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Iterated Forcing with ${}^{\omega}\omega$-bounding and Semiproper Preorders

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Abstract

Assume the Continuum Hypothesis (CH) in the ground model. If we iteratively force with preorders which are ${}^{\omega}\omega$-bounding and semiproper taking suitable limits, then so is the final preorder constructed. Therefore we may show that the Cofinal Branch Principle (CBP) of [F] is strictly weaker than the Semiproper Forcing Axiom (SPFA).

§0 Introduction

We formulate a theory of iterated forcing and propose a construction of a limit stage in [M2]. We call it the *simple* limit. This limit is designed to preserve the semiproperness of preorders. Namely, if we construct an iterated forcing of semiproper preorders which takes this limit at every limit stage, then so is the iterated forcing thus constructed. We move on to consider the ${}^{\omega}\omega$-boundingness together with the semiproperness in this note. Assuming CH in the ground model, we show the preservation of these two properties combined. It is known that the ${}^{\omega}\omega$-boundingness together with the properness is preserved under countable support. Our proof is a straightforward modification of [B] in the current context.

§1 Preliminary

Our approach to forcing is based on the notion of preorders. We say $(P, \leq, 1)$ is a *preorder*, if $\leq$ is a reflexive and transitive binary relation on P with a greatest element 1. So a preorder may not be anti-symmetric. A preorder P (we use this type of abbreviations) is *separative*, if for any $p, q \in P$, we have $q \leq p$ iff $q \Vdash_P$"$p \in \dot{G}$", where $\dot{G}$ denotes the canonical P-name of the generic filters. For $p, q \in P$, we write $p \equiv q$ for short, if $p \leq q \leq p$. Since P is a preorder, this relation $\equiv$ is an equivalence relation. But we never take the quotient.

We review the following technical but important structures and notions from [M1] and [M2]. For more details, we may consult [M1] and [M2].

- *Iterations* $I = \langle P_\alpha \mid \alpha < \nu \rangle$ and *dynamical stages* $\langle \dot{\delta}_k \mid k < \omega \rangle$.
- The *simple limit* and *simple iterations.*
- *Nested antichains* T.
- (T, I)*-nice sequences.*
- A *fusion structure* $\mathcal{F}$ and its *fusions.*

Let us first recall our axiomatic approach to the theory of iterated forcing. The actual construction of a relevant iterated forcing $\langle P_\alpha \mid \alpha < \nu \rangle$ together with the P_α-names $\dot{Q}_\alpha$ is done by recursion.

1.1 Definition. Let $I = \langle (P_\alpha, \leq_\alpha, 1_\alpha) \mid \alpha < \nu \rangle$ be a sequence of separative preorders s.t. for any $\alpha < \nu$, P_α is a set of sequences of length α. We say I is an *iteration*, if for all $\alpha \leq \beta < \nu$, we have

- $1_\beta \lceil \alpha = 1_\alpha$ and if $p \in P_\beta$, then $p \lceil \alpha \in P_\alpha$.
- If $p \in P_\beta$, $a \in P_\alpha$ and $a \leq_\alpha p \lceil \alpha$, then $a^\frown p \lceil [\alpha, \beta) \in P_\beta$ and $a^\frown p \lceil [\alpha, \beta) \leq_\beta p$.
- If $p \leq_\beta q$, then $p \lceil \alpha \leq_\alpha q \lceil \alpha$ and $p \leq_\beta p \lceil \alpha^\frown q \lceil [\alpha, \beta)$.
- If β is a limit ordinal, then for any $x, y \in P_\beta$, we have $y \leq_\beta x$ iff $\forall \alpha < \beta \; y \lceil \alpha \leq_\alpha x \lceil \alpha$.

Let $I = \langle P_\alpha \mid \alpha < \nu \rangle$ (we allow this type of abbreviations) be an iteration and $\alpha \leq \beta < \nu$. Let G_α be a P_α-generic filter over the ground model V. In $V[G_\alpha]$, the *quotient* preorder $P_{\alpha\beta}$ is defined as follows:

- $P_{\alpha\beta} = \{x \lceil [\alpha, \beta) \mid x \in P_\beta \text{ with } x \lceil \alpha \in G_\alpha\}$.
- For $t, s \in P_{\alpha\beta}$, we set $t \leq_{\alpha\beta} s$, if there is $a \in G_\alpha$ s.t. $a^\frown t \leq a^\frown s$ in P_β.

We consider conditions in the limit stages which have a sort of their own Boolean-valued ω-stages. The stages are dependent to given generic objects in the manner given below.

1.2 Definition. Let ν be a limit ordinal and $J = \langle P_\alpha \mid \alpha \leq \nu \rangle$ be an iteration. We say $p \in P_\nu$ has P_ν*-dynamical stages* $\langle \dot{\delta}_k \mid k < \omega \rangle$, if for all

- $\dot{\delta}_k$ is a P_ν-name s.t.
    - $\Vdash_{P_\nu}$ "$\dot{\delta}_k \leq \dot{\delta}_{k+1} \leq \nu$".
    - For any $x \in P_\nu$, if $x \Vdash_{P_\nu}$ "$\dot{\delta}_k = \check{\xi}$", then $x\lceil\xi^\frown 1_\nu\lceil[\xi,\nu) \Vdash_{P_\nu}$ "$\dot{\delta}_k = \check{\xi}$".
- $p \Vdash_{P_\nu}$ "$\dot{\delta}_k < \nu$".
- $\Vdash_{P_\nu}$ "if $\dot{\delta} = \sup\{\dot{\delta}_k \mid k < \omega\}$ and $p\lceil\dot{\delta} \in \dot{G}_\nu\lceil\dot{\delta}$, then $p \in \dot{G}_\nu$".

The conditions with dynamical stages constitute a limit stage of an iterated forcing.

1.3 Definition. Let ν be a limit ordinal and $I = \langle P_\alpha \mid \alpha < \nu\rangle$ be an iteration. We write I^* for the inverse limit of I. By this we mean $I^* = \{x \mid x$ is a sequence of length ν s.t. for all $\alpha < \nu$, we demand $x\lceil\alpha \in P_\alpha\}$. The *simple* limit X of I is a subpreorder of I^* and its universe is defined by $X = \{x \in I^* \mid$ there exist I^*-dynamical stages $\langle\dot{\alpha}_k \mid k < \omega\rangle$ for $x\}$.

In the following the former requirement is what we call the *fullness of successor stages.* This requirement is satisfied by the usual recursive constructions. We demand to take the simple limit at every relevant limit stages.

