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Uniqueness and Regularity of solutions to the Navier-Stokes equations.

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Introduction

The purpose of this article is to give a survey on the recent development of well-posedness on the Navier-Stokes equations. We are mainly concerned with the results given by the author. Consider the Navier-Stokes equations in $\mathbb{R}^n (n \geq 2)$:

$$\text{(N-S)} \qquad \begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + u \cdot \nabla u + \nabla p = 0, & x \in \mathbb{R}^n, t \in (0,T), \\ \operatorname{div} u = 0 \quad x \in \mathbb{R}^n, t \in (0,T), \\ u\,|_{t=0} = a, \end{cases}$$

where $u = u(x,t) = (u^1(x,t), \cdots, u^n(x,t))$ and $p = p(x,t)$ denote the unknown velocity vector and the pressure of the fluid at the point $(x,t) \in \mathbb{R}^n \times (0,T)$, respectively, while $a = a(x) = (a^1(x), \cdots, a^n(x))$ is the given initial velocity vector field. For simplicity, we assume that the external force has a scalar potential and is included into the pressure gradient.

Let us first introduce some function spaces. We denote by $C_{0,\sigma}^{\infty}$ the set of all C^{∞} vector functions $\phi = (\phi^1, \cdots, \phi^n)$ with compact support in $\mathbb{R}^n$, such that div $\phi = 0$. L_{σ}^r is the closure of $C_{0,\sigma}^{\infty}$ with respect to the L^r-norm $\|\cdot\|_r$. $(\cdot,\cdot)$ denotes the duality pairing between L^r and $L^{r'}$, where $1/r + 1/r' = 1$. L^r stands for the usual (vector-valued) L^r-space over $\mathbb{R}^n$, where $1 < r < \infty$. $H_{0,\sigma}^1$ denotes the closure of $C_{0,\sigma}^{\infty}$ with respect to the norm

$$\|\phi\|_{H^1} = \|\phi\|_2 + \|\nabla\phi\|_2,$$

where $\nabla\phi = (\partial\phi^i/\partial x_j), i,j = 1, \cdots, n$. For an interval I in $\mathbb{R}^1$ and a Banach space X, $L^p(I;X)$ and $C^m(I;X)$ denote the usual Banach spaces of functions of L^p and C^m-class on I with values in X, respectively, where $1 \leq p \leq \infty$, $m = 0, 1, \cdots$.

Our definition of a weak solution of (N-S) now reads

Definition 0.1 *Let $a \in L_{\sigma}^2$. A measurable function u on $\mathbb{R}^n \times (0,T)$ is called a weak solution of (N-S) on $(0,T)$ if*

(i) $u \in L^{\infty}(0,T; L_{\sigma}^2) \cap L^2(0,T; H_{0,\sigma}^1)$;

(ii) *For every* $\Phi \in H^1(0,T;H^1_{0,\sigma} \cap L^n)$ *with* $\Phi(T) = 0$,

$$\int_0^T \{-(u, \partial_t\Phi) + (\nabla u, \nabla\Phi) + (u \cdot \nabla u, \Phi)\}dt = (a, \Phi(0)). \tag{0.1}$$

Concerning existence of the weak solutions, we have Leray [13] and Hopf [7].

Theorem 0.2 (Leray-Hopf) *For every* $a \in L^2_\sigma$, *there exists at least one weak solution* u *of (N-S) on* $(0,\infty)$ *such that*

$$\|u(t)\|_2^2 + 2\int_0^t \|\nabla u(\tau)\|_2^2 d\tau \le \|a\|_2^2, \quad 0 \le t < \infty, \tag{0.2}$$

and

$$\|u(t) - a\|_2 \to 0 \quad \text{as } t \to +0.$$

We are interested in the following problems on well-posedness to (N-S);

Problems.

(I) Uniqueness and regularity of weak solutions

(II) Global existence of regular solutions for large data a

(III) Blow-up; dose there exist $T_* < \infty$ such that

$$u(t) \in C^\infty(\mathbb{R}^n) \quad \text{for } 0 < t < T_*, \text{ but } u(T_*) \notin C^\infty(\mathbb{R}^n) \ ?$$

1 Uniqueness and regularity

Let us introduce the class $L^s(0,T;L^r)$ with the norm $\|\cdot\|_{L^s(0,T;L^r)}$;

$$\|u\|_{L^s(0,T;L^r)} = \left(\int_0^T \|u(t)\|_r^s dt\right)^{1/s} = \left(\int_0^T (\int_{\mathbb{R}^n} |u(x,t)|^r dx)^{s/r} dt\right)^{1/s}$$

The classical result on uniqueness and regularity of weak solutions in the class $L^s(0,T;L^r)$ was given by Foias-Serrin-Masuda [3], [16], [17], [14]:

Theorem 1.1 (Foias-Serrin-Masuda) *Let* $a \in L^2_\sigma$.

