \cap a)}{\mu(a)} \cdot \frac{\mu(a)}{\mu(a \cap b)} \\ &= \frac{\mu(c \cap b)}{\mu(a)} \cdot \frac{\mu(a)}{\mu(b)} \\ &= \frac{\mu(c \cap b)}{\mu(b)} = (\mu||b)(c)\end{aligned}$$

(2): It is enough to show that the both sides of the equation have the same value for elements of $B_{(u_1 \cup u_2)}$ of the form $c_1 \cap c_2$ for some $c_1 \in B_{(u_1)}$ and $c_2 \in B_{(u_2)}$. This can be shown as follows:

$$\begin{aligned}((\mu_1 \otimes \mu_2)||(a_1 \cap a_2))(c_1 \cap c_2) &= \frac{(\mu_1 \otimes \mu_2)(c_1 \cap c_2 \cap a_1 \cap a_2)}{(\mu_1 \otimes \mu_2)(a_1 \cap a_2)} \\ &= \frac{\mu_1(c_1 \cap a_1) \cdot \mu_2(c_2 \cap a_2)}{\mu_1(a_1) \cdot \mu_2(a_2)} \\ &= (\mu_1||a_1)(c_1) \cdot (\mu_2||a_2)(c_2) \\ &= ((\mu_1||a_1) \otimes (\mu_2||a_2))(c_1 \cap c_2)\end{aligned}$$

□ (Lemma 2.4)

For $u \subseteq v$, $B_{(u)}$ can be embedded canonically into $B_{(v)}$ by identifying each $a \in B_{(u)}$ with the element of $B_{(v)}$ with the same “definition”. In the following we always regard $B_{(u)}$ as the subalgebra of $B_{(v)}$ by this canonical embedding. In particular, e.g. if $a \in B_{(u)}$ and $b \in B_{(v)}$, then we mean with $a \cap b$ the intersection of b with the element of $B_{(v)}$ which corresponds to a by this embedding.

m_u denotes the Borel measure on $B_{(u)}$, i.e. the $[0,1]$-measure which makes $[\{\langle x, 0\rangle\}]$, $x \in u$ independent events with $m_u([\{\langle x,0\rangle\}]) = \frac{1}{2}$ where $[t]$, or $[t]_{B_{(u)}}$ for $t \in \mathrm{Fn}(u,2)$ denotes the basic clopen set: $\{f \in {}^u 2 : t \subseteq f\}$. Let $\mathcal{B}_u = B_{(u)}/null(m_u)$. $null(m_\omega)$ is also denoted simply by $null$.

Concerning forcing, we use "the reverse Jerusalem notation". I.e., in a p.o.-set P, a condition $p \in P$ is stronger than another condition $q \in P$ if $p \leq_P q$. P-names are denoted by $\underset{\sim}{X}$, $\underset{\sim}{Y}, \ldots$, $\underset{\sim}{f}$, $\underset{\sim}{g}, \ldots$, etc. We assume that P-names are constructed as in [8].

For ground model set X, $\check{X}$ denotes its standard P-name. V denotes the ground model and $\underset{\sim}{G}$ the standard name of V generic set over the p.o.-set.

3 Free amalgamation of measures

The following theorems are used in later sections.

Theorem 3.1 (D. Fremlin; an instance of 456N in [3]) *Suppose that $u_0 = u_1 \cap u_2$, and $\mu_1 : B_{(u_1)} \to [0,1]$, $\mu_2 : B_{(u_2)} \to [0,1]$ are $[0,1]$-measures such that $\mu_1 \restriction B_{(u_0)} = \mu_2 \restriction B_{(u_0)}$. Then there is a $[0,1]$-measure $\tilde{\mu} : B_{(u_1 \cup u_2)} \to [0,1]$ extending both μ_1 and μ_2 such that $\tilde{\mu}(a_1 \cap a_2) = \mu_1(a_1) \cdot \mu_2(a_2)$ for any $a_1 \in B_{(u_1)}$ and $a_2 \in B_{(u_2)}$ which are independent events over $B_{(u_0)}$ with respect to μ_1 and μ_2 respectively.*

Proof Let $\mathcal{R}$ be the subslgebra of $B_{(u_1 \cup u_2)}$ consisting of finite union of elements of the form:

$(*)$ $a_0 \cap a_1 \cap a_2$ where $a_0 \in B_{(u_0)}$, $a_1 \in B_{(u_1 \setminus u_0)}$ and $a_2 \in B_{(u_2 \setminus u_0)}$.

For $c \in \mathcal{R}$, let Δ_c be the set of the partitions of c consisting of elements of the form $(*)$. We consider Δ_c as a partial ordering with the ordering:

$$P \leq P' \quad \Leftrightarrow \quad P \text{ is a refinement of } P'$$

for P, $P' \in \Delta_c$.

Now, for $c \in \mathcal{R}$ and $P \in \Delta_c$, let

$$(\dagger) \qquad \mu^*_P(c) = \sup\Big\{ \sum_{a_0 \cdot a_1 \cdot a_2 \in P', \mu_1(a_0) \neq 0} \frac{\mu_1(a_0 \cap a_1) \cdot \mu_2(a_0 \cap a_2)}{\mu_1(a_0)} : P' \in \Delta_c,\ P' \leq P\Big\}.$$

Finally, for $c \in \mathcal{R}$, let

$$(\ddagger) \qquad \mu^*(c) = \text{the inverse limit of } \mu^*_P(c),\ P \in \Delta_c.$$

We can show that μ^* is σ-additive measure on $\mathcal{R}$. Hence, by Hopf's Extension Theorem, μ^* can be extended to a $[0,1]$-measure $\tilde{\mu} : B_{(u_1 \cup u_2)} \to [0,1]$ with the desired property. ❑ (Theorem 3.1)

We call $\tilde{\mu}$ as constructed in the proof of Theorem 3.1, the free amalgamation of μ_1 and μ_2 over u_0 and denote it with $\mu_1 \otimes_{u_0} \mu_2$. Note that, if $u_0 = \emptyset$, then $\mu_1 \otimes_{u_0} \mu_2$ is just the usual product measure of μ_1 and μ_2.

Theorem 3.1 can be extended to the following amalgamation theorem for infinitely may measures.

Theorem 3.2 *Suppose that $\mathcal{S}$ is a Δ-system with the root u^* and, for each $u \in \mathcal{S}$, let $\mu_u : B_{(u)} \to [0,1]$ be a $[0,1]$-measure such that $\mu_u \restriction B_{(u^*)} = \mu_{u'} \restriction B_{(u^*)}$ for any u, $u' \in \mathcal{S}$. Then there is a $[0,1]$-measure $\tilde{\mu} : B_{(\bigcup \mathcal{S})} \to [0,1]$ extending each of μ_u, $u \in \mathcal{S}$ such that for any $u_0, \ldots, u_{n-1} \in \mathcal{S}$ and $a_0 \in B_{(u_0)}, \ldots, a_{n-1} \in B_{(u_{n-1})}$, if $a_0, \ldots, a_{n-1}$ are independent over $B_{(0)}, \ldots, B_{(n-1)}$ with respect to $\mu_{u_0}, \ldots, \mu_{u_{n-1}}$ respectively, then $\tilde{\mu}(a_0 \cap \cdots \cap a_{n-1}) = \mu_{u_0}(a_0) \cdot \cdots \cdot \mu_{u_{n-1}}(a_{n-1})$.*

Proof The construction of measures in the proof of Theorem 3.1 can be extended to amalgamation of n measures for all $n \geq 2$ (see $(\dagger_n)$ and $(\ddagger_n)$ below). By this construction, we obtain a system $\{\mu_U : U \in [\mathcal{S}]^{<\aleph_0}\}$ of $[0,1]$-measures where $\mu_U : B_{\bigcup U} \to [0,1]$ such that

(1) μ_U, $U \in [\mathcal{S}]^{<\aleph_0}$ are pairwise compatible;

(2) each μ_U extends μ_u for all $u \in U$;

(3) for any $u_0, \ldots, u_{n-1} \in \mathcal{S}$ and $a_0 \in B_{(u_0)}, \ldots, a_{n-1} \in B_{(u_{n-1})}$, if $a_0, \ldots, a_{n-1}$ are independent over $B_{(0)}, \ldots, B_{(n-1)}$ with respect to $\mu_{u_0}, \ldots, \mu_{u_{n-1}}$ respectively, then $\tilde{\mu_U}(a_0 \cap \cdots \cap a_{n-1}) = \mu_{u_0}(a_0) \cdot \cdots \cdot \mu_{u_{n-1}}(a_{n-1})$ for $U \in [\mathcal{S}]^{<\aleph_0}$ with $u_0, \ldots, u_{n-1} \in U$.