1.4 Definition. We say an iteration $I = \langle P_\alpha \mid \alpha < \nu\rangle$ is *simple*, if

- For any α with $\alpha + 1 < \nu$ and any P_α-name τ, if $a \Vdash_{P_\alpha}$ "$\tau \in P_{\alpha+1}$ with $\tau\lceil\alpha \in \dot{G}_\alpha$", then there is $b \in P_{\alpha+1}$ s.t. $b\lceil\alpha = a$ and $a \Vdash_{P_\alpha}$ "$b\lceil[\alpha, \alpha+1) \equiv \tau\lceil[\alpha, \alpha+1)$ in $P_{\alpha\alpha+1}$".
- For any limit $\alpha < \nu$, P_α is the simple limit of $I\lceil\alpha$.

We sum up what we got here. The following is 3.5 Lemma and 3.12 Proposition in [M2]. The last two items say we have some freedom when we choose dynamical stages in the simple limit. We caution that to define the simple limit we use I^*-dynamical stages. But once we got the limit X, then every $x \in X$ has X-dynamical stages. This takes some proof and these two should not be confused.

1.5 Lemma. *Let $I = \langle P_\alpha \mid \alpha < \nu\rangle$ be a simple iteration. For any limit ordinal $j < \nu$, let us write $(I\lceil j)_*$ for the direct limit of $I\lceil j$ and $(I\lceil j)^*$ for the inverse limit of $I\lceil j$. Then we have*

- $(I\lceil j)_* \subseteq P_j \subseteq (I\lceil j)^*$.
- *For any $x \in (I\lceil j)^*$, we have $x \in P_j$ iff x has $(I\lceil j)^*$-dynamical stages.*
- *And so if $cf(j) = \omega$, then $P_j = (I\lceil j)^*$. If $cf(j) \geq \omega_1$, then P_j might be bigger than $(I\lceil j)_*$.*
- *For any $x, y \in P_j$, we have $x \leq_j y$ iff $\forall \alpha < j\ x\lceil\alpha \leq_\alpha y\lceil\alpha$.*
- *For any $i < j$ and $x \in P_j$, we may construct P_j-dynamical stages $\langle\dot{\delta}_k \mid k < \omega\rangle$ for x s.t. $\Vdash_{P_j}$ "$\dot{\delta}_0 = \check{i}$".*
- *For any $x, y \in P_j$ with $y \leq_j x$, if $\langle\dot{\delta}_k(x) \mid k < \omega\rangle$ are P_j-dynamical stages for x and $\langle\dot{\delta}_k(y) \mid k < \omega\rangle$ are P_j-dynamical stages for y, then we may assume for all $k < \omega$, $\Vdash_{P_j}$ "$\dot{\delta}_{k+1}(x) \leq \dot{\delta}_k(y)$".*

The simple iterations enjoy the fullness not only at the successor stages but at the limit stages. We state this precisely in 1.11 Lemma. And for that we introduce a kind of generalized names.

1.6 Definition. Let ν be a limit ordinal and $I = \langle P_\alpha \mid \alpha < \nu\rangle$ be an iteration. A *nested antichain* T in I is $(\langle T_n \mid n < \omega\rangle, \langle \mathrm{suc}_T^n \mid n < \omega\rangle)$ s.t. for all $n < \omega$ and all $a \in T_n$, we have

- $T_0 = \{a_0\}$ for some $\alpha_0 < \nu$ and $a_0 \in P_{\alpha_0}$.
- $T_n \subseteq \bigcup\{P_\alpha \mid \alpha < \nu\}$ and $\mathrm{suc}_T^n : T_n \to P(T_{n+1})$.
- For any $b \in \mathrm{suc}_T^n(a)$, we have $l(a) \leq l(b)$ and $b\lceil l(a) \leq a$.
- $\{b\lceil l(a) \mid b \in \mathrm{suc}_T^n(a)\}$ is a maximal antichain below a in $P_{l(a)}$.
- For any $b, b' \in \mathrm{suc}_T^n(a)$, if $b\lceil l(a) = b'\lceil l(a)$, then $b = b'$.
- $T_{n+1} = \bigcup\{\mathrm{suc}_T^n(a) \mid a \in T_n\}$.

We consider a relation between conditions and nested antichains. We intend to identify conditions and nested antichains in the simple iterations via this relation.

1.7 Definition. Let ν be a limit ordinal and $I = \langle P_\alpha \mid \alpha < \nu\rangle$ be an iteration. Let T be a nested antichain in I. We say $x \in I^*$ (the inverse limit of I) is *(T, I)-nice*, if the following hold.

- $x\lceil l(a_0) \equiv a_0$, where $\{a_0\} = T_0$.
- For any $a \in T_n$, we have $a \le x\lceil l(a)$.
- For any $a \in T_n$ and any $b \in \mathrm{suc}^n_T(a)$, we have $b \equiv b\lceil l(a)^\frown x\lceil[l(a), l(b))$.
- For any $\alpha < \nu$ and $w \in P_\alpha$ with $w \le x\lceil\alpha$, if $w \Vdash_{P_\alpha}$ "there is a sequence $\langle a_n \mid n < \omega\rangle$ s.t. for all $n < \omega$, we have $a_n \in T_n$, $a_{n+1} \in \mathrm{suc}^n_T(a_n)$, $l(a_n) \le \alpha$ and $a_n \in \dot{G}_\alpha\lceil l(a_n)$", then we have $w^\frown x\lceil[\alpha,\nu) \equiv w^\frown 1_\nu\lceil[\alpha,\nu)$.

In particular, in this case, for any limit Y of I with $x \in Y$, we have

- $\Vdash_Y$ "$x \in \dot{G}_Y$ iff there is a sequence $\langle a_n \mid n < \omega\rangle$ s.t. for all $n < \omega$, we have $a_n \in T_n$, $a_{n+1} \in \mathrm{suc}^n_T(a_n)$ and $a_n \in \dot{G}_Y\lceil l(a_n)$".

We are going to perform a diagonal argument by constructing a nested antichain with associated objects. The following is from [M1].