(i) *Let* u *and* v *be two weak solutions of (N-S) on* $(0,T)$. *Suppose that* u *satisfies*

$$u \in L^s(0,T;L^r) \quad \text{for } 2/s + n/r = 1 \text{ with } n < r \le \infty. \tag{1.1}$$

Assume that v *fulfills the energy inequality (0.2) for* $0 \le t < T$. *Then we have* $u \equiv v$ *on* $[0,T)$.

(ii) *Eery weak solution* u *of (N-S) in the class (1.1) satisfies*

$$\frac{\partial u}{\partial t}, \quad \frac{\partial^{\alpha_1+\cdots+\alpha_n} u}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}} \in C(\mathbb{R}^n \times (0,T)) \tag{1.2}$$

for all multi-indices $\alpha = (\alpha_1,\cdots,\alpha_n)$ *with* $|\alpha| = \alpha_1 + \cdots + \alpha_n \le 2$.

Remark 1.2 (i) In Theorem 1.1 (i) , v need not belong to the class (1.1). On the other hand, every weak solution u with (1.1) fulfills the energy identity

$$(1.3)\qquad \|u(t)\|_2^2 + 2\int_0^t \|\nabla u(\tau)\|_2^2 d\tau = \|a\|_2^2, \quad 0 \le t \le T.$$

It seems to be an interesting question whether every weak solution satisfies the energy inequality (0.2).

(ii) If u is merely in the Leray-Hopf class, then there exists s_0, r_0 with $2/s_0 + n/r_0 = n/2$ such that $u \in L^{s_0}(0,T;L^{r_0})$. For example, we may take $s_0 = 2$ and $r_0 = 2n/(n-2)$. In particular, by Therem 1.1 with the aid of interpolation inequality

$$\|u\|_{L^{r_0}(\mathbb{R}^2)} \le C\|u\|_{L^2(\mathbb{R}^2)}^{r_0/2}\|\nabla u\|_{L^2(\mathbb{R}^2)}^{1-r_0/2}, \quad 2 < r_0 < \infty \quad \text{for all } u \in H^1(\mathbb{R}^2),$$

we see that every weak solution of (N-S) in the 2-dimensional case is unique and regular, so Problems (I), (II) and (III) are completely solved in $\mathbb{R}^2$. Notice that if u is regular, then s and r can be taken arbitrarily large, which makes the quantity $2/s + n/r$ *smaller*.

(iii) The class (1.1) is important from viewpoint of the scaling invariance. It can be easily seen that if $\{u, p\}$ is a pair of the solution to (N-S) on $\mathbb{R}^n \times (0,\infty)$, then so is the family $\{u_\lambda, p_\lambda\}_{\lambda>0}$, where

$$u_\lambda(x,t) \equiv \lambda u(\lambda x, \lambda^2 t), \quad p_\lambda(x,t) \equiv \lambda^2 p(\lambda x, \lambda^2 t).$$

Scaling invariance means that there holds

$$\|u_\lambda\|_{L^s(0,\infty;L^r)} \left(= \lambda^{1-(\frac{2}{s}+\frac{n}{r})}\|u\|_{L^s(0,\infty;L^r)}\right) = \|u\|_{L^s(0,\infty;L^r)} \quad \text{for all } \lambda > 0$$

if and only if

$$2/s + n/r = 1.$$

The solution $\{u, p\}$ with the property that $u_\lambda(x,t) = u(x,t)$, $p_\lambda(x,t) = p(x,t)$ for all $\lambda > 0$ is called a *self-similar* solution. For (N-S), the self-similar solution has the form such as

$$u(x,t) = \frac{1}{\sqrt{t}}U(\frac{x}{\sqrt{t}}), \quad p(x,t) = \frac{1}{t}P(\frac{x}{\sqrt{t}}),$$

where $U = (U^1(y), \cdots, U^n(y))$, $P = P(y)$ is the functions for $y = (y_1, \cdots, y_n) \in \mathbb{R}^n$. More presisely, the above solution is called a *forward self-similar* solution.

We shall next deal with the critical case with $s = \infty$ and $r = n$ in (1.1).