Applying Corollary 2.2 to these μ_U's, we obtain a $[0,1]$-measure as desired. ❑ (Theorem 3.2)

We shall call the $[0,1]$-measure $\tilde{\mu}$ constructecd as in the proof of Theorem 3.2 the free amalgamation of μ_u, $u \in \mathcal{S}$ over u^*. The free amalgamation $\tilde{\mu}$ is characterized by the following equations which correspond $(\dagger)$ and $(\ddagger)$:

Let $\mathcal{R}'$ be the subset of $B_{(\bigcup \mathcal{S})}$ consisting of finite unions of elements of the form

(**) $\quad a_* \cap a_{u_0} \cap \cdots \cap a_{u_{n-1}}$ for $n \in \omega$ where $u_0, \ldots, u_{n-1} \in \mathcal{U}$ and $a_* \in B_{(u^*)}$, $a_{u_0} \in B_{(u_0 \setminus u^*)}, \ldots, a_{u_{n-1}} \in B_{(u_{n-1} \setminus u^*)}$.

For $c \in \mathcal{R}'$, let Δ_c be the set of partitions of c consisting of elements of the form (**). $P \leq P'$ for P, $P' \in \Delta_c$, let $P' \leq P$ be defined just as before.

For $c \in \mathcal{R}'$ of the form (**) and $P \in \Delta_c$, let

$$(\dagger_n) \quad \mu^*_P(c) = \sup\Big\{ \sum_{\substack{a'_* \cap a'_{u_0} \cap \cdots \cap a'_{u_{n-1}} \in P', \\ \mu_1(a'_*) \neq 0}} \frac{\mu_{u_0}(a'_* \cap a_0) \cdot \cdots \cdot \mu_{u_{n-1}}(a'_* \cap a'_{n-1})}{\mu_1(a'_*)} : P' \in \Delta_P,\ P' \leq P \Big\}.$$

Then

$$(\ddagger_n) \qquad \mu^*(c) = \text{the inverse limit of } \mu^*_P(c),\ P \in \Delta_c.$$

4 The model V^Q

For cardinals $\aleph_1 < \kappa \leq \lambda$, let $Q = Q_{\kappa,\lambda}$ be the p.o.-set defined as follows:

(A) $p \in Q \;\Leftrightarrow\; p = \langle u^p, \mu^p \rangle$ where:

(a) $u^p \in [\lambda]^{<\kappa}$.

(b) $|\, u \cap \theta^+ \,| < \theta$ for all strongly inaccessible θ.

(c) μ^p is a $[0,1]$-measure on $B_{(u^p)}$.

(B) For p, $q \in Q$, $q \leq_Q p \;\Leftrightarrow\;$ there is an $r \in Q$ such that $q \leq_{apr} r \leq_{pr} p$ where $\leq_{pr}$ and $\leq_{apr}$ are defined as follows where "pr" and "apr" stand for "pure" and "anti-pure" respectively:

(C) For p, $q \in Q$,

(a) $q \leq_{pr} p \;\Leftrightarrow\; u^p \subseteq u^q$ and μ^q extends μ^p. For $\gamma = \sup u^p$, $\mu^q \restriction B_{((u^q \cap \gamma) \cup u^p)}$ is the free product of μ^p and $\mu^q \restriction B_{((u^q \setminus u^p) \cap \gamma)}$.

(b) $q \leq_{apr} p \;\Leftrightarrow\; u^p = u^q$ and $\mu^q = \mu^p || a$ for some $a \in B_{(u^p)} \setminus null(\mu^p)$.

We shall call a as in (C)(b) *a witness of* $q \leq_{apr} p$. Note that the witness of $q \leq_{apr} p$ is unique upto elements of $null(\mu^p)$ by Lemma 2.3. We shall also say that a witnesses $q \leq_{apr} p$ and denote $q = p||a$.

For $p \in Q$, let $\mathcal{B}_{(p)} = B_{(u^p)}/null(\mu^p)$. For $a \in B_{(u^p)}$, $[a]_p$ (or $a/null(\mu^p)$) denotes the equivalence class of a modulo $null(\mu^p)$.

The following lemma shows that the relation $\leq_Q$ is a partial ordering on Q:

Lemma 4.1 (1) *$\leq_{pr}$ and $\leq_{apr}$ are transitive relations.*

(2) *For p, q, $r \in Q$, if $r \leq_{pr} q \leq_{apr} p$, then there is $q' \in Q$ such that $r \leq_{apr} q' \leq_{pr} p$.*

(3) *$\leq_Q$ is a partial ordering on Q.*

Proof (1) is clear by definition of $\leq_{pr}$ and $\leq_{apr}$. For (2), suppose that $r \leq_{pr} q \leq_{apr} p$. Let $a \in B_{(u^p)} \setminus null(\mu^p)$ witness $q \leq_{apr} p$. Without loss of generality, we may assume that $-a \in B_{(u^p)} \setminus null(\mu^p)$ as well. Put $u^{q'} = u^r$. Let $\mu^* : B_{(u^r)} \restriction (-a) \to [0, \mu^p(-a)]$ be any $[0, \mu^p(-a)]$-measure extending $\mu^p \restriction (B_{(u^p)} \restriction (-a))$ freely and let $\mu^{q'} : B_{(u^{q'})} \to [0, 1]$ be defined by

$$\mu^{q'}(b) = \mu^r(b \cap a) \cdot \mu^p(a) + \mu^*(b \setminus a)$$

for $b \in B_{(u^r)}$. Then, letting $q' = \langle u^{q'}, \mu^{q'} \rangle$, we have $q' \leq_{pr} p$ and $r \leq_{apr} q'$ by $r = q' || a$.

(3): It is enough to show the transitivity of $\leq_Q$. Suppose that p, q, $r \in P$ are such that $p \leq_Q q \leq_Q r$. By definition of $\leq_Q$, there are s, $s' \in Q$ such that

$$p \leq_{apr} s \leq_{pr} q \leq_{apr} s' \leq_{pr} r.$$

By (2) we can find an $s'' \in Q$ such that

$$p \leq_{apr} s \leq_{apr} s'' \leq_{pr} s' \leq_{pr} r.$$

It follows by (1) that $p \leq_{apr} s'' \leq_{pr} r$. Thus $p \leq_Q r$ as desired. □ (Lemma 4.1)

For $q \in Q$, let

$$AP_q^Q = \{p \in Q : p \leq_{apr} q\}.$$

Lemma 4.2 *Suppose that p, $q \in Q$ with $p \leq_{pr} q$ and $a \in B_{(u^q)} \setminus null(\mu^q)$. Then $p||a \leq_{pr} q||a$.*

Proof Let $\gamma = \sup u^q$ and $u = (u^p \setminus u^q) \cap \gamma$. It is enough to show that $(q||a) \restriction B_{((u^q \cap \gamma) \cup u^p)}$ is the free product of $p||a$ and $(q||a) \restriction B_{(u \cap \gamma)}$.

Suppose that $b_1 \in B_{(u^p)}$ and $b_2 \in B_{(u \cap \gamma)}$. Then by Lemma 2.4,(2) we have:

$$\begin{aligned}
(q||a)(b_1 \cap b_2) &= (q \restriction B_{((u^q \cap \gamma) \cup u^p)})||a(b_1 \cap b_2) \\
&= (p \otimes (q \restriction B_{(u \cap \gamma)}))||a(b_1 \cap b_2) \\
&= ((p||a) \otimes ((q \restriction B_{(u \cap \gamma)})||a))(b_1 \cap b_2) \\
&= p||a(b_1) \cdot (q \restriction B_{(u \cap \gamma)})||a(b_2).
\end{aligned}$$

□ (Lemma 4.2)

Lemma 4.3 (1) *For $q \in Q$, $\langle AP^Q_q, \leq_{apr}\rangle$ is equivalent to the forcing by the measure algebra of Maharam type $\leq |\, u^q\,|$.*

(2) $\langle AP^Q_q, \leq_{apr}\rangle \lessdot \langle Q, \leq_Q\rangle \restriction q$.

Proof (1): The mapping

$$\Phi : AP^Q_q \to \mathcal{B}_{(q)};\ q||a \mapsto [a]_q$$

is well-defined and a dense embedding of AP^Q_q into $\mathcal{B}_{(q)}$.

(2): First we show that, p, $p' \in AP^Q_q$ are compatible in $\langle Q, \leq_Q\rangle$ if and only if they are compatible in $\langle AP^Q_q, \leq_{apr}\rangle$.

If p, $p' \in AP^Q_q$ are compatible in $\langle AP^Q_q, \leq_{apr}\rangle$ then clearly they are also compatible in $\langle Q, \leq_Q\rangle$.

Conversely, suppose that p, $p' \in AP^Q_q$ are compatible in $\langle Q, \leq_Q\rangle$ and let $r^* \in Q$ be such that $r^* \leq_Q p$, p'. Let a and a' be the witnesses of $p \leq_{apr} q$ and $p' \leq_{apr} q$ respectively. Let r, $r' \in Q$ be such that $r^* \leq_{apr} r \leq_{pr} p$ and $r^* \leq_{apr} r' \leq_{pr} p'$, and let b, $b' \in B_{u^{r^*}}$ be witnesses of $r^* \leq_{apr} r$ and $r^* \leq_{apr} r'$ respectively. Then $\mu^{r^*}(a \cap b) = \mu^{r^*}(a' \cap b') = 1$. It follows that $a \cap a' \in B_{(u^q)} \setminus null(\mu^q)$. So letting $r^\dagger = q||(a \cap a')$, we have $r^\dagger \in AP^Q_q$ and $r^\dagger \leq_{apr} p$, p'. Thus p and p' are compatible in $\langle AP^Q_q, \leq_{apr}\rangle$.