1.8 Definition. Let ν be a limit ordinal and $J = \langle P_\alpha \mid \alpha \le \nu\rangle$ be an iteration. Let T be a nested antichain in $I = J\lceil\nu$. We say a structure $\mathcal{F} = \langle (a,n) \mapsto (x^{(a,n)}, T^{(a,n)}) \mid a \in T_n, n < \omega\rangle$ is a *fusion structure* in J, if for all $n < \omega$ and all $a \in T_n$

- $T^{(a,n)}$ is a nested antichain in I.
- $x^{(a,n)} \in P_\nu$ is $(T^{(a,n)}, I)$-nice.
- $a \le x^{(a,n)}\lceil l(a)$ and if $r \in T^{(a,n)}_0$, then $l(r) = l(a)$.
- For any $b \in \mathrm{suc}^n_T(a)$, we have $x^{(b,n+1)} \le x^{(a,n)}$ and $T^{(b,n+1)} \angle T^{(a,n)}$. By this we mean that for all $i < \omega$ and all $e \in T^{(b,n+1)}_i$, there is $f \in T^{(a,n)}_{i+1}$ s.t. $l(f) \le l(e)$ and $e\lceil l(f) \le f$.

We are interested in the identification between a condition and the nested antichain T which served as indices of the fusion structure $\mathcal{F}$. The condition, if any, is to work kind of a master condition by generically descending through T.

1.9 Definition. Let $\mathcal{F}$ be a fusion structure in J, $y \in P_\nu$ and write $I = J\lceil\nu$. We say y is a *fusion* of $\mathcal{F}$, if y is (T, I)-nice.

We have the following as in 4.9 Lemma of [M2].

1.10 Lemma. *Let ν be a limit ordinal and $J = \langle P_\alpha \mid \alpha \leq \nu \rangle$ be a simple iteration. Let us write $I = J\lceil\nu$. If $\mathcal{F} = \langle (a,n) \mapsto (x^{(a,n)}, T^{(a,n)}) \mid a \in T_n, n < \omega \rangle$ is a fusion structure in J with fusion $y \in P_\nu$, then $y \Vdash_{P_\nu}$ "there is $\langle \dot{a}_n \mid n < \omega \rangle$ s.t. for all $n < \omega$, we have $\dot{a}_n \in \dot{T}_n$, $\dot{a}_{n+1} \in suc^n_T(\dot{a}_n)$, $\dot{a}_n \in \dot{G}_\nu \lceil l(\dot{a}_n)$ and $x^{(\dot{a}_n,n)} \in \dot{G}_\nu$".*

The following are 4.10 Theorem and Claim 1 of 6.4 Theorem in [M2]. The former says every nested antichain is equivalent to a condition in the limit stage. The latter says the conditions which have equivalent nested antichains are dense in the limit stage. Hence we may assume every condition in any limit stage of any simple iteration has not only dynamical stages but also an associated nested antichain. Whenever we want a condition in the simple limit, we construct a nested antichain through the iteration with associated objects, i.e., a fusion structure.

1.11 Lemma. *Let ν be a limit ordinal and $J = \langle P_\alpha \mid \alpha \leq \nu \rangle$ be a simple iteration. Let us write $I = J\lceil\nu$. Then we have*

(1) For every nested antichain T in I, there is $x \in P_\nu$ s.t. x is (T, I)-nice.

(2) For every $p \in P_\nu$, there is a nested antichain T in I s.t. if $q \in P_\nu$ is (T, I)-nice, then $q \leq p$.

We quote the preservation theorem for the semiproperness with the simple iterations from [M2].

1.12 Theorem. *Let $I = \langle P_i \mid i < \nu \rangle$ be a simple iteration of semiproper preorders, i.e., I is a simple iteration and for all i with $i+1 < \nu$, we assume $\Vdash_{P_i}$ "P_{ii+1} is semiproper", θ be a sufficiently large regular cardinal and N be a countable elementary substructure of H_θ with $I \in N$.*

Suppose we have $\alpha \leq \beta < \nu$, P_α-names $\dot{M}$, $\dot{y}$ and $d \in P_\alpha$ s.t.

(1) $d \Vdash_{P_\alpha}$ "$N \cup \{\dot{G}_\alpha\} \subseteq \dot{M} \prec H_\theta^{V[\dot{G}_\alpha]}$, $\dot{M}$ is countable and $\beta \in \dot{M}$"

(2) $d \Vdash_{P_\alpha}$ "$\dot{y} \in \dot{M} \cap P_\beta$ and $\dot{y}\lceil\alpha \in \dot{G}_\alpha$"

Then there is $d^* \in P_\beta$ s.t.

(3) $d^*\lceil\alpha = d$.

(4) $d^* \Vdash_{P_\beta}$ "$\dot{y} \in \dot{G}_\beta$".

(5) $d^* \Vdash_{P_\beta}$ "$\dot{M}[\dot{G}_\beta \lceil [\alpha, \beta)] \cap \omega_1^V = \dot{M} \cap \omega_1^V$".

Proof. We consider a maximal antichain below d to decide the values of $\dot{y}$. Then take a mixture of conditions gotten by 6.2 Theorem in [M2] to form d^*. □

§2 Main Theorem

Let us recall that a preorder P is *${}^\omega\omega$-bounding*, if for any P-generic filter G over V and any $f \in V[G]$ with $f : \omega \to \omega$, there is $g \in V$ with $g : \omega \to \omega$ s.t. $f <^* g$ holds. Namely, there is $k < \omega$ s.t. for all n with $k \leq n < \omega$, we have $f(n) < g(n)$. We write this $f <_k g$ for short. So $f <_0 g$ means for all $n < \omega$, we have $f(n) < g(n)$. We may abusively use this notation to finite sequences of natural numbers.

We also recall that a preorder P is *semiproper*, if for all sufficiently large regular cardinals θ, all countable elementary substructures N of H_θ with $P \in N$ and all $p \in P \cap N$, there is $q \leq p$ s.t. we have $q \Vdash_P$ "$N \cap \omega_1^V = N[\dot{G}] \cap \omega_1^V$". For a cardinal $\lambda \geq \omega_1$ and $S \subseteq [\lambda]^\omega$, we say S is *semistationary*, if $\{Y \in [\lambda]^\omega \mid \exists X \in S\ X \subseteq Y \text{ with } X \cap \omega_1 = Y \cap \omega_1\}$ is stationary in $[\lambda]^\omega$. It is known that P is semiproper iff P preserves not only ω_1 but also every semistationary subset $S \subseteq [\lambda]^\omega$ for every cardinal $\lambda \geq \omega_1$.