Theorem 1.3 **(Masuda [14], Kozono-Sohr [11], [12])** *Let $a \in L^2_\sigma$.*

(i)*(uniqueness) Let u and v be weak solutions of* (N-S). *Suppose that $u \in L^\infty(0,T;L^n)$ and that v satisfies the energy inequality (0.2) for $0 \le t < T$. Then we have $u \equiv v$ on $[0,T)$.*

(ii) *(regularity) There exists a positive constant* ε_0 *such that if* u *is a weak solution of* (N-S) *in* $L^\infty(0,T;L^n)$ *with the property*

$$\limsup_{t\to t_*-0}\|u(t)\|_n^n \le \|u(t_*)\|_n^n + \varepsilon_0 \quad \textit{for } t_* \in (0,T), \tag{1.4}$$

then u *satisfies*

$$\frac{\partial u}{\partial t},\ \frac{\partial^{\alpha_1+\cdots+\alpha_n}u}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}} \in C(\mathbb{R}^n\times(t_*-\rho,t_*+\rho)) \quad \textit{for some } \rho>0, \tag{1.5}$$

where $\alpha=(\alpha_1,\cdots,\alpha_n)$ *is an arbitrary multi-index with* $|\alpha|=\alpha_1+\cdots+\alpha_n\le 2$. *In particular, if* u *has the property (1.4) for every* $t_*\in(0,T)$, *then* u *is regular on* $\mathbb{R}^n\times(0,T)$ *as in (1.2).*

Remark 1.4 (i) Masuda [14] proved that if $u\in L^\infty(0,T;L^n)$ is continuous from the right on $[0,T)$ in the norm of L^n, then there holds $u\equiv v$ on $[0,T)$. Later on, Kozono-Sohr [11] showed that every weak solution u in $L^\infty(0,T;L^n)$ of (N-S) on $(0,T)$ becomes necessarily continuous from the right in the norm of L^n.

(ii) By the above theorem, every weak solution in $C([0,T);L^n)$ is unique and regular. This was proved by Giga [5] and von Wahl [20]. In Section 2, we shall give another proof by a different method.

(iii) Recently, Hishida-Izumida [8] improved the condition (1.4). They proved regularity of u under the weaker assumption that

$$\liminf_{t\to t_*-0}\|u(t)\|_n^n \le \|u(t_*)\|_n^n + \varepsilon_0.$$

It seems to be an interesting question whether or not *every* weak solution $u\in L^\infty(0,T;L^n)$ is regular.

Finally in this section, we investigate the size of singular sets of weak solutions in the 3-dimensional case. For a weak solution u in $\mathbb{R}^3\times(0,T)$ we denote by $S(u)$ the singular set defined by

$$S(u)\equiv\{(x,t)\in\mathbb{R}^3\times(0,T); u\notin L^\infty(B_\rho(x,t)) \quad \text{for } \forall\rho>0\},$$

where $B_\rho(x,t)=\{(y,s)\in\mathbb{R}^3\times(0,T); |y-x|<\rho, |s-t|<\rho\}$. For each $t\in(0,T)$ we set $S_t(u)=\{x\in\mathbb{R}^3;(x,t)\in S(u)\}$.

Theorem 1.5 (Neustupa [15]) *Let* $n=3$. *There is an absolute constant* $\varepsilon_0>0$ *such that every weak solution* u *in* $L^\infty(0,T;L^3)$ *fulfills*

$$\sharp S_t(u)\le\left(\frac{1}{\varepsilon_0}\cdot\sup_{0<\tau<T}\|u(\tau)\|_3\right)^3$$

for all $t\in(0,T)$. *Here* $\sharp S$ *denotes the number of elements of the set* S.

Remark 1.6 (i) Caffarelli-Kohn-Nirenberg [2] showed if the weak solution u satisfies the *generalized* energy inequality

$$(1.6)\quad 2\iint_{\mathbb{R}^3\times(0,T)} |\nabla u|^2\phi\,dxdt \le \iint_{\mathbb{R}^3\times(0,T)} [|u|^2(\partial_t\phi+\Delta\phi)+(|u|^2+2p)u\cdot\nabla\phi]dxdt$$

for all $\phi\in C_0^\infty(\mathbb{R}^3\times(0,T))$ with $\phi\ge 0$, then $H^1(S)=0$, where $H^1(S)$ denotes the one-dimensional Hausdorff measure of the set S in the space-time $\mathbb{R}^3\times(0,\infty)$.

(ii) Taniuchi [19] found a class of weak solutions satisfying (1.6). His class is larger than that of Serrin's (1.1).

Finally in this section, we investigate *local* properties of weak solutions in $\mathbb{R}^3$. Let u be a weak solution of (N-S) on $(0,T)$. We call $(x_0,t_0)\in\mathbb{R}^3\times(0,T)$ a *regular* point if there are $\delta>0$ and $\rho>0$ such that

$$\frac{\partial u}{\partial t}, \frac{\partial^{\alpha_1+\cdots+\alpha_n}u}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}} \in C(B_\delta(x_0)\times(t_0-\rho,t_0+\rho))$$

for all multi-indices $\alpha=(\alpha_1,\cdots,\alpha_n)$ with $|\alpha|=\alpha_1+\cdots+\alpha_n\le 2$. Here $B_\delta(x_0)=\{y\in\mathbb{R}^3; |y-x_0|<\delta\}$. The point (x_0,t_0) is called *singular* unless it is regular. u is called *regular* on a space-time $Q=D\times(a,b)$ if every point of Q is a regular one.