Now suppose that $I \subseteq AP^Q_q$ is predense in $\langle AP^Q_q, \leq_{apr}\rangle$. It is enough to show that I is predense below q in $\langle Q, \leq_Q\rangle$.

So suppose $r \leq_Q q$. We show that r is compatible with an element of I. Let $s \in Q$ be such that $r \leq_{apr} s \leq_{pr} q$ and let $b \in B_{(u^s)}$ be such that $r = s||b$. Since I is predense below q, we have

$$\sum\nolimits^{\mathcal{B}_{(q)}}\{[a]_q\ :\ q||a \in I\} = 1_{\mathcal{B}_{(q)}}.$$

Hence there is $a \in B_{(u^q)}$ such that $q||a \in I$ and $c = a \cap b \in B_{(u^s)} \setminus null(\mu^s)$. Now $s||a \leq_{pr} q||a$ by Lemma 4.2. Hence, by Lemma 2.4,(1), $s||c = (s||a)||c \leq_Q q||a$. Similarly, $s||c = (s||a)||c = r||c \leq_{apr} r$. ❑ (Lemma 4.3)

Lemma 4.4 *Suppose that p, q, $r \in Q$, $q \leq_{pr} p$ and $r \leq_{apr} p$. Then q and r are compatible in Q. Furthermore, there is an $s \in Q$ such that $s \leq_{apr} q$ and $s \leq_{pr} r$.*

Proof Let $a \in B_{(u^p)}$ be a witness of $r \leq_{apr} p$. Then $q||a \leq_{pr} p||a = r$ by Lemma 4.2 and $q||a \leq_{apr} q$. Thus $s = q||a$ is as desired. ❑ (Lemma 4.4)

Lemma 4.5 *Suppose that p, $q \in Q$ and $u_0 = u^p \cap u^q$. If $\mu^p \restriction B_{(u_0)} = \mu^q \restriction B_{u_0}$ then p and q are compatible.*

Proof Let $u^r = u^p \cup u^q$ and $\mu^r = \mu^p \otimes_{u_0} \mu^q$. Then $r \leq_{pr} p$, q. $\square$ (Lemma 4.5)

For p, $q \in Q$ with $\mu^p \restriction B_{(u_0)} = \mu^q \restriction B_{u_0}$ for $u_0 = u^p \cap u^q$, r as in the proof of Lemma 4.5 is denoted by $p \otimes q$.

For $p \in Q$ and $X \subseteq \lambda$ let $p\restriction X = \langle u^p \cap X, \mu^p \restriction B_{(u^p \cap X)} \rangle$. $p\restriction X \in Q$ and $p\restriction X \leq_{pr} p$ for any $p \in Q$ and $X \subseteq \lambda$. For $X \subseteq \lambda$, let

$$Q \downarrow X = \{p \in Q \, : \, u^p \subseteq X\}.$$

Clearly $Q \downarrow X = \{p\restriction X \, : \, p \in Q\}$.

Lemma 4.6 *Suppose that $\alpha < \lambda$.*

(1) *For $p \in Q$ and $q \in Q \downarrow \alpha$, if $q \leq_Q p\restriction \alpha$, then p and q are compatible.*

(2) *If α is a strong limit or a successor of a strong limit then $Q \downarrow \alpha \lessdot Q$.*

Proof (1): Let $q' \in Q \downarrow \alpha$ be such that $q \leq_{apr} q' \leq_{pr} p\restriction \alpha$. Let $a \in B_{(u^{q'})}$ be such that $q = q'||a$, $u^r = u^q \cup u^p$ and let μ^r be the free amalgamation of $\mu^{q'}$ and μ^p over $u^p \cap \alpha$ (see Theorem 3.1). Let $r = \langle u^r, \mu^r \rangle$ and $r' = r||a$. Then $r' \leq_{pr} q$ by Lemma 4.2 and $r' \leq_{apr} r \leq_{pr} p$.

(2): We first show that if q, $q' \in Q \downarrow \alpha$ are compatible in Q then they are compatible in $Q \downarrow \alpha$. So suppose that $r \in Q$ is such that $r \leq_Q q$, q'. Let s, $s' \in Q$ be such that $r \leq_{apr} s \leq_{pr} q$ and $r \leq_{apr} s' \leq_{pr} q'$ with $a \in B_{u^s} \setminus null(\mu^s)$ and $a' \in B_{u^{s'}} \setminus null(\mu^{s'})$ witnessing $r \leq_{apr} s$ and $r \leq_{apr} s'$ respectively. Let $u \subseteq u^r$ be such that $|\,u\,| = |\,u^q\,| + |\,u^{q'}\,| + \aleph_0$, $u^q \cup u^{q'} \subseteq u$ and a, $a' \in B_{(u)}$. Then we have $r\restriction u \leq_{apr} s\restriction u \leq_{pr} q$ and $r\restriction u \leq_{apr} s'\restriction u \leq_{pr} q'$. By (A)(b) we have $|\,u\,| = |\,u^q\,| + |\,u^{q_0}\,| + \aleph_0 < \alpha$. Hence we can find $\bar{u} \subseteq \alpha$ such that $u^q \cup u^{q'} \subseteq \bar{u}$, $|\,u \setminus (u^q \cup u^{q'})\,| = |\,\bar{u} \setminus (u^q \cup u^{q'})\,|$, $\bar{u}$ is an end-extension of $u^q \cup u^{q'}$ and so that $\bar{u}$ satisfy the requirement (A)(b). Let $f : u \to \bar{u}$ be a bijection with $f\restriction (u^q \cup u^{q'}) = id_{u^q \cup u^{q'}}$. f induces an isomorphism of $r\restriction u$, $s\restriction u$ and $r\restriction u$ to some $\bar{r}$, $\bar{s}$, $\bar{s}'$ such that $u^{\bar{r}} = u^{\bar{s}} = u^{\bar{s}'} = \bar{u}$, $\bar{r} \leq_{apr} \bar{s} \leq_{pr} q$ and $\bar{r} \leq_{apr} \bar{s}' \leq_{pr} q'$. This shows that q and q' are compatible in $Q \downarrow \alpha$.

Now suppose that $I \subseteq Q \downarrow \alpha$ is predense in $Q\restriction \alpha$. We show that I is also predense in Q. Let $p \in Q$ be arbitrary. By the assumption there is $q \in I$ such that $p\restriction \alpha$ and q are compatible. Let $r \in Q \downarrow \alpha$ be such that $r \leq_Q p\restriction \alpha$, q. By (1), r and p are compatible. Hence q and p are compatible. $\square$ (Lemma 4.6)

Lemma 4.7 *Suppose that $p \Vdash_Q$ "$\underset{\sim}{\tau} \in V$" for some $p \in Q$ and Q-name $\underset{\sim}{\tau}$. Then there is a $q \in Q$, $q \leq_{pr} p$ such that $I = \{r \in AP^Q_q \, : \, r$ decides $\underset{\sim}{\tau}\}$ is predense below*

Proof Let q_i, $r_{i+1} \in Q$, $i < \omega_1$ be defined inductively such that

(0) $q_0 = p$;

(1) $\langle q_i : i < \omega_1 \rangle$ is a decreasing sequence with respect to $\leq_{pr}$;

(2) For limit $\gamma < \omega_1$, let $q_\gamma = \langle u^{q_\gamma}, \mu^{q_\gamma} \rangle$ where $u^{q_\gamma} = \bigcup_{i<\gamma} u^{q_i}$ and μ^{q_γ} is the measure on $B_{(u^{q_\gamma})}$ generated from $\bigcup_{i<\gamma} \mu^{q_i}$;

(3) For all $i < \omega_1$, $r_{i+1} \leq_{apr} q_{i+1}$, r_{i+1} decides $\underset{\sim}{\tau}$ and r_{i+1} is incompatible with all r_{j+1}, $j < i$ provided that there are such q_{i+1}, r_{i+1}; otherwise we let $q_{i+1} = q_i$ and $r_{i+1} = \langle \emptyset, \emptyset \rangle$.

Note that (2) is possible by Corollary 2.2.