We consider a simple iteration $I = \langle P_i \mid i < \nu \rangle$ of semiproper preorders. By this we mean that for every $i < \nu$, we assume $\Vdash_{P_i}$ "P_{ii+1} is semiproper". Namely, we force with a semiproper preorder each time. We know that the ${}^\omega\omega$-boundingness is trivially preserved by the 2-step iterations. And we have

2.1 Lemma. (CH) *Let ν be a limit ordinal and $J = \langle P_i \mid i \leq \nu \rangle$ be a simple iteration of semiproper preorders. If for all $i < \nu$, P_i is ${}^\omega\omega$-bounding, then so is P_ν.*

So we immediately have

2.2 Theorem. (CH) *If $I = \langle P_i \mid i < \nu \rangle$ is a simple iteration of ${}^\omega\omega$-bounding and semiproper preorders, i.e., for all i with $i + 1 < \nu$, we assume $\Vdash_{P_i}$ "P_{ii+1} is ${}^\omega\omega$-bounding and semiproper", then for all $i < \nu$, so are P_i.*

Proof of 2.1 Lemma. Let $\dot{f}$ be a P_ν-name s.t. $p \Vdash_{P_\nu}$ "$\dot{f} : \omega \to \omega$". We want to find $q \leq p$ in P_ν and $g : \omega \to \omega$ s.t. $q \Vdash_{P_\nu}$ "$\dot{f} <^* \check{g}$". To this end we define $D = D(J, p, \dot{f})$ and fix an operation $\Pi = \Pi(J, p, \dot{f})$ in the ground model. We write $I = J \lceil \nu$ for short in the following.

$X_0 = (\alpha, x, T^x, \dot{f}_0, \langle \dot{s}_k \mid k < \omega \rangle) \in D$, if

(1) $\alpha < \nu$.

(2) $x \leq p$ in P_ν and T^x is a nested antichain in I s.t. x is (T^x, I)-nice and for $a_0^x \in T_0^x$, we assume $l(a_0^x) = \alpha$.

(3) $\dot{f}_0$ is a P_α-name s.t.

- $x \lceil \alpha \Vdash_{P_\alpha}$ "$\dot{f}_0 : \omega \to \omega$".

(4) For all $k < \omega$, $\dot{s}_k$ are P_α-names s.t. given any P_α-generic filter G_α over V with $x \lceil \alpha \in G_\alpha$, we have the following in $V[G_\alpha]$. We first calculate $f_0 = \dot{f}_0[G_\alpha]$ and $s_k = \dot{s}_k[G_\alpha]$. We then have two items

- $s_0 = x$, $s_{k+1} \leq s_k$ in P_ν and $s_k \lceil \alpha \in G_\alpha$.
- $s_k \Vdash^V_{P_\nu}$ "$\dot{f} \lceil \check{k} = (f_0 \lceil k)\check{}$".

So we are looking at $\dot{f}$ standing in $V[G_\alpha]$ and it looked like $f_0 \lceil k$ with the condition s_k.

For $X_0 \in D$ and $a_1^x \in T_1^x$, we consider

$$\begin{aligned}&\Pi(\alpha, a_1^x, x, T^x, \dot{f}_0, \langle \dot{s}_k \mid k < \omega \rangle) \\ &\quad = (l(a_1^x), x, S^x, \dot{f}_1, \langle \dot{t}_k \mid k < \omega \rangle, \langle \dot{w}_k, \dot{g}_k \mid k < \omega \rangle, \dot{g}_\omega)\end{aligned}$$

Let us write $\beta = l(a_1^x)$ for short. Then concerning the image of Π, we demand

(*) $(\beta, x, S^x, \dot{f}_1, \langle \dot{t}_k \mid k < \omega \rangle) \in D$.

(5) Given any P_β-generic filter G_β over V with $x \lceil \beta \in G_\beta$, let us calculate $f_1 = \dot{f}_1[G_\beta]$ and $t_k = \dot{t}_k[G_\beta]$. Let us write $G_\alpha = G_\beta \lceil \alpha$ and calculate $f_0 = \dot{f}_0[G_\alpha]$, $s_k = \dot{s}_k[G_\alpha]$. Then

- If $s_k \lceil \beta \in G_\beta$, then $t_k = s_k$ and so $f_0 \lceil k = f_1 \lceil k$.

(6) For all $k < \omega$, $\dot{w}_k$, $\dot{g}_k$ and $\dot{g}_\omega$ are P_α-names. Given any P_α-generic filter G_α over V with $a_1^x \lceil \alpha \in G_\alpha$, let us calculate $s_k = \dot{s}_k[G_\alpha]$, $w_k = \dot{w}_k[G_\alpha]$, $g_k = \dot{g}_k[G_\alpha]$ and $g_\omega = \dot{g}_\omega[G_\alpha]$. Then we have the following in $V[G_\alpha]$.

- $w_k \leq s_k \lceil \beta$, a_1^x and $w_k \lceil \alpha \in G_\alpha$.
- $g_k \in ({}^\omega\omega)^V$ and $w_k \Vdash^V_{P_\beta}$ "$\dot{f}_1 <_0 \check{g}_k$".
- $w_k \lceil \alpha$ decides the value of $\dot{s}_k$.
- $g_\omega : \omega \to \omega$ s.t. $g_\omega(m) > \max\{g_k(m) \mid k \leq m\}$ for all $m < \omega$ and so $g_k <_m g_\omega$ for all m with $k \leq m < \omega$.

So we have the following three Facts in the generic extension $V[G_\alpha]$ as in (6) for all $k < \omega$ and all $g \in ({}^\omega\omega)^V$.

Fact 1. *$w_k \lceil \alpha \Vdash^V_{P_\alpha}$ "$\dot{s}_k = \check{s}_k$" and so $w_k \Vdash^V_{P_\beta}$ "$\dot{t}_k = \check{s}_k$".*

Proof. Since $w_k \lceil \alpha$ decides the value of $\dot{s}_k$, $w_k \lceil \alpha \in G_\alpha$ and $\dot{s}_k[G_\alpha] = s_k$, we must have $w_k \lceil \alpha \Vdash^V_{P_\alpha}$ "$\dot{s}_k = \check{s}_k$".