Theorem 1.7 (Kozono [10]) *Let $n=3$. There is an absolute constant $\varepsilon_0>0$ with the following property. If u is a weak solution of (N-S) on $(0,T)$ and if u satisfies at $(x_0,t_0)\in\mathbb{R}^3\times(0,T)$*

$$(1.7)\qquad \sup_{t_0-\rho<t<t_0+\rho}\|u(t)\|_{L^3_{\mathrm{w}}(B_\delta(x_0))}\le\varepsilon_0$$

for some $\delta>0$ and $\rho>0$, then (x_0,t_0) is a regular point. Here $\|\cdot\|_{L^3_{\mathrm{w}}(B_\delta(x_0))}$ denotes the weak L^3-norm $\|u\|_{L^3_{\mathrm{w}}(B_\delta(x_0))}=\sup_{R>0}R\mu\{x\in B_\delta(x_0); |u(x)|>R\}^{\frac{1}{3}}$ (μ; Lebesgue measure).

Corollary 1.8 (Removable Singularities) *Let $n=3$. There is an absolute constant ε_0 with the following property. Suppose that u is a weak solution of (N-S) on $(0,T)$. If (x_0,t_0) is an isolated singular point of u satisfying*

$$(1.8)\qquad \limsup_{x\to x_0,t\to t_0}|x-x_0||u(x,t)|<\varepsilon_0,$$

then (x_0,t_0) is a regular point.

In particular, if u behaves at (x_0,t_0) like

$$(1.9)\qquad u(x,t)=o(|x-x_0|^{-1})\quad as\ x\to x_0$$

uniformly with respect to t in some neighbourhood of t_0, then (x_0,t_0) is a regular point.

Remark 1.9 (i) Serrin [16] and Takahashi [18] showed that every weak solution u of (N-S) satisfying

$$\int_a^b \left(\int_D |u(x,t)|^r dx\right)^{\frac{s}{r}} dt < \infty \quad \text{on a cylinder } D \times (a,b) \subset \Omega \times (0,T),$$

for $2/s + 3/r \le 1$ with $r > 3$ is of class C^∞ in the space variables. Our theorem deals with the marginal case when $s = \infty$ and $r = 3$. Furthermore, our weak space $L^3_{\mathrm{w}}(D)$ is larger than the usual $L^3(D)$. Under the condition (1.7), we obtain interior regularity of u not only in the space but also in the *space-time* variables, while Serrin [16] imposed the additional assumption that

$$\partial_t u \in L^s(a,b;L^2(D)) \quad \text{for some } s \ge 1.$$

(ii) Caffarelli-Kohn-Nirenberg [2] gave an absolute constant ε_1 with the following property. Let u be a weak solution of (N-S) on $(0,T)$ with the *generalized* energy inequality (1.6). Suppose that u and its the associated pressure p satisfy

$$R^{-2} \iint_{Q_R(x_0,t_0)} (|u|^3 + |u||p|)dxdt + R^{-13/4} \int_{t_0-R^2}^{t_0} \left(\int_{|x-x_0|<R} |p|dx\right)^{5/4} dt$$

$$(1.10) \quad \le \quad \varepsilon_1,$$

where $Q_R(x_0,t_0) = \{(x,t); |x-x_0| < R, t_0 - R^2 < t < t_0\}$ denotes the parabolic cylinder. Then u is regular in $Q_{R/2}(x_0,t_0)$. In Theorem 1.7, we do not need any energy inequality and show that the condition on the pressure p is redundant. Moreover, the advantage of our theorem enables us to handle the singularity (x_0,t_0) of u such as

$$u(x,t) = o(|x-x_0|^{-1}) \quad \text{as } x \to x_0$$

uniformly with respect to t in some neighbourhood of t_0, the case of which is excluded in their paper because for such (x_0,t_0) we have in (1.10)

$$\iint_{Q_R(x_0,t_0)} |u(x,t)|^3 dxdt = \infty.$$

2 Local existence and uniqueness of strong solutions

In this section, we investigate the solution with (1.1). To this end, we define the strong solutions.

Definition 2.1 *Let $a \in L^n_\sigma$. A measurable function u defined on $\mathbb{R}^n \times (0,T)$ is called a strong solution of (N-S) on $(0,T)$ if*

(i)

$$(2.1) \qquad u \in C([0,T);L^n_\sigma), \quad \frac{\partial u}{\partial t}, Au \in C((0,T);L^n_\sigma);$$

(ii) *u satisfies*

$$\text{(2.2)} \qquad \begin{cases} \frac{\partial u}{\partial t} + Au + P(u \cdot \nabla u) = 0, & \text{in } L^n_\sigma \text{ for } 0 < t < T, \\ u(0) = a. \end{cases}$$

In the above definition, P denotes the Helmholtz-Weyl projection from L^r onto L^r_σ for $1 < r < \infty$. More precisely, $P = \{P_{jk}\}_{j,k=1,\cdots,n}$ can be represented as $P_{jk} = \delta_{jk} + R_j R_k$, where δ_{jk} is the Kronecker symbol and $R_j = F^{-1}(\frac{\sqrt{-1}\xi_j}{|\xi|^2} F)$, $j = 1, \cdots, n$ are the Riesz transforms(F; Fourier transform). $A = -P\Delta$ is the Stokes operator.