Claim 4.7.1 *There is a $\delta < \omega_1$ such that $r_{\delta+1} = \langle \emptyset, \emptyset \rangle$.*

$\vdash$ Otherwise r_{i+1}, $i < \omega_1$ are pairwise incompatible. For each $i < \omega_1$ let a_{i+1} be the witness of $r_{i+1} \leq_{apr} q_{i+1}$. Let $u^* = \bigcup_{i<\omega_1} u^{q_i}$ and $\mu^* = \bigcup_{i<\omega_1} \mu^{q_i}$. Then μ^* is a $[0,1]$-measure on $B_{(u^*)}$ and $a_{i+1}/null(\mu^*)$, $i < \omega_1$ are pairwise disjoint non-zero elements of $B_{(u^*)}/null(\mu^*)$. This is a contradiction to the c.c.c. of $B_{(u^*)}/null(\mu^*)$. $\dashv$ (Claim 4.7.1)

Now, let $\delta^* < \omega_1$ be minimal with $r_{\delta^*+1} = \langle \emptyset, \emptyset \rangle$ and let $q = q_{\delta^*}$. We show that this q is as required. By Lemma 4.4, for each $i < \delta^*$, we can find $r^+_{i+1} \in Q$ such that $r^+_{i+1} \leq_{apr} q$ and $r^+_{i+1} \leq_{pr} r_{i+1}$. Let $I' = \{r^+_{i+1} : i < \delta^*\}$. By (3), every elements of I' decides $\underset{\sim}{\tau}$.

Hence it is enough to show that I' is predense below q.

Suppose not. Then by Lemma 4.3,(2), there is $r' \leq_{apr} q$ such that r is incompatible with every r^+_{i+1}. It follows that r' is also incompatible with every r_{i+1}. Let $r \leq_Q r'$ be such that $\underset{\sim}{\tau}$ and $q' \in Q$ be such that $r \leq_{apr} q' \leq_{pr} q$. Then we could have choosen q' and r at $\delta^* + 1$'st stage of construction as q_{δ^*+1} and r_{δ^*+1}. This is a contradiction to $r_{\delta^*+1} = \langle \emptyset, \emptyset \rangle$. $\square$ (Lemma 4.7)

Proposition 4.8 *Q preserves all cardinals $\leq \kappa$.*

Proof Suppose not. Then there are $q \in Q$, $\delta < \kappa$ and a Q-name $\underset{\sim}{f}$ such that

$$q \Vdash_Q \text{“} \underset{\sim}{f} : \check{\delta}^+ \to \check{\delta} \text{ is a one to one mapping”}$$

where δ^+ denotes the successor cardinal of δ in the ground model. Let

$$\nu = \sup(\delta \cap \{\theta, \theta^+ : \theta \text{ is strongly inaccessible}\}).$$

Let q_i, $s_i \in Q$, $a_i \in B_{(u^{q_i})}$ for $i < \delta^+$ be such that:

(0) $q_0 = q$;
(1) $\langle q_i \restriction (\lambda \setminus \nu) : i < \delta^+ \rangle$ is a decreasing sequence with respect to $\leq_{pr}$;
(2) $s_i \leq_{apr} q_i$ and s_i decides $\underset{\sim}{f}(i)$;
(3) a_i witnesses $s_i \leq_{apr} q_i$.

Now since $| Q \downarrow \nu | \leq \delta$, there are $X \in [\delta^+]^{\delta^+}$, $p \in Q \downarrow \nu$ and $n \in \omega \setminus \{0\}$ such that

(4) $q_i \restriction \nu = p$ for all $i \in X$;
(5) $\mu^{q_i}(a_i) \geq \dfrac{1}{n}$ for all $i \in X$.

Let $f : X \to \delta$ be defined by $f(i) = j$ for $i \in X$ and $j \in \delta$ such that $s_i \Vdash_A$ "$\underset{\sim}{f}(i) = j$". Then by (4) and (5), f is $\leq n$ to 1. But this is impossible since $| X | > \delta$. □ (Proposition 4.8)

Lemma 4.9 *Suppose that κ is strongly incaccessible. Then Q has the strong κ^+-c.c.*

Proof Suppose that $q_i \in Q$, for $i < \kappa^+$ By Δ-system lemma, there is an $S \in [\kappa^+]^{\kappa^+}$ such that u^{q_i}, $i \in S$ form a Δ-system, say with root u^*. Since $| {}^{B(u^*)}[0,1] | < \kappa$, there is $S' \in [S]^{\kappa^+}$ and μ^* such that $\mu^{q_i} \restriction B_{(u^*)} = \mu^*$ for all $i \in S'$.

For i, $i' \in S'$, let $\tilde{u} = u^{q_i} \cup u^{q_{i'}}$ and $\tilde{\mu}$ be the $[0,1]$-measure on $B_{(\tilde{u})}$ obtained as the free product of μ^{q_i} and $\mu^{q_{i'}}$ over u^*. Then, for $\tilde{q} = \langle \tilde{u}, \tilde{\mu} \rangle$, we have $\tilde{q} \leq_{pr} q_i$, $q_{i'}$.
□ (Lemma 4.9)

5 Real-valued measurability in V^Q

We show that the p.o.-set Q introduced in the last section forces real-valued measurability of κ provided that κ in V has enough large cardinal property.

Let us begin with introducing the following notation: Suppose that $p \in Q$ and φ is a formula in the forcing language over Q. Let

$$I_{p,\varphi} = \{[a]_p : a \in B_{(u^p)} \setminus null(\mu^p),\ p \| a \Vdash_Q \text{“}\varphi\text{”}\}.$$

Let $a^* \in B_{(u^p)}$ be such that

$$(*) \qquad [a^*]_p = \sum\nolimits^{\mathcal{B}(p)} I_{p,\varphi}$$

and

$$[\![\varphi]\!]_p = \mu^p(a^*).$$

Note that $[\![\varphi]\!]_p$ does not depend on the choice of a^*.

For a Q-name $\underset{\sim}{\tau}$ of an element of V and $p \in Q$, let

$$[\![\underset{\sim}{\tau}]\!]^*_p = \mu^p(a^\dagger)$$

for $a^\dagger \in B_{(u^p)}$ with $[a^\dagger]_p = \sum^{\mathcal{B}(p)}\{[a]_p : a \in B_{(u^p)} \setminus null(\mu^p),\ p||a \text{ desides } \underset{\sim}{\tau}\}$.

We identify each formula φ in the forcing language for Q with the Q-name which gives the truth value of the formula (i.e. either 0 or 1 depending on whether φ is forced or not.) and consider $[\![\varphi]\!]^*_p$ under this identification. Thus $[\![\varphi]\!]^*_p = [\![\varphi]\!]_p + [\![\neg\varphi]\!]_p$.

Lemma 5.1 *Suppose that $p \in Q$ and φ is a formula in the forcing language over Q. Let a^* be as in $(*)$ above. If $a^* \notin null(\mu^p)$, then*

$$p||a^* \Vdash_Q \text{“}\varphi\text{”}.$$

Proof Otherwise there is some $q \leq_Q p||a^*$ such that $q \Vdash_Q \text{“}\neg\varphi\text{”}$. Let q' be such that $q \leq_{apr} q' \leq_{pr} p||a^*$ and let $b \in B_{(u^{q'})}$ be such that $q = q'||b$. Since $b \cap a^* \notin null(\mu^{q'})$, there is $a_0 \in I_{p,\varphi}$ such that $b \cap a_0 \notin null(\mu^{q'})$ by $(*)$. Hence by the same argument as in the proof of Lemma 4.3,(2), we can show that $p||a_0$ and q are compatible. But $p||a_0 \Vdash_Q \text{“}\varphi\text{”}$. This is a contradiction. ❑ (Lemma 5.1)

Lemma 5.2 *For p, $p' \in Q$, if $p \leq_{pr} p'$ then $[\![\varphi]\!]_{p'} \leq [\![\varphi]\!]_p$.*

Proof Suppose $[a]_{p'} \in I_{p',\varphi}$. Then $p'||a \Vdash_Q \text{“}\varphi\text{”}$. Since $p||a \leq_{pr} p'||a$ by Lemma 4.2, $p||a \Vdash_Q \text{“}\varphi\text{”}$. As $p||a \leq_{apr} p$, it follows that $[a]_p \in I_{p,\varphi}$. Thus $[\![\varphi]\!]_{p'} \leq [\![\varphi]\!]_p$.
❑ (Lemma 5.2)

Theorem 5.3 *Suppose that $\lambda = \kappa^+$ and $j : V \to M$ is an elementary embedding with $crit(j) = \kappa$ and ${}^{2^\kappa>}M \subseteq M$. Then*

$$\Vdash_Q \text{“}\kappa \textit{ is real-valued measurable}\text{”}.$$

Proof In M, we have $Q = j(Q) \downarrow \lambda$ by (A)(b). Hence, by Lemma 4.6, $Q \lessdot j(Q)$. Let $\langle \underset{\sim}{\tau}_k : k < 2^\kappa \rangle$ be an emumeration of Q-names of mappings from κ to some $\gamma < \kappa$.