To observe the latter half, let $\overline{G_\beta}$ be any P_β-generic filter over V with $w_k \in \overline{G_\beta}$. Let us write $\overline{G_\alpha}$ for $\overline{G_\beta} \lceil \alpha$. Since $w_k \in \overline{G_\beta}$, we have $w_k \lceil \alpha \in \overline{G_\alpha}$. Since $w_k \lceil \alpha \Vdash^V_{P_\alpha}$ "$\dot{s}_k = \check{s}_k$", we have $w_k \leq s_k \lceil \beta = \dot{s}_k[\overline{G_\alpha}] \lceil \beta$ and so $\dot{s}_k[\overline{G_\alpha}] \lceil \beta \in \overline{G_\beta}$. Hence by the definition of $\dot{t}_k$, we have $\dot{t}_k[\overline{G_\beta}] = \dot{s}_k[\overline{G_\alpha}] = s_k$. □

Fact 2. *If $f_0 \lceil k <_0 g \lceil k$ and $g_\omega <_k g$, then $w_k \Vdash^V_{P_\beta}$ "$\dot{f}_1 <_0 \check{g}$".*

Proof. Suppose $f_0 \lceil k <_0 g \lceil k$ and $g_\omega <_k g \in ({}^\omega\omega)^V$. Let $\overline{G_\beta}$ be any P_β-generic filter over V with $w_k \in \overline{G_\beta}$. Since $w_k \Vdash^V_{P_\beta}$ "$\dot{f}_1 <_0 \check{g}_k$", we have $\dot{f}_1[\overline{G_\beta}] <_0 g_k <_k g_\omega <_k g$. Since $w_k \Vdash^V_{P_\beta}$ "$\dot{t}_k = \check{s}_k$", we have $\dot{t}_k[\overline{G_\beta}] = s_k$. Since $w_k \leq x \lceil \beta \in \overline{G_\beta}$, we have $\dot{t}_k[\overline{G_\beta}] \Vdash^V_{P_\nu}$ "$\dot{f} \lceil k = (\dot{f}_1[\overline{G_\beta}] \lceil k)\check{}$". Since $s_k \Vdash^V_{P_\nu}$ "$\dot{f} \lceil k = (f_0 \lceil k)\check{}$", we have $\dot{f}_1[\overline{G_\beta}] \lceil k = f_0 \lceil k <_0 g \lceil k$. So we have $\dot{f}_1[\overline{G_\beta}] <_0 g$. □

Fact 3. *If $f_0 \lceil k <_0 g \lceil k$, then $w_k {}^\frown s_k \lceil [\beta, \nu) \Vdash^V_{P_\nu}$ "$\dot{f} \lceil k <_0 (g \lceil k)\check{}$".*

Proof. Suppose $f_0 \lceil k <_0 g \lceil k$. Let $\overline{G_\nu}$ be any P_ν-generic filter over V with $w_k {}^\frown s_k \lceil [\beta, \nu) \in \overline{G_\nu}$. Since $s_k \in \overline{G_\nu}$ and $s_k \Vdash^V_{P_\nu}$ "$\dot{f} \lceil k = (f_0 \lceil k)\check{}$", we have $\dot{f}[\overline{G_\nu}] \lceil k = f_0 \lceil k <_0 g \lceil k$. □

On (5): Given G_β with $x\lceil\beta \in G_\beta$, we descend along the $s_k\lceil\beta$ as long as $s_k\lceil\beta \in G_\beta$. And we set $t_k = s_k$. But once we hit $k < \omega$ s.t. $s_k\lceil\beta \notin G_\beta$, then we forget about s_k and start to construct the rest of the t_k.

On (6): Suppose we have fixed $\dot{f}_1$ and $\langle \dot{t}_k \mid k < \omega\rangle$. Since we assume P_β is ${}^\omega\omega$-bounding, it is routine to get $\dot{w}_k$ and $\dot{g}_k$.

The following is from 2.11 Lemma in [M1].

Lemma. *Let $x \in P_\nu$ be (T^x, I)-nice, $a^x_1 \in T^x_1$, $\beta = l(a^x_1)$, $y \in P_\nu$ and $y \le a^x_1{}^\frown x\lceil[\beta,\nu)$. Then there is a nested antichain T^y in I s.t. y is (T^y, I)-nice, $T^y_0 = \{a^y_0\}$, $l(a^y_0) = \beta$, $a^y_0 \le a^x_1$ and $T^y \angle T^x$. Namely, we mean for all $n < \omega$ and all $e \in T^y_n$, there is $f \in T^x_{n+1}$ s.t. $l(f) \le l(e)$ and $e\lceil l(f) \le f$.* □

The following is concerned with a closure property of D.

Subclaim. *Let $x \in P_\nu$ be (T^x, I)-nice and $a^x_1 \in T^x_1$. Let us write β for $l(a^x_1)$. Let $X_1 = (\beta, x, S^x, \dot{f}_1, \langle \dot{t}_k \mid k < \omega\rangle) \in D$, $k^* < \omega$, $w \in P_\beta$ and $t \in P_\nu$ s.t.*

- $w \le a^x_1$.
- $w \Vdash_{P_\beta}$ "$\dot{t}_{k^*} = \check{t}$".

Then there are y, T^y, $\langle \dot{u}_k \mid k < \omega\rangle$ s.t. $Y_0 = (\beta, y, T^y, \dot{f}_1, \langle \dot{u}_k \mid k < \omega\rangle) \in D$ and for all $k < \omega$

- $y = w^\frown t\lceil[\beta,\nu) \le a^x_1{}^\frown x\lceil[\beta,\nu)$, t.
- $y\lceil\beta \Vdash_{P_\beta}$ "$\dot{u}_k\lceil[\beta,\nu) = \dot{t}_{k^*+k}\lceil[\beta,\nu)$".
- $T^y \angle T^x$.

Proof. Since $w \le t\lceil\beta$, we may consider $y = w^\frown t\lceil[\beta,\nu) \in P_\nu$. We have $y \le a^x_1{}^\frown x\lceil[\beta,\nu)$, t. So by Lemma, we may take a nested antichain T^y in I s.t. y is (T^y, I)-nice, $\{a^y_0\} = T^y_0$, $l(a^y_0) = \beta$, $a^y_0 \le a^x_1$ and $T^y \angle T^x$. To define $\langle \dot{u}_k \mid k < \omega\rangle$, let G_β be any P_β-generic filter over V with $y\lceil\beta \in G_\beta$. We define a sequence $\langle u_k \mid k < \omega\rangle$ by recursion in $V[G_\beta]$. Let $t_k = \dot{t}_k[G_\beta]$ and $f_1 = \dot{f}_1[G_\beta]$. So we have $t_{k^*} = t$. We first set $u_0 = y$. So we clearly have $u_0\lceil[\beta,\nu) = t\lceil[\beta,\nu) = t_{k^*+0}\lceil[\beta,\nu)$. Suppose we have constructed u_k so that $u_k \le y$, $u_k\lceil\beta \in G_\beta$ and $u_k\lceil[\beta,\nu) = t_{k^*+k}\lceil[\beta,\nu)$. Since $t_{k^*+k+1}\lceil\beta$, $u_k\lceil\beta \in G_\beta$, we have a common extension $e \in G_\beta$. Let $u_{k+1} = e^\frown t_{k^*+k+1}\lceil[\beta,\nu) \in P_\nu$. Then we have $u_{k+1} \leq u_k$, t_{k^*+k+1}. And so $u_{k+1} \Vdash^V_{P_\nu}$ "$\dot{f}\lceil k+1 = (f_1\lceil \check{k}+1)$". This completes the construction. Now back in V, let $\dot{u}_k$ be a P_β-name of u_k. This $\langle \dot{u}_k \mid k < \omega\rangle$ together with y work. □

Let θ be a sufficiently large regular cardinal and N be a countable elementary substructure of H_θ with $p, \dot{f}, J, D, \Pi \in N$. Let us fix $g : \omega \to \omega$ so that $N \cap {}^\omega\omega <^* g$ which is meant that for any $h \in N \cap {}^\omega\omega$, we have $h <^* g$.