Remark 2.2 It is easy to see that every strong solution u of (N-S) on $(0, T)$ is regular as in (1.2).

Concerning the existence and uniqueness of the strong solution, we have

Theorem 2.3 (Kato [9], Giga-Miyakawa [6], Brezis [1]) *For $n < r < \infty$, there is a constant $\gamma = \gamma(n, r) > 0$ with the following property. If the initial data $a \in L^n_\sigma$ and $T_* > 0$ satisfy*

$$\text{(2.3)} \qquad \sup_{0<t\le T_*} t^{\frac{n}{2}(\frac{1}{n}-\frac{1}{r})} \|e^{-tA}a\|_r < \gamma$$

then there exists a unique strong solution $u(t)$ of (N-S) on $[0, T_*)$. Moreover, such a solution u has the property $t^{\frac{n}{2}(\frac{1}{n}-\frac{1}{r})} u(\cdot) \in C([0, T_*); L^r)$ with

$$\text{(2.4)} \qquad \lim_{t\to+0} t^{\frac{n}{2}(\frac{1}{n}-\frac{1}{r})} \|u(t)\|_r = 0.$$

If, in addition, $a \in L^n_\sigma \cap L^2_\sigma$ satisfies (2.3), then u is also a weak solution of (N-S) on $(0, T_*)$.

Under the condition (2.3) we can construct a strong solution u on the interval $(0, T_*)$ by the successive approximation. To verify (2.3), we make use of the following $L^p - L^r$-estimates for the Stokes semigroup $\{e^{-tA}\}_{t\ge0}$;

$$\text{(2.5)} \qquad \begin{cases} \|e^{-tA}a\|_r \le Ct^{-\frac{n}{2}(1/p-1/r)} \|a\|_p, & 1 \le p \le r \le \infty, \\ \|\nabla e^{-tA}a\|_r \le Ct^{-\frac{n}{2}(1/p-1/r)-1/2} \|a\|_p, & 1 \le p \le r < \infty \end{cases}$$

hold for all $a \in L^p_\sigma$ and all $t > 0$, where $C = C(n, p, r)$. Hence, if $a \in L^n_\sigma \cap L^r$ for some $n < r < \infty$, then (2.3) can be achieved in such a way that

$$\text{(2.6)} \qquad T_* = \left(\frac{\gamma}{C\|a\|_r}\right)^{\frac{2r}{r-n}}$$

with the same constant C as in (2.5). If $a \in L^n_\sigma$, by the density argument, for every $\varepsilon > 0$, we can take $\tilde{a} \in C^\infty_{0,\sigma}$ so that $\|a - \tilde{a}\|_n < \varepsilon$. Hence by (2.5) with $p = n$, we have

$$\text{(2.7)} \qquad \begin{aligned} t^{\frac{n}{2}(\frac{1}{n}-\frac{1}{r})} \|e^{-tA}a\|_r &\le t^{\frac{n}{2}(\frac{1}{n}-\frac{1}{r})} \|e^{-tA}(a - \tilde{a})\|_r + t^{\frac{n}{2}(\frac{1}{n}-\frac{1}{r})} \|e^{-tA}\tilde{a}\|_r \\ &\le C\|a - \tilde{a}\|_n + Ct^{\frac{n}{2}(\frac{1}{n}-\frac{1}{r})} \|\tilde{a}\|_r \\ &\le C\varepsilon + Ct^{\frac{n}{2}(\frac{1}{n}-\frac{1}{r})} \|\tilde{a}\|_r, \end{aligned}$$

which yields $\limsup_{t\to+0} t^{\frac{n}{2}(\frac{1}{n}-\frac{1}{r})}\|e^{-tA}a\|_r \le C\varepsilon$. Since ε is arbitrary, we obtain

$$\lim_{t\to+0} t^{\frac{n}{2}(\frac{1}{n}-\frac{1}{r})}\|e^{-tA}a\|_r = 0 \tag{2.8}$$

which ensures existence of T_* in (2.3) for $a \in L^n_\sigma$. However, this convergence is *not uniform* for a in any fixed bounded subset of L^n_σ. So, it is not clear whether the interval T_* for existence of strong solution with the initial data $a \in L^n_\sigma$ can be characterized in terms of the L^n-norm of a such as (2.6). To overcome this difficulty, Brezis [1] considered a class of precompact subsets in L^n_σ.