Let $\eta : 2^\kappa \to Q \times 2^\kappa;\ i \mapsto \langle \eta_0(i), \eta_1(i) \rangle$ be a surjection such that $|\,\eta^{-1}{}''\{\langle q, i\rangle\}\,| = 2^\kappa$ for all $\langle q, i \rangle \in Q \times 2^\kappa$. Let $\langle q_i : i < 2^\kappa \rangle$ be a sequence of elements of $j(Q)$ such that

(a) $q_i \in j(Q) \downarrow (j(\lambda) \setminus \lambda)$ for all $i < 2^\kappa$ and $\langle q_i : i < 2^\kappa \rangle$ is a decreasing sequence with respect to $\leq_{pr}$;

(b) for all $i < 2^\kappa$, there is $\tilde{p}_i \in Q$, $\tilde{p}_i \leq_{pr} \eta_0(i)$ such that, in M, for any $p' \in Q$ and $q' \in j(Q) \downarrow j(\lambda) \setminus \lambda$ with $p' \leq_{pr} \tilde{p}_i$ and $q' \leq_{pr} q_{i+1}$

$$[\![j(\underset{\sim}{\tau}_{\eta_1(i)})(\kappa)]\!]^*_{\tilde{p}_i \otimes q_{i+1}} = [\![j(\underset{\sim}{\tau}_{\eta_1(i)})(\kappa)]\!]^*_{p' \otimes q'}.$$

The construction is possible by closedness of M (for (b), a construction similar to the one in the proof of Lemma 4.7 is to be applied).

For a Q-name $\underset{\sim}{X}$ of a subset of κ and $q \in Q$, let

$$\mu_q(\underset{\sim}{X}) = \max\{[\![\check{\kappa} \in j(\underset{\sim}{X})]\!]_{q' \otimes q_i} : q' \in Q,\ q' \leq_{pr} q,\ i < 2^\kappa\}.$$

The following is immediate from Lemma 5.1 and the definition of μ_q:

Claim 5.3.1 *Suppose that $p \in Q$, $\underset{\sim}{X}$ is a Q-name of a subset of κ and $i^* < \omega_1$. If*

$$[\![\check{\kappa} \in j(\underset{\sim}{X})]\!]^*_{p \otimes q_{i^*}} = [\![\check{\kappa} \in j(\underset{\sim}{X})]\!]^*_{p' \otimes q'}$$

for any $p' \in Q$ and $q' \in j(Q) \downarrow (j(\lambda \setminus \lambda))$ with $p' \leq_{pr} p$ and $q' \leq_{pr} q_{i^*}$, then we have $\mu_p(\underset{\sim}{X}) = [\![\check{\kappa} \in j(\underset{\sim}{X})]\!]_{p \otimes q_{i^*}}$. ⊣ (Claim 5.3.1)

Claim 5.3.2 (1) *For $X \subseteq \kappa$, if $\kappa \in j(X)$, then $\mu_q(\check{X}) = 1$ for all $q \in Q$.*

(2) *For $X \subseteq \kappa$, if $\kappa \notin j(X)$, then $\mu_q(\check{X}) = 0$ for all $q \in Q$.*

(3) *For $q \in Q$ and Q-names $\underset{\sim}{X}$, $\underset{\sim}{Y}$ of subsets of κ, if $q \Vdash_Q$ "$\underset{\sim}{X} \subseteq \underset{\sim}{Y}$", then $\mu_q(\underset{\sim}{X}) \leq \mu_q(\underset{\sim}{Y})$.*

(4) *For $q \in Q$, $\gamma < \kappa$ and Q-names $\underset{\sim}{X}$, $\underset{\sim}{X}_\ell$, $\ell < \kappa$, if*

$$q \Vdash_Q \text{“}\underset{\sim}{X} = \dot{\bigcup}\{\underset{\sim}{X}_\ell : \ell < \gamma\}\text{”},$$

then for any $q' \leq_Q q$ there is $q'' \leq_{pr} q'$ such that $\mu_{q''}(\underset{\sim}{X}) = \sum_{\ell<\gamma} \mu_{q''}(\underset{\sim}{X}_\ell)$.

⊢ (1): If $\kappa \in j(X)$, then $q \otimes q_i \Vdash_{j(Q)}$ "$\check{\kappa} \in j(\check{X})$" for any $i < 2^\kappa$. Hence $1 \geq \mu_q(X) \geq \mu^{q \otimes q_i}(u^q \cup u^{q_i}) = 1$.

(2): Simlilarly to (1).

(3): Clear.

(4): Suppose that $q' \leq_Q q$. Let $k < 2^\kappa$ be such that

$$\Vdash_Q \text{“}\underset{\sim}{\tau}_k : \kappa \to \gamma + 1\text{”}$$

and

$$q \Vdash_Q \text{“}\forall i < \kappa\ (\underset{\sim}{\tau}_k(i) = \ell \leftrightarrow i \in \underset{\sim}{X}_\ell)\text{”}$$

for all $\ell < \gamma$.

Let $i^* < 2^\kappa$ be such that $\eta_0(i^*) = q'$ and $\eta_1(i^*) = k$. By (b) in the definition of $\langle q_i : i < 2^\kappa \rangle$, the assumption of Claim 5.3.1 is satisfied for $p'' = p_{i^*}$ and each of $\underset{\sim}{X}$ and $\underset{\sim}{X}_\ell$, $\ell < \gamma$. Hence

$$\mu_{p''}(\underset{\sim}{X}) = [\![\check{\kappa} \in j(\underset{\sim}{X})]\!]_{p'' \otimes q_{i^*+1}} = \sum_{\ell<\gamma} [\![\check{\kappa} \in j(\underset{\sim}{X}_\ell)]\!]_{p'' \otimes q_{i^*+1}} = \sum_{\ell<\gamma} \mu_{p''}(\underset{\sim}{X}_\ell).$$

$\dashv$ (Claim 5.3.2)

From this point on, the proof is very similar to the one for real-valued measurability in a random model by Solovay [9]. We shall follow closely the version of the proof given in Kanamori [7] (the proof of Theorem 17.5 in [7]). The last paragraphs of our proof in particular are almost identical with the corresponding part of the proof in [7]. Nevertheless, we also include them for convenience of the reader.

Claim 5.3.3 *Suppose that $p \in Q$, $\underset{\sim}{X}$ is a Q-name of a subset of κ and $r \in \mathbb{R}$ with $0 \le r \le 1$. If $Q \models \forall q \le_{pr} p\ \exists s \le_Q q\ (\mu_s(\underset{\sim}{X}) \ge r)$ then we have $\mu_p(\underset{\sim}{X}) \ge r$.*

$\vdash$ Suppose that $Q \models \forall q \le_{pr} p\ \exists s \le_Q q\ (\mu_s(\underset{\sim}{X}) \ge r)$. We show that $\mu_p(\underset{\sim}{X}) \ge r$.

For $i < \omega_1$, let $p_i \in Q$, $j_i < \omega_1$, a_{i+1}, $b_{i+1} \in B_{u^{p_{i+1} \otimes q_{j_{i+1}}}}$ be defined inductively such that

(0) $p_0 = p$;

(1) $\langle p_i : i < \omega_1 \rangle$ is a decreasing sequence in Q with respect to $\le_{pr}$;

(2) For a limit $\gamma < \omega_1$, $p_\gamma = \langle u, \mu \rangle$ where $u = \bigcup_{i<\gamma} u^{p_i}$ and μ is a $[0,1]$-measure on $B_{(u)}$ extending $\bigcup_{i<\gamma} \mu^{p_i}$;

(3) $\langle j_i : i < \omega_1 \rangle$ is a continuously increasing sequence of ordinals $< \omega_1$;

(4) If we cannot find p_{i+1}, j_{i+1}, a_{i+1}, b_{i+1} satisfying the conditions (5) $\sim$ (7) in the following, then we let $p_{i+1} = p_i$, $j_{i+1} = j_i$ and $a_{i+1} = b_{i+1} = \emptyset$; otherwise:

(5) $a_{i+1} \cap a_{i'+1}$ is a null set with respect to $\mu^{p_{i+1} \otimes q_{j_{i+1}}}$ for all $i' < i$;

(6) $b_{i+1} \subseteq a_{i+1}$, $b_{i+1} \in B_{(u^{p_{i+1} \otimes q_{j_{i+1}}})} \setminus null(\mu^{p_{i+1} \otimes q_{j_{i+1}}})$ and

$$\frac{\mu^{p_{i+1} \otimes q_{j_{i+1}}}(b_{i+1})}{\mu^{p_{i+1} \otimes q_{j_{i+1}}}(a_{i+1})} \ge r;$$

(7) $\langle u^{p_{i+1} \otimes q_{j_{i+1}}}, \mu^{p_{i+1} \otimes q_{j_{i+1}}} \| b_{i+1} \rangle \Vdash_{j(Q)}$ "$\check{\kappa} \in j(\underset{\sim}{X})$".