The following construction is crutial.

Main Claim. *Suppose we have given $X_0 = (\alpha, x, T^x, \dot{f}_0, \langle \dot{s}_k \mid k < \omega\rangle) \in D$, $\dot{M}$, $a \in P_\alpha$ and $K < \omega$ s.t.*

(7) $a \leq x\lceil\alpha$.

(8) $\dot{M}$ *is a* P_α*-name s.t.*

- $a \Vdash_{P_\alpha}$ "$N \cup \{\dot{G}_\alpha, \check{X}_0\} \subseteq \dot{M} \prec H_\theta^{V[\dot{G}_\alpha]}$ *and* $\dot{M}$ *is countable*".
- $a \Vdash_{P_\alpha}$ "$N \cap \omega_1^V = \dot{M} \cap \omega_1^V$".

Since we assume CH, an enumeration of ${}^\omega\omega$ *in the order type* ω_1 *exists in* N*. Since we assume* P_α *is* ${}^\omega\omega$*-bounding, we consequently have*

- $a \Vdash_{P_\alpha}$ "$N \cap ({}^\omega\omega)^V = \dot{M} \cap ({}^\omega\omega)^V$ *and* $\dot{M} \cap ({}^\omega\omega)^{V[\dot{G}_\alpha]} <^* \check{g}$".

(9) $x\lceil\alpha \Vdash_{P_\alpha}$ "$\dot{f}_0 <_0 \check{g}$".

Then the following $d \in P_\alpha$ *are dense below* a.

There are $Y_0 = (\beta, y, T^y, \dot{f}_1, \langle \dot{u}_k \mid k < \omega\rangle) \in D$, $a_1^x \in T_1^x$, $\dot{M}_1$, $d^* \in P_\beta$ *and* k^* *s.t.*

(0) $y \leq a_1^{x\frown} x\lceil[\beta,\nu)$ *in* P_ν *and that*

- *If* $\{a_0^y\} = T_0^y$, *then* $\alpha \leq l(a_0^y) = l(a_1^x) = \beta$ *and* $T^y \angle T^x$.

(0⁺) $d \Vdash_{P_\alpha}$ "$\check{y}, \check{\beta} \in \dot{M}$".

(0⁺⁺) $d^*\lceil\alpha = d \leq y\lceil\alpha \leq a_1^x\lceil\alpha$.

(7) $d^* \leq y\lceil\beta$.

(8) $\dot{M}_1$ is a P_β-name s.t.

- $d^* \Vdash_{P_\beta}$ "$N \cup \{\dot{G}_\beta, \check{Y}_0\} \subseteq \dot{M}_1 = \dot{M}[\dot{G}_\beta\lceil[\alpha,\beta)] \prec H_\theta^{V[\dot{G}_\beta]}$ *and* $\dot{M}_1$ *is countable*".
- $d^* \Vdash_{P_\beta}$ "$N \cap \omega_1^V = \dot{M} \cap \omega_1^V = \dot{M}_1 \cap \omega_1^V$".

(9) $y\lceil\beta \Vdash_{P_\beta}$ "$\dot{f}_1 <_0 \check{g}$".

(10) $K \leq k^* < \omega$ and $y \Vdash_{P_\nu}$ "$\dot{f}\lceil k^* <_0 \check{g}\lceil k^*$".

Proof. Take any d (we abusively denote it) below a. Let us write $T_0^x = \{a_0^x\}$. Fix any P_α-generic filter G_α over V with $d \in G_\alpha$. We argue in $V[G_\alpha]$.

(**Step 1**) Since $d \leq a \leq x\lceil\alpha \equiv a_0^x$, we have $a_0^x \in G_\alpha$. Since $\{b\lceil\alpha \mid b \in \mathrm{suc}^0_{T^x}(a_0^x)\}$ is a maximal antichain below a_0^x in P_α, there is $a_1^x \in \mathrm{suc}^0_{T^x}(a_0^x)$ s.t. $a_1^x\lceil\alpha \in G_\alpha$. Let us write $M = \dot{M}[G_\alpha]$. Since $T^x, a_0^x, G_\alpha \in M$, we have $a_1^x \in M$ and so $\beta = l(a_1^x) \in M$.

(**Step 2**) Let $X = (\alpha,\ a_1^x,\ x,\ T^x,\ \dot{f}_0,\ \langle \dot{s}_k \mid k < \omega\rangle)$ and

$$Y = \Pi(X) = (\beta, x, S^x, \dot{f}_1, \langle \dot{t}_k \mid k < \omega\rangle, \langle \dot{w}_k, \dot{g}_k \mid k < \omega\rangle, \dot{g}_\omega).$$

Since $\Pi \in N \cup \{X_0\} \subseteq M$, we have $X, Y \in M$.

(**Step 3**) Let us write $f_0 = \dot{f}_0[G_\alpha]$, $s_k = \dot{s}_k[G_\alpha]$, $w_k = \dot{w}_k[G_\alpha]$, $g_k = \dot{g}_k[G_\alpha]$, $g_\omega = \dot{g}_\omega[G_\alpha]$. Since $X, Y, G_\alpha \in M$, we have $f_0, \langle s_k \mid k < \omega\rangle, \langle w_k, g_k \mid k < \omega\rangle, g_\omega \in M$. Since $g_\omega \in M \cap ({}^\omega\omega)^{V[G_\alpha]}$, there is k^* s.t. $K \leq k^* < \omega$ and $g_\omega <_{k^*} g$. Since $f_0 <_0 g$, we have

- $f_0\lceil k^* <_0 g\lceil k^*$ and $g_\omega <_{k^*} g$. Hence by Facts 1 through 3, we have
- $w_{k^*} \Vdash^V_{P_\beta}$ "$\dot{f}_1 <_0 \check{g}$".
- $w_{k^*}^\frown s_{k^*}[\beta,\nu) \Vdash^V_{P_\nu}$ "$\dot{f}\lceil k^* <_0 (g\lceil k^*)\check{}$". We set $y = w_{k^*}^\frown s_{k^*}\lceil[\beta,\nu)$. Then we have
- $y \in M$. By Subclaim, we have T^y and $\langle \dot{u}_k \mid k < \omega\rangle$ s.t.