Proposition 2.4 (Brezis) *Let $n < r < \infty$. For every precompact set K in L^n_σ there exists a monotone non-decreasing and uniformly bounded function $\delta_r(t;K)$ of $t > 0$ with $\lim_{t\to+0}\delta_r(t;K) = 0$ such that*

$$t^{\frac{n}{2}(\frac{1}{n}-\frac{1}{r})}\|e^{-tA}a\|_r \le \delta_r(t;K) \tag{2.9}$$

holds for all $a \in K$ and all $t > 0$. In particular, we can take $T_* = T_*(K)$ so that (2.3) holds for all $a \in K$.

Proof. $\delta_r(t;K)$ can be given by the following definition

$$\delta_r(t;K) \equiv \sup_{a\in K}\left(\sup_{0<\tau\le t} \tau^{\frac{n}{2}(\frac{1}{n}-\frac{1}{r})}\|e^{-\tau A}a\|_r\right).$$

Indeed, since K is precompact in L^n_σ, it is bounded. Hence there is a constant $L > 0$ such that $\|a\|_n \le L$ for all $a \in K$. By (2.5) we see that the right hand side of the above definition is finite and that $\delta_r(t;K)$ is well-defined with

$$\delta_r(t;K) \le CL, \quad \forall t > 0.$$

This implies uniform boudedness. Obviously by definition, $\delta_r(t;K)$ is a monotone non-decreasing function of $t > 0$. Now, it suffices to show that

$$\lim_{t\to+0}\delta_r(t;K) = 0.$$

Let $U_\varepsilon(a) = \{b \in L^n_\sigma; \|b-a\|_n < \varepsilon\}$. For any $\varepsilon > 0$, there holds $\bar{K} \subset \bigcup_{a\in\bar{K}} U_\varepsilon(a)$. Since $\bar{K}$ is compact, we can select finitely many points $a_1(\varepsilon), a_2(\varepsilon), \cdots, a_m(\varepsilon) \in \bar{K}$ such that $\bar{K} \subset \bigcup_{j=1}^m U_\varepsilon(a_j(\varepsilon))$. Since $C^\infty_{0,\sigma}$ is dense in L^n_σ, we may assume that $a_j(\varepsilon) \in C^\infty_{0,\sigma}$ for all $1 \le j \le m$. Define $M_\varepsilon \equiv \mathrm{Max}\{\|a_1(\varepsilon)\|_r, \cdots, \|a_m(\varepsilon)\|_r\}$. For any $a \in K$ there is some $1 \le j_0 \le m$ such that $a \in U_\varepsilon(a_{j_0}(\varepsilon))$. For such j_0 we have in the same way as in (2.7) with the aid of (2.5)

$$\begin{aligned}
\tau^{\frac{n}{2}(\frac{1}{n}-\frac{1}{r})}\|e^{-\tau A}a\|_r &\le C\|a - a_{j_0}(\varepsilon)\|_n + \tau^{\frac{n}{2}(\frac{1}{n}-\frac{1}{r})}\|e^{-\tau A}a_{j_0}(\varepsilon)\|_r \\
&\le C\varepsilon + C\tau^{\frac{n}{2}(\frac{1}{n}-\frac{1}{r})}\|a_{j_0}(\varepsilon)\|_r \\
&\le C\varepsilon + CM_\varepsilon t^{\frac{n}{2}(\frac{1}{n}-\frac{1}{r})}
\end{aligned}$$

for all $0 < \tau \leq t$. Taking the supremum of the above estimate for $\tau \in (0, t]$ and $a \in K$, we obtain

$$\delta_r(t; K) \leq C\varepsilon + CM_\varepsilon t^{\frac{n}{2}(\frac{1}{n}-\frac{1}{r})}.$$

Letting $t \to +0$ in both sides of the above, we have $\limsup_{t\to+0} \delta_r(t; K) \leq C\varepsilon$. Since $\varepsilon > 0$ is arbitrary, this implies that

$$\lim_{t\to+0} \delta_r(t; K) = 0.$$

□

Proposition 2.4 has two applications. One is refinement of the classical theorem on uniqueness of strong solutions, and another is simplification of the proof of regularity criterion on weak solutions in $C([0,T); L^n)$. Although both of them are relatively well known for the experts of the Navier-Stokes equations, we give here a sketch of proofs. In particular, we should notice that our investigation is closely related to the question on regularity given by Remark 1.4 (iii).

First, we consider uniqueness of strong solutions in Theorem 2.3. In the classical result of Fujita-Kato [4] and Kato [9], they imposed the restriction (2.4) on the behaviour near $t = 0$ of $\|u(t)\|_r$ for $n < r < \infty$. Later on, Brezis [1] showed that (2.4) is redundant by proving that every strong solution u of (N-S) necessarily fulfills (2.4).