Now, as in the proof of Lemma 4.7, there is the minimal $\delta^* < \omega_1$ such that $a_{\delta^*+1} = \emptyset$. Let $j^* = \sup\{j_i : i < \delta^*\}$. $\{[a_{i+1}]_{p_{\delta^*} \otimes q_{j^*}} : i < \delta^*\}$ is a maximal antichain in $\mathcal{B}_{(p_{\delta^*}\otimes q_{j^*})}$ by the assumption of the Claim. Let $b \in B_{(u^{p_{\delta^*}\otimes q_{j^*}})}$ be such that

$$[b]_{p_{\delta^*}\otimes q_{j^*}} = \sum\nolimits^{\mathcal{B}_{(p_{\delta^*}\otimes q_{j^*})}}\{[b_{i+1}]_{p_{\delta^*}\otimes q_{j^*}} \; : \; i < \delta^*\}.$$

Then we have $\mu^{p_{\delta^*}\otimes q_{j^*}}(b) \geq r$ by (6) and $\langle u^{p_{\delta^*}\otimes q_{j^*}}, \mu^{p_{\delta^*}\otimes q_{j^*}} \| b\rangle \Vdash_{j(Q)}$ "$\check{\kappa} \in j(\underset{\sim}{X})$" by (7) and Lemma 5.1. Since $p_{\delta^*} \leq_{pr} p$, it follows that $\mu_p(\underset{\sim}{X}) \geq r$. $\dashv$ (Claim 5.3.3)

Now, for $q \in Q$ and Q-name $\underset{\sim}{X}$ of a subset of κ, let

$$\mu^*_p(\underset{\sim}{X}) = \inf\left(\{\mu_q(\underset{\sim}{X}) \; : \; q \leq_Q p\}\right).$$

By this definition the following is clear:

Claim 5.3.4 *For p, $p' \in Q$ and Q-name $\underset{\sim}{X}$ of a subset of κ, if $p \leq_Q p'$, then $\mu^*_p(\underset{\sim}{X}) \geq \mu^*_{p'}(\underset{\sim}{X})$.* $\dashv$

Let $\underset{\sim}{\mu}$ be the Q-name such that $\Vdash_Q$ "$\underset{\sim}{\mu} : \mathcal{P}(\kappa) \to [0,1]$" and

$$\Vdash_Q \text{“}\underset{\sim}{\mu}(\underset{\sim}{X}) = \sup\left(\{\mu^*_p(\underset{\sim}{X}) \; : \; p \in \underset{\sim}{G}\}\right)\text{”}$$

for Q-name $\underset{\sim}{X}$ of subset of κ and the standard Q-name $\underset{\sim}{G}$ of the generic set over Q.

The well-definedness of $\underset{\sim}{\mu}$ is yet to be established in course of the proof: At the moment we consider each "$\underset{\sim}{\mu}(\underset{\sim}{X})$" merely as an abbreviation of a Q-name $\underset{\sim}{r}$ of a real such that

$$\Vdash_Q \text{“}\underset{\sim}{r} = \sup\left(\{\mu^*_p(\underset{\sim}{X}) \; : \; p \in \underset{\sim}{G}\}\right)\text{”}.$$

Claim 5.3.5 *Suppose that $q \in Q$, $\underset{\sim}{X}$ is a Q-name of a subset of κ and $r \in [0,1]$. Then we have*

$$\mu^*_q(\underset{\sim}{X}) \geq r \quad \Leftrightarrow \quad q \Vdash_Q \text{“}\underset{\sim}{\mu}(\underset{\sim}{X}) \geq \check{r}\text{”}.$$

$\vdash$ ($\Rightarrow$): This is clear by definition of $\underset{\sim}{\mu}^*(\underset{\sim}{X})$ and since $q \Vdash_Q$ "$q \in \underset{\sim}{G}$".

($\Leftarrow$): Suppose that $q \Vdash_Q$ "$\underset{\sim}{\mu}(\underset{\sim}{X}) \geq \check{r}$" and let $r_0 \in \mathbb{R}$ be such that $0 < r_0 < r$. It is enough to show that $\mu^*_q(\underset{\sim}{X}) \geq r_0$. By definition of $\underset{\sim}{\mu}(\underset{\sim}{X})$, we have

$$q \Vdash_Q \text{“}\exists s \in \underset{\sim}{G} \; (\mu^*_s(\underset{\sim}{X}) \geq \check{r}_0)\text{”}.$$

By Claim 5.3.4, it follows that

$$\forall p \leq_Q q \; \exists s \leq_Q p \; (\mu^*_s(\underset{\sim}{X}) \geq r_0).$$

Hence by definition of μ_s^* we have $\forall p \leq_Q q\ \exists s \leq_Q p\ (\mu_s(\underset{\sim}{X}) \geq r_0)$. In particular for any $q' \leq_Q q$ we have $\forall p \leq_Q q'\ \exists s \leq_Q p\ (\mu_s(\underset{\sim}{X}) \geq r_0)$. Hence by Claim 5.3.3, $\mu_{q'}(\underset{\sim}{X}) \geq r_0$ for any $q' \leq_Q q$. It follows that $\mu_q^*(\underset{\sim}{X}) \geq r_0$. $\dashv$ (Claim 5.3.5)

In the rest of the proof, we show that $\underset{\sim}{\mu}$ is well-defined and

$$\Vdash_Q \text{“}\underset{\sim}{\mu} \text{ is a } \kappa\text{-additive } [0,1]\text{-measure on } \mathcal{P}(\kappa)\text{”}.$$

For any $X \subseteq \kappa$, if $\kappa \in j(X)$, then we have $\Vdash_Q$ “$\underset{\sim}{\mu}(\check{X}) = 1$” by Claim 5.3.2(1); otherwise, i.e. if $\kappa \notin j(X)$ then $\Vdash_Q$ “$\underset{\sim}{\mu}(\check{X}) = 0$” by Claim 5.3.2(2). In particular, $\Vdash_Q$ “$\underset{\sim}{\mu}(\check{\kappa}) = 1$” and $\Vdash_Q$ “$\underset{\sim}{\mu}(\emptyset) = 0$”.

If $p \Vdash_Q$ “$\underset{\sim}{X} \subseteq \underset{\sim}{Y}$” then $\mu_q(\underset{\sim}{X}) \leq \mu_q(\underset{\sim}{Y})$ for any $q \leq_Q p$ by Claim 5.3.2(3). Hence $p \Vdash_Q$ “$\underset{\sim}{\mu}(\underset{\sim}{X}) \leq \underset{\sim}{\mu}(\underset{\sim}{X})$” : Otherwise we can find $q^* \leq_Q p$ and $r \in \mathbb{Q}$ such that $q^* \Vdash_Q$ “$\underset{\sim}{\mu}(\underset{\sim}{X}) > \check{r} > \underset{\sim}{\mu}(\underset{\sim}{Y})$”. But then we have $\mu_{q^*}^*(\underset{\sim}{X}) > r \geq \mu_{q^*}^*(\underset{\sim}{Y})$ by Claim 5.3.5. It follows that $\mu_{q^{**}}(\underset{\sim}{X}) > \mu_{q^{**}}(\underset{\sim}{Y})$ for some $q^{**} \leq_Q q$. This is a contradiction.

In particular, $p \Vdash_Q$ “$\underset{\sim}{\mu}(\underset{\sim}{X}) = \underset{\sim}{\mu}(\underset{\sim}{Y})$” whenever $p \Vdash_Q$ “$\underset{\sim}{X} = \underset{\sim}{Y}$”. This shows the well-definedness of $\underset{\sim}{\mu}$.

Next, we show that $\Vdash_Q$ “$\underset{\sim}{\mu}$ is finitely additive”. Suppose that $\Vdash_Q$ “$\underset{\sim}{X}, \underset{\sim}{Y} \subseteq \kappa\ \wedge\ \underset{\sim}{X} \cap \underset{\sim}{Y} = \emptyset$”. We show that $\Vdash_Q$ “$\underset{\sim}{\mu}(\underset{\sim}{X} \cup \underset{\sim}{Y}) = \underset{\sim}{\mu}(\underset{\sim}{X}) + \underset{\sim}{\mu}(\underset{\sim}{Y})$” follows.

Let $p \in Q$ and $r_1, r_2 \in \mathbb{Q}$ be such that $p \Vdash_Q$ “$\underset{\sim}{\mu}(\underset{\sim}{X}) \geq \check{r}_1\ \wedge\ \underset{\sim}{\mu}(\underset{\sim}{Y}) \geq \check{r}_2$”. Then, by Claim 5.3.2(4), Claim 5.3.3 and by definition of μ_p^*, we have

$$\mu_p^*(\underset{\sim}{X} \cup \underset{\sim}{Y}) \geq r_1 + r_2.$$

Hence by Claim 5.3.5, $p \Vdash_Q$ “$\underset{\sim}{\mu}(\underset{\sim}{X} \cup \underset{\sim}{Y})$” $\geq \check{r}_1 + \check{r}_2$. By density argument it follows that $\Vdash_Q$ “$\forall r_1, r_2 \in \mathbb{Q}\ \big((r_1 < \underset{\sim}{\mu}(\underset{\sim}{X})\ \wedge\ r_2 < \underset{\sim}{\mu}(\underset{\sim}{Y})) \ \rightarrow\ r_1 + r_2 \leq \underset{\sim}{\mu}(\underset{\sim}{X} \cup \underset{\sim}{Y})\big)$”. Hence $\Vdash_Q$ “$\underset{\sim}{\mu}(\underset{\sim}{X} \cup \underset{\sim}{Y})$” $\geq \underset{\sim}{\mu}(\underset{\sim}{X}) + \underset{\sim}{\mu}(\underset{\sim}{Y})$.