- $Y_0 = (\beta, y, T^y, \dot{f}_1, \langle \dot{u}_k \mid k < \omega \rangle) \in D \cap M$.
- $y \leq a_1^x {}^\frown x \lceil [\beta, \nu)$ and $y \lceil \alpha \in G_\alpha$.
- $T^y \angle T^x$.

($\bar{9}$) $y \lceil \beta \Vdash^V_{P_\beta}$ "$\dot{f}_1 <_0 \check{g}$".

(10) $K \leq k^* < \omega$ and $y \Vdash^V_{P_\nu}$ "$\dot{f} \lceil k^* <_0 (g \lceil k^*)\check{}$".

(Step 4) Now back in V, by extending d, we may decide the values of a_1^x, k^* and $Y_0 \in D \cap M$. So we may assume

- $d \Vdash_{P_\alpha}$ "$\check{\beta}, \check{y} \in \dot{M}$ and $\check{y} \lceil \check{\alpha} \in \dot{G}_\alpha$".

Then by the iteration theorem for the semiproperness, we have $d^* \in P_\beta$ s.t.

- $d^* \lceil \alpha = d$, $d^* \leq y \lceil \beta$.
- $d^* \Vdash_{P_\beta}$ "$\dot{M} \cap \omega_1^V = \dot{M}[\dot{G}_\beta \lceil [\alpha, \beta)] \cap \omega_1^V$".

(Step 5) Let $\dot{M}_1$ be a P_β-name s.t.

- $d^* \Vdash_{P_\beta}$ "$\dot{M}_1 = \dot{M}[\dot{G}_\beta \lceil [\alpha, \beta)]$".

Then we certainly have ($\bar{8}$). In particular,

- $d^* \Vdash_{P_\beta}$ "$N \cup \{\dot{G}_\beta, \check{Y}_0\} \subseteq \dot{M}_1 \prec H_\theta^{V[\dot{G}_\beta]}$ and $\dot{M}_1$ is countable".
- $d^* \Vdash_{P_\beta}$ "$N \cap \omega_1^V = \dot{M}_1 \cap \omega_1^V$".

This establishes Main Claim. □

We now carry a recursive construction of a fusion structure

$$\mathcal{F} = \langle (a, n) \mapsto (x^{(a,n)}, T^{(a,n)}) \mid a \in T_n, n < \omega \rangle$$

together with

$$\langle (a, n) \mapsto (x^{(a,n)}, \dot{f}^{(a,n)}, \langle \dot{s}_k^{(a,n)} \mid k < \omega \rangle, \dot{M}^{(a,n)}) \mid a \in T_n, n < \omega \rangle$$

such that if we denote $X^{(a,n)} = (l(a), x^{(a,n)}, T^{(a,n)}, \dot{f}^{(a,n)}, \langle \dot{s}_k^{(a,n)} \mid k < \omega \rangle)$ for short, then

(∗) $X^{(a,n)} \in D$.

- For any $b \in \mathrm{suc}^n_T(a)$, we have $x^{(b,n+1)} \leq x^{(a,n)}$ and $T^{(b,n+1)} \angle T^{(a,n)}$.

(7) $a \leq x^{(a,n)} \lceil l(a)$.

(8) $\dot{M}^{(a,n)}$ is a $P_{l(a)}$-name s.t.

- $a \Vdash_{P_{l(a)}}$ "$N \cup \{\dot{G}_{l(a)}, \check{X^{(a,n)}}\} \subseteq \dot{M}^{(a,n)} \prec H_\theta^{V[\dot{G}_{l(a)}]}$ and $\dot{M}^{(a,n)}$ is countable".
- $a \Vdash_{P_{l(a)}}$ "$N \cap \omega_1^V = \dot{M}^{(a,n)} \cap \omega_1^V$".

(9) $x^{(a,n)} \lceil l(a) \Vdash_{P_{l(a)}}$ "$\dot{f}^{(a,n)} <_0 \check{g}$".

(10) For $b \in \mathrm{suc}^n_T(a)$, we have $x^{(b,n+1)} \Vdash_{P_\nu}$ "$\dot{f} \lceil \check{n} <_0 \check{g} \lceil \check{n}$".

If q is a fusion of $\mathcal{F}$, then $q \leq p$ in P_ν and $q \Vdash_{P_\nu}$ "$\dot{f} <_0 \check{g}$". So we would be done.

Here is the construction. We first take $f : \omega \to \omega$ and $\langle s_k \mid k < \omega \rangle$ s.t. $s_0 = p$ and for all $k < \omega$, we have $s_{k+1} \leq s_k$ and $s_k \Vdash_{P_\nu}$ "$\dot{f} \lceil k = (\check{f \lceil k})$". We may assume $f, \langle s_k \mid k < \omega \rangle \in N$. We may also fix a nested antichain T^p s.t. p is (T^p, I)-nice and $T^p_0 = \{\emptyset\}$. We may assume $T^p \in N$.

Now we let $T_0 = \{\emptyset\}$. And set $x^{(\emptyset,0)} = p$, $T^{(\emptyset,0)} = T^p$, $\dot{f}^{(\emptyset,0)} = \check{f}$, $\langle s_k^{(\emptyset,0)} \mid k < \omega \rangle = \langle \check{s_k} \mid k < \omega \rangle$ and $M^{(\emptyset,0)} = \check{N}$.

Since we may assume $g : \omega \to \omega$ s.t. $f <_0 g$ and $N \cap {}^\omega\omega <^* g$, we are done at T_0.

For successor stages, suppose we have gotten T_n and the associated objects. Fix any $a \in T_n$ and apply Main Claim to $X^{(a,n)}$, $\dot{M}^{(a,n)}$, a and n. It is routine to form $\mathrm{suc}^n_T(a)$ and attach associated objects to each $b \in \mathrm{suc}^n_T(a)$. This completes the construction.

□

§3 Applications

We consider an application of our main theorem. We pick up a forcing axiom from [F].