By Duhamel's principle, (2.2) can be reduced to the following integral equation.

$$u(t) = e^{-tA}a - \int_0^t e^{-(t-\tau)A}P(u\cdot\nabla u)(\tau)d\tau, \quad 0 < t < T. \tag{2.10}$$

The classical result on existence uniqueness reads as follows.

Theorem 2.5 (**Fujita-Kato** [4], **Kato** [9]) *Let $a \in L^n_\sigma$ and let $n < r < \infty$.*

(i) *If a and T_* satisfy (2.3), then we can construct a solution $u(t)$ of (2.10) on $[0, T_*)$ in the class $C([0,T_*); L^n_\sigma) \cap C((0,T_*); L^r)$ with the property (2.4).*

(ii) *Suppose that u is a solution of (2.10) in $C([0,T); L^n_\sigma) \cap C((0,T); L^r)$. If u satisfies (2.4), then u is the only solution of (2.10) .*

To show that (2.4) is redundant for uniqueness, we need

Proposition 2.6 *Let K be a precompact set in L^n_σ and let $n < r < \infty$. Suppose that $\delta_r(t; K)$ is the same function of $t > 0$ as in Proposition 2.4. Then there exists $T_* > 0$ such that for every $a \in K$ we can construct a solution $u(t)$ of (2.10) on $[0, T_*)$ in the class $C([0,T_*); L^n_\sigma) \cap C((0,T_*); L^r)$. Moreover, such a solution satisfies*

$$t^{\frac{n}{2}(\frac{1}{n}-\frac{1}{r})}\|u(t)\|_r \leq 2\delta_r(t; K) \quad \text{for all } 0 < t < T_*. \tag{2.11}$$

In particular, u fulfills (2.4).

Remark 2.7 This proposition asserts that the time-interval T_* of existence of solutions to (2.10) can be taken *uniformly* on each precompact subset K of the initial data in L^n_σ.

Proof of Proposition 2.6. Since $\lim_{t\to+0}\delta_r(t;K)=0$, we can choose $T_*>0$ so that $\delta_r(T_*;K)<\gamma$, where γ is the same constant as in (2.3). Since $\delta_r(t;K)$ is a monotone non-decreasing of t, we have by (2.9) that

$$\sup_{0<t<T_*} t^{\frac{n}{2}(\frac{1}{n}-\frac{1}{r})}\|e^{-tA}a\|_r<\gamma \quad \text{for all } a\in K.$$

Then it follows from Theorem 2.5 (i) that for every $a\in K$ there is a solution $u(t)$ of (2.10) on $[0,T_*)$ in the class $C([0,T_*);L^n_\sigma)\cap C((0,T_*);L^r)$ with the property (2.4). Let us define $M(t)$ by

$$M(t)\equiv\sup_{0<\tau\le t}\tau^{\frac{n}{2}(\frac{1}{n}-\frac{1}{r})}\|u(\tau)\|_r.$$

By (2.4), we see that $M\in C([0,T_*))$. Then by (2.5) and (2.10) there holds

$$\begin{aligned}\|u(t)\|_r &\le \|e^{-tA}a\|_r+\int_0^t\|P\nabla\cdot e^{-(t-\tau)A}(u\otimes u)(\tau)\|_r d\tau\\ &\le \|e^{-tA}a\|_r+C\int_0^t(t-\tau)^{-\frac{n}{2r}-\frac{1}{2}}\|u(\tau)\|_r^2d\tau\\ &\le \|e^{-tA}a\|_r+CM(t)^2\int_0^t(t-\tau)^{-\frac{n}{2r}-\frac{1}{2}}\tau^{\frac{n}{r}-1}d\tau\\ &\le \|e^{-tA}a\|_r+C\beta M(t)^2t^{-\frac{n}{2}(\frac{1}{n}-\frac{1}{r})}, \qquad 0<t<T_*,\end{aligned}$$

where $\beta=B(1/2-n/2r,n/r)$, $C=C(n,r)$. Applying Proposition 2.4 to the above estimate, we have

$$t^{\frac{n}{2}(\frac{1}{n}-\frac{1}{r})}\|u(t)\|_r\le\delta_r(t;K)+C\beta M(t)^2, \quad 0<t<T_*.$$

Since both $\delta_r(t;K)$ and $M(t)$ are non-decreasing functions of $t>0$, this implies

(2.12) $$M(t)\le\delta_r(t;K)+C\beta M(t)^2, \quad 0<t<T_*.$$

Since $\lim\limits_{t\to+0}\delta_r(t;K)=0$, we may assume T_* satisfies also

$$\delta_r(T_*;K)<\frac{1}{4C\beta}.$$

Hence by (2.12), there holds

(2.13) $$M(t)\le\frac{1-\sqrt{1-4C\beta\delta_r(t;K)}}{2C\beta}\ (\le 2\delta_r(t;K))$$

or

(2.14) $$M(t)\ge\frac{1+\sqrt{1-4C\beta\delta_r(t;K)}}{2C\beta}\left(\ge\frac{1}{2C\beta}\right)$$

for all $0<t<T_*$. Since $M(t)$ is continuous on $[0,T_*)$ with $\lim_{t\to+0}M(t)=0$ (see (2.4)), the latter case (2.14) cannot occur. Hence we obtain from (2.13)

$$M(t)\le 2\delta_r(t;K), \quad 0<\forall t<T_*$$

This proves Proposition 2.6.