For the opposite inequality, assume that $\Vdash_Q$ “$\underset{\sim}{\mu}(\underset{\sim}{X} \cup \underset{\sim}{Y}) > \underset{\sim}{\mu}(\underset{\sim}{X}) + \underset{\sim}{\mu}(\underset{\sim}{Y})$”. Then we can find $p \in Q$ and $r_3, r_4, r_5, r_6 \in \mathbb{Q}$ such that

$$p \Vdash_Q \text{“}\underset{\sim}{\mu}(\underset{\sim}{X}) < \check{r}_3 < \check{r}_4\ \wedge\ \underset{\sim}{\mu}(\underset{\sim}{Y}) < \check{r}_5 < \check{r}_6\ \wedge\ \underset{\sim}{\mu}(\underset{\sim}{X} \cup \underset{\sim}{Y}) \geq \check{r}_4 + \check{r}_6\text{”}.$$

For all $q \leq_Q p$ we have $\mu_q^*(\underset{\sim}{X}) < r_3$ by Claim 5.3.5. Hence there is an $s \leq_Q q$ such that $\mu_s(\underset{\sim}{X}) < r_3$. By Claim 5.3.3 with “$\geq$” replaced by “$\leq$”, it follows that $\mu_p(\underset{\sim}{X}) \leq r_3$. Similarly, we get $\mu_p(\underset{\sim}{Y}) \leq r_5$. We may assume that p is taken so that the additivity in Claim 5.3.2(4) holds for “$\underset{\sim}{X} \dot{\cup} \underset{\sim}{Y}$”. So

$$\mu_p^*(\underset{\sim}{X} \cup \underset{\sim}{Y}) \leq \mu_p(\underset{\sim}{X} \cup \underset{\sim}{Y}) = \mu_p(\underset{\sim}{X}) + \mu_p(\underset{\sim}{Y}) \leq r_3 + r_5 < r_4 + r_6.$$

But by Claim 5.3.5, this is a contradiction to $p \Vdash_Q$ “$\underset{\sim}{\mu}(\underset{\sim}{X} \cup \underset{\sim}{Y}) \geq \check{r}_4 + \check{r}_6$”.

Finally, to show κ-additivity, suppose that $\gamma < \kappa$ and $\underset{\sim}{X}$, $\underset{\sim}{X}_\ell$, $\ell < \gamma$ be Q-names of subests of κ such that $\Vdash_Q$ "$\underset{\sim}{X} = \dot{\bigcup}\{\underset{\sim}{X}_\ell : \ell < \gamma\}$". By finite additivity, we have

$$\Vdash_Q \text{"} \underset{\sim}{\mu}(\underset{\sim}{X}) \geq \sum_{\ell<\gamma} \underset{\sim}{\mu}(\underset{\sim}{X}_\ell) \text{"}.$$

To show the opposite inequality, suppose that there are $p \in Q$ and $r_1, r_2 \in \mathbb{Q}$ such that

$$p \Vdash_Q \text{"} \sum_{\ell<\gamma} \underset{\sim}{\mu}(\underset{\sim}{X}_\ell) < \check{r}_1 < \check{r}_2 \leq \underset{\sim}{\mu}(\underset{\sim}{X}) \text{"}.$$

By finite additivity we have $p \Vdash_B$ "$\underset{\sim}{\mu}(\sum_{\ell\in t} X_\ell) < \check{r}_1$" for all $t \in [\gamma]^{<\aleph_0}$. Hence by Claim 5.3.5, $\mu_p(\sum_{\ell\in t} \underset{\sim}{X}_\ell) \leq r_1$ for any $t \in [\gamma]^{<\aleph_0}$. Without loss of generality we may assume that p is taken so that the additivity in Claim 5.3.2(4) holds for "$\underset{\sim}{X} = \dot{\bigcup}\{\underset{\sim}{X}_\ell : \ell < \gamma\}$". Then we have $\mu_p(\underset{\sim}{X}) \leq r_1$. Hence $\mu_p^*(\underset{\sim}{X}) \leq r_1$. But this is a contradiction to $p \Vdash_Q$ "$\check{r}_2 \leq \underset{\sim}{\mu}(\underset{\sim}{X})$" by Claim 5.3.5. $\square$ (Theorem 5.3)

6 Stick and club principles

For cardinals $\nu < \kappa$ the stick principle $\dagger^\nu_\kappa$ is defined as follows:

$$(\dagger^\nu_\kappa) \quad \exists X \subseteq [\kappa]^\nu \ \big(|X| = \kappa \wedge \forall Y \in [\kappa]^\kappa \ \exists x \in X \ (x \subseteq Y)\big).$$

For a stationary $E \subseteq \kappa$, the club principle $\clubsuit_\kappa(E)$ is:

$$(\clubsuit_\kappa(E)) \quad \exists \langle x_i : i \in E\rangle \ \big(\forall i \in E \ (x_i \text{ is a cofinal subset of } i) \\ \forall Y \in [\kappa]^\kappa \ \exists i \in E \ (x_i \subseteq Y)\big).$$

Clearly $\diamondsuit_\kappa(E)$ implies $\clubsuit_\kappa(E)$ and $\clubsuit(E^\nu_\kappa)$ implies $\dagger^\nu_\kappa$ where $E^\nu_\kappa = \{i < \kappa : \mathrm{cf}(i) = \nu\}$. For $\nu < \nu' < \kappa$, $\dagger^{\nu'}_\kappa$ implies $\dagger^\nu_\kappa$.

Random forcing destroys stick principle for every cardinal $<$ the Maharam type of the forcing. For uncountable ν this is because of the following result by J. Brendle.

For $\nu \leq \kappa$, let

$$r(\nu, \kappa) = \min\{|\mathcal{F}| : \mathcal{F} \subseteq [\kappa]^\nu, \forall A \subseteq \lambda \ \exists b \in \mathcal{F} \ (b \subseteq A \vee b \subseteq \lambda \setminus A)\}.$$

It is clear that $\dagger^\nu_\kappa$ implies that $r(\nu, \kappa) \leq \kappa$.

Theorem 6.1 (J. Brendle, [1]) *For uncountable $\nu \leq \kappa$, we habe $cov(null) \leq$*

The random forcing $\mathcal{B}_\lambda$ forces the value of $cov(null)$ to be $\geq \lambda$ hence $\Vdash_{\mathcal{B}_\lambda}$ "$\neg \dagger^{\nu}_{\kappa}$" for every uncountable $\nu \leq \kappa < \lambda$.

For $\nu = \aleph_0$ we have the following:

Lemma 6.2 (Folklore) *Suppose that* $\aleph_0 < \kappa < \lambda$. *Then* $\Vdash_{\mathcal{B}_\lambda}$ " $\dagger^{\aleph_0}_{\kappa}$ does not hold".

Proof Suppose that $\underset{\sim}{x}_i$, $i < \kappa$ are $\mathcal{B}_\lambda$-names of countable subsets of κ. we show that $\Vdash_{\mathcal{B}_\lambda}$ "$\{\underset{\sim}{x}_i : i < \kappa\}$ is not a $\dagger^{\aleph_0}_{\kappa}$-set". By the c.c.c. of $\mathcal{B}_\lambda$, there is an $S \in [\lambda]^\kappa$ such that $\underset{\sim}{x}_i$, $i < \kappa$ are all $\mathcal{B}_S$-names.

Let $\xi_j \in \lambda \setminus S$, $j < \kappa$ be pairwise distinct and

$$a_j = [\{\langle \xi_j, 1\rangle\}], \quad b_j = [\{\langle \xi_j, 0\rangle\}]$$

for $j < \kappa$. Let

$$\underset{\sim}{X} = \{\langle \check{j}, a_j\rangle : j < \kappa\}.$$

Then $\Vdash_{\mathcal{B}_\lambda}$ "$\underset{\sim}{X} \in [\kappa]^\kappa$". Hence the following claim implies that $\{\underset{\sim}{x}_i : i < \kappa\}$ is forced not to be a $\dagger^{\aleph_0}_{\kappa}$-set.

Claim 6.2.1 $\Vdash_{\mathcal{B}_\lambda}$ "$\underset{\sim}{x}_i \not\subseteq \underset{\sim}{X}$" *for all* $i < \kappa$.

$\vdash$ Let G be a generic filter over $\mathcal{B}_S$. In $V[G]$, let $B = \mathcal{B}_\lambda / G$. m_λ induces a finitely additive measure m on B. Let b'_j, $j < \kappa$ be the elements of B corresponding to b_j, $j < \kappa$ respectively. Note that b'_j, $j < \kappa$ are independent events in $\langle B, m\rangle$ of measure $1/2$. By Borel-Cantelli theorem (which holds also for finitely additive measure) we have

$$\sum_{j \in \underset{\sim}{x}_i^G} b'_j = 1.$$

Hence for any $b \in B^+$ there is $j \in x_i$ such that $b \cdot b'_j \neq 0$. But $b'_j \Vdash_B$ "$j \notin \underset{\sim}{X}$". Hence $b \cdot b'_j \Vdash_B$ "$\underset{\sim}{x}_i \not\subseteq$ "$\tilde{X}$. This shows $\Vdash_B$ "$\underset{\sim}{x}_i \not\subseteq \underset{\sim}{X}$" and, since G was arbitrary, $\Vdash_{\mathcal{B}_\lambda}$ "$\underset{\sim}{x}_i \not\subseteq \underset{\sim}{X}$". $\dashv$ (Claim 6.2.1)

$\square$ (Lemma 6.2)

Stick or club principle at the cardinality of the partial ordering depends rather on the ground model:

Proposition 6.3 (Folklore) *Suppose that* κ *is a regular cardinal and* S *a stationary subset of* κ *such that* $\diamondsuit_\kappa(S)$. *If* P *is a* κ-*c.c. p.o.-set of cardinality* $\leq \kappa$, *then* $\Vdash_P$ "$\diamondsuit_\kappa(S)$".