3.1 Definition. ([F]) The *Cofinal Branch Principle* (CBP) says: Every tree of height ω_1 which preserves every stationary subset of ω_1 has a cofinal branch.

We have the following.

3.2 Proposition. ([F]) *The Semiproper Forcing Axiom (SPFA) implies CBP.*

We give a forcing construction which establishes the consistency of CBP and more.

3.3 Theorem. (CH) *Let κ be a supercompact cardinal. We may construct a notion of forcing P_κ which is semiproper, has the κ-c.c. and we have the following in V^{P_κ}.*

- *For any $\dot{f} : \omega \to \omega$, there exists $g \in ({}^{\omega}\omega)^V$ s.t. $\dot{f} <^* g$.*
- *CBP holds.*

3.4 Corollary. *CBP does not imply SPFA.*

To give a proof to the theorem, we make preparations.

3.5 Lemma. *The following are equivalent.*

(1) CBP.

(2) Every tree of height ω_1 which is semiproper has a cofinal branch.

Proof. Since semiproper preorders preserve every stationary subset of ω_1, we know that (1) implies (2). To show the converse the following suffice.

Claim 1. *(2) implies the Strong Reflectin Principle (SRP).*

Claim 2. *SRP implies that if a preorder P preserves every stationary subset of ω_1, then P is semiproper.*

Proof of Claim 1. By p. 58 of [Be], the notion of forcing which forces a strong reflection sequence is a tree of height ω_1 and is semiproper. Hence (2) implies SRP. □

Proof of Claim 2. Let $\lambda \geq \omega_1$ be a cardinal and $S \subseteq [\lambda]^\omega$ be a semistationary subset. We show $\Vdash_P$"S remains semistationary in $[\lambda]^\omega$". To this end suppose $p \Vdash_P$"$\dot\pi : {}^{<\omega}\lambda \to \lambda$". We want to find $q \le p$, $X \in S$ and $\dot Y$ s.t. $q \Vdash_P$"$X \subseteq \dot Y \in [\lambda]^\omega$, $X \cap \omega_1^V = \dot Y \cap \omega_1^V$ and $\dot Y$ is $\dot\pi$-closed".

Let θ be a sufficiently large regular cardinal. Then by SRP, we have $\langle N_i \mid i < \omega_1\rangle$ s.t.

- N_i is a countable elementary substructure of H_θ.
- $P, \dot\pi \in N_0$.
- If $j < \omega_1$, then $\langle N_i \mid i \le j\rangle \in N_{j+1}$.
- If $j < \omega_1$ is a limit ordinal, then $N_j = \bigcup\{N_i \mid i < j\}$.
- Either there is $X \in S$ s.t. $X \subseteq N_i \cap \lambda$ and $X \cap \omega_1 = N_i \cap \omega_1$,
- Or, for any countable elementary substructure N of H_θ s.t. $N_i \subseteq N$ and $N_i \cap \omega_1 = N \cap \omega_1$, we have no $X \in S$ s.t. $X \subseteq N \cap \lambda$ and $X \cap \omega_1 = N \cap \omega_1$.

Subclaim. *$S_0 = \{i < \omega_1 \mid \exists X \in S\ X \subseteq N_i \cap \lambda$ and $X \cap \omega_1 = N_i \cap \omega_1\}$ is stationary in ω_1.*

Proof. By contradiction. Suppose $\omega_1 \setminus S_0$ contained a club subset C_0 of ω_1. Since S is semistationary, we may take a sufficiently large regular cardinal χ and a countable elementary substructure M of H_χ so that $C_0, H_\theta, \langle N_i \mid i < \omega_1\rangle \in M$ and there is $X \in S$ s.t. $X \subseteq M \cap \lambda$ and $X \cap \omega_1 = M \cap \omega_1$. Let $i_1 = M \cap \omega_1$. Since C_0 is a club and $C_0 \in M$, we have $i_1 \in C_0$. Since $\langle N_i \mid i < \omega_1\rangle \in M$, each N_i is countable and continuously increasing, we have $N_{i_1} = \bigcup\{N_i \mid i < i_1\} \subseteq M \cap H_\theta \prec H_\theta$ and $N_{i_1} \cap \omega_1 = i_1 = (M \cap H_\theta) \cap \omega_1$. Since we have $X \in S$ and $X \subseteq (M \cap H_\theta) \cap \lambda$ and $X \cap \omega_1 = (M \cap H_\theta) \cap \omega_1$ and since $\langle N_i \mid i < \omega\rangle$ is a strong reflection sequence, we must have some $\bar X \in S$ s.t. $\bar X \subseteq N_{i_1} \cap \lambda$ and $\bar X \cap \omega_1 = N_{i_1} \cap \omega_1$. But $i_1 \in C_0 \subseteq \omega_1 \setminus S_0$, so $i_1 \notin S_0$. This contradicts to the definition of S_0. This establishes Subclaim.

□

Now since P preserves every stationary subset of ω_1, we have $p \Vdash_P$"S_0 remains stationary". However, $p \Vdash_P$"$\dot C = \{i < \omega_1 \mid N_i \cap \omega_1 = i = N_i[\dot G] \cap \omega_1\}$ is a club". Therefore, p forces the following. There is $i_0 \in S_0 \cap \dot C$. Since

$i_0 \in S_0$, we have $X \in S$ s.t. $X \subseteq N_{i_0} \cap \lambda$ and $X \cap \omega_1 = N_{i_0} \cap \omega_1$. Since $i_0 \in \dot{C}$, we have $N_{i_0} \cap \omega_1 = i_0 = N_{i_0}[\dot{G}] \cap \omega_1$. Let $\dot{Y} = N_{i_0}[\dot{G}] \cap \lambda$. Then $X \subseteq \dot{Y}$, $X \cap \omega_1 = \dot{Y} \cap \omega_1$ and $\dot{Y}$ is $\dot{\pi}$-closed. So we may take $q \leq p$ which decides the value of $X \in S$. This establishes Claim 2 and 3.5 Lemma. □

Proof of 3.3 Theorem. Let κ be a supercompact cardinal. We may assume CH. We construct a simple iteration $J = \langle P_\alpha \mid \alpha \leq \kappa \rangle$. The construction is exactly the same as when we force SPFA. So P_κ is semiproper and has the κ-c.c. However, since we force with σ-Baire preorders each time in this contruction, we are iterating with ${}^\omega\omega$-bounding and semiproper preorders. Therefore by our main theorem, P_κ is ${}^\omega\omega$-bounding. □

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