Proof Without loss of generality, we may assume that the underlying set of P is κ. Let $\langle x_i : i \in S\rangle$ be a $\diamondsuit_\kappa(S)$-sequence which guesses subsets of $\kappa \times \kappa$. I.e. $x_i \subseteq i \times i$ and for every $X \subseteq \kappa\times\kappa$, $\{i \in S : X \cap (i \times i) = x_i\}$ is stationary. Each x_i can be naturally associated with the P-name $\underset{\sim}{x}_i = \{\langle\check{i}, p\rangle : \langle i, p\rangle \in x_i\}$. We show that $\Vdash_P$ "$\langle \underset{\sim}{x}_i : i \in S\rangle$ is a $\diamondsuit_\kappa(S)$-sequence".

Let $\underset{\sim}{X}$ be a P-name of a subset of κ. We may assume that $\underset{\sim}{X} \subseteq \{\langle\check{i}, p\rangle : i \in \kappa, p \in P\}$ and that $\{p \in P : \langle\check{i}, p\rangle \in \underset{\sim}{X}\}$ is incompatible for all $i \in \kappa$. By the κ-c.c. of P this set then has cardinality $< \kappa$. Let $\underset{\sim}{C}$ be a P-name of a club subset of κ. Again by the κ-c.c. of P, there is a club set $C \subseteq \kappa$ such that $\Vdash_P$ "$C \subseteq \underset{\sim}{C}$".

It is enough to show that there is an $i \in C$ such that $\Vdash_P$ "$\underset{\sim}{x}_i = \underset{\sim}{X} \cap i$". Let $X = \{\langle i, p\rangle : \langle\check{i}, p\rangle \in \underset{\sim}{X}\}$. Then

$$D = \{i < \kappa : \forall j < i \forall \gamma < \kappa\big(\langle j, \gamma\rangle \in X \ \rightarrow\ \gamma < i\big)\}.$$

is a club subset of κ. For $i \in C \cap D \cap S$ such that $x_i = X \cap (i \times i)$, we have $\Vdash_P$ "$\underset{\sim}{x}_i = \underset{\sim}{X} \cap i$". □ (Proposition 6.3)

In contrast to the situation in the generic extension by a random algebra discribed in Lemma 6.2, it is possible to have stick and club principles for κ in the generic extension by the p.o.-set Q.

Theorem 6.4 *Assume that κ is strongly Mahlo cardinal and $\nu < \kappa \leq \lambda$. Then for $Q = Q_{\kappa,\lambda}$:*

(a) $\Vdash_Q$ "⫯^{ν}_{κ}".

(b) *if $S_* \subseteq E^\nu_\kappa$ is stationary and $V \models \diamondsuit_\kappa(S_*)$ then $\Vdash_Q$ "$\clubsuit_\kappa(S_*)$".*

Proof (a): We show that $[\kappa]^\nu$ in the ground model is forced to be a ⫯^{ν}_{κ}-set (i.e. a set as X in the definition of ⫯^{ν}_{κ}).

Suppose that $p^* \in Q$ and $\underset{\sim}{Y}$ is a Q-name such that $p^* \Vdash_Q$ "$\underset{\sim}{Y} \in [\kappa]^\kappa$", say

$$p^* \Vdash_Q \text{“}\langle \underset{\sim}{\alpha}_i : i < \kappa\rangle \text{ is an increasing sequence and } \underset{\sim}{Y} = \{\underset{\sim}{\alpha}_i : i < \kappa\}\text{”}.$$

For $i < \kappa$, let ξ_i be the i'th strongly inaccessible cardinal. By assumption $\{\xi_i : i < \kappa\}$ is stationary subset of κ.

For $i < \kappa$, let q_i, a_i, γ_i be chosen inductively so that

(1) $q_i \leq_{pr} p^*$;

(2) $\gamma_i < \kappa$;

(3) $a_i \in B_{(u^{q_i})} \setminus null(\mu^{q_i})$;

(4) $q_i || a_i \Vdash_Q$ "$\alpha_i = \check{\gamma}_i$".

By Fodor's lemma, there are $I \subseteq \kappa$, $\alpha^{**} < \kappa$, $p^{**} \in Q$ such that

(5) I is cofinal in κ;

(6) $q_i \restriction \xi_i = p^{**}$ for all $i \in I$;

(7) $\sup(u^{q_i}) < \xi_j$ for all i, $j \in I$ with $i < j$;

(9) $otp(u^{q_i}) = \alpha^{**}$ for all $i \in I$;

(10) the order preserving mapping $p_{i,j} : u^{q_i} \to u^{q_j}$ induces an isomorphism from $\langle B_{(u^{q_i})}, \mu^{q_i}\rangle$ to $\langle B_{(u^{q_j})}, \mu^{q_j}\rangle$ sending a_i to a_j for all i, $j \in I$.

Let $\langle i_n : n < \nu\rangle$ be an increasing sequence in I. Then $\mu^{q_{i_n}}||a_{i_n}$, $n < \nu$ are compatible to each other. Hence by Theorem 3.2, there is the free amalgamation μ^{q^*} of $\mu^{q_{i_n}}||a_{i_n}$, $n < \nu$ over $u^{p^{**}}$. Let

$$q^* = \langle \bigcup_{n<\nu} u^{q_{i_n}}, \mu^{q^*}\rangle.$$

Then we have $q^* \leq_{pr} q_{i_n}||a_{i_n}$ for all $n < \nu$. In particular $q^* \leq_Q p^*$ by (1).

By (4), it follows that γ_{i_n}, $n < \nu$ are all distinct and

$$q^* \Vdash_Q \text{“}\{\check{\gamma}_{i_n} : n < \nu\}\text{”} \subseteq \underset{\sim}{Y}.$$

This shows that

$$\Vdash_Q \text{“}\forall Y \subseteq [\kappa]^\kappa\, \exists x \in \check{X}(x \subseteq Y)\text{”}$$

for $X = [\kappa]^\nu$ (in V).

(b): The proof is dsimilar to (1). Using q_i, a_i, γ_i for $i < \kappa$ as well as $I \subseteq \kappa$, and $p^{**} \in Q$ as in the proof of (1), we can show that the $\diamondsuit_\kappa(S^*)$-sequence in the ground model becomes a $\clubsuit_\kappa(S^*)$-sequence in the generic extension. □ (Theorem 6.4)

7 Concluding remarks

We can put together the results obtained in the previous sections to get:

Theorem 7.1 *If*

ZFC + “there exists a measurable cardinal”

is consistent, then so is the theory:

$$\text{ZFC} + \text{“there is a real-valued measurable cardinal } \kappa < 2^{\aleph_0}\text{”}$$
$$+ \text{“}\clubsuit(E_\kappa^{\aleph_\nu}) \textit{ for all } \nu < \kappa\text{”} + \text{“}cov(null) = \kappa\text{”}.$$

Proof Let κ be a measurable cardinal. By moving to the inner model of measurability $V[U]$, we may assume that $2^\kappa = \kappa^+$. Let $Q = Q_{\kappa,\kappa^+}$ be the p.o.-set introduced in section 4.

By Proposition 4.8 and Lemma 4.9, Q preserves all cardinals. By Theorem 5.3 (here we need $2^\kappa = \kappa^+$), Q forces κ to be a real-valued-measurable cardinal. By Theorem 6.4, $\clubsuit_\kappa(E^\nu_\kappa)$ is forced for all $\nu < \kappa$ — actually also $\clubsuit_\kappa(E)$ for all $E \subseteq \kappa$ such that $\diamondsuit_\kappa(E)$ holds in the ground model. By Lemma 4.3, $cov(null) \geq \kappa$. By Lemma 4.3 and $|\,Q\,| = \kappa^+$, $2^{\aleph_0} = \kappa^+$. By $\clubsuit(E^{\omega_1}_\kappa)$ and Theorem 6.1, it follows that $cov(null) = \kappa$. $\square$ (Theorem 7.1)

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Authors' addresses

Sakaé Fuchino

Kitami Institute of Technology, Kitami, Japan.

`fuchino@math.cs.kitami-it.ac.jp`

Saharon Shelah

Institute of Mathematics, The Hebrew University of Jerusalem, 91904 Jerusalem, Israel.

Department of Mathematics, Rutgers University, New Brunswick, NJ 08854, U.S.A.

`shelah@math.huji.ac.il`