Because of Theorem 2.5 (ii), to prove assertion on uniqueness in Theorem 2.3, we may show the following lemma.

Lemma 2.8 **(Brezis [1])** *Let $a \in L^n_\sigma$ and let $n < r < \infty$. Every solution u of (2.10) in the class $C([0,T); L^n_\sigma) \cap C((0,T); L^r)$ fulfills (2.4).*

Proof. We first define K as

$$K \equiv \{u(t); 0 < t < T/2\}.$$

Since $u \in C([0,T); L^n_\sigma)$, K is a precompact subset of L^n_σ. For this K, we take the function $\delta_r(t; K)$ given by Proposition 2.4. Furthermore, by Proposition 2.6 we can take $T_* > 0$ and a solution $\tilde{u}(t)$ of (2.10) on $(0, T_*)$ for every initial data $\tilde{a} \in K$. Let us denote this $\tilde{u}(t)$ by

$$\tilde{u}(t) \equiv S(t)\tilde{a}, \quad 0 < t < T_*$$

By (2.11), there holds

$$t^{\frac{n}{2}(\frac{1}{n}-\frac{1}{r})}\|S(t)\tilde{a}\|_r \leq 2\delta_r(t; K), \quad 0 < t < T_* \tag{2.15}$$

for all $\tilde{a} \in K$. Let us take s arbitrarily as $0 < s < \mathrm{Min.}\{T/2, T_*\}$. Then we have $u(s) \in K$. Since $u \in C((0,T); L^r)$, we see $\lim_{t \to +0} t^{\frac{n}{2}(\frac{1}{n}-\frac{1}{r})}\|u(t+s)\|_r = 0$. Hence it follows from Theorem 2.5 (ii) and definition of the map $S(t)$ that

$$u(t+s) = S(t)u(s), \quad 0 < t < T_*.$$

From (2.15) we obtain

$$t^{\frac{n}{2}(\frac{1}{n}-\frac{1}{r})}\|u(t+s)\|_r = t^{\frac{n}{2}(\frac{1}{n}-\frac{1}{r})}\|S(t)u(s)\|_r \leq 2\delta_r(t; K), \quad 0 < t \leq T_*.$$

Since $u \in C((0,T); L^r)$, by letting $s \to 0$ in the above estimate we have

$$t^{\frac{n}{2}(\frac{1}{n}-\frac{1}{r})}\|u(t)\|_r \leq 2\delta_r(t; K), \quad 0 < \forall t < T_*.$$

Since $\lim_{t \to +0} \delta_r(t; K) = 0$, this yields

$$\lim_{t \to +0} t^{\frac{n}{2}(\frac{1}{n}-\frac{1}{r})}\|u(t)\|_r = 0.$$

□

We shall next apply Proposition 2.4 to the proof of regularity of weak solutions in $C([0,T); L^n)$.

Theorem 2.9 **(Giga [5], von Wahl [20])** *Let $a \in L^2_\sigma$. Every weak solution u of (N-S) in $C([0,T); L^n)$ is regular as in (1.2).*

Proof. Let us define the set K by

$$K = \{u(t); 0 < t < T\}.$$

Since $u \in C([0,T); L^n)$ with div $u = 0$, K is a precompact subset of L^n_σ. We take some $n < r < \infty$. Then it follows from Proposition 2.4 that there exists $T_* = T_*(K,r)$ such that

$$\sup_{0<t<T_*} t^{\frac{n}{2}(\frac{1}{n}-\frac{1}{r})}\|e^{-tA}a\|_r \leq \delta_r(T_*; K) < \gamma, \tag{2.16}$$

for all $a \in K$ where γ is the same constant as in (2.3). Let $\rho \equiv T_*/2$. For every $t_* \in (0,T)$ we have by (2.16) that

$$\sup_{0<t<T_*} t^{\frac{n}{2}(\frac{1}{n}-\frac{1}{r})}\|e^{-tA}u(t_* - \rho)\|_r < \gamma.$$

By Theorem 2.3 and Remark 2.2, there exists a strong solution v of (N-S) with $v|_{t=t_*-\rho} = u(t_* - \rho)$ such that

$$v \in C([t_* - \rho, t_* + \rho); L^n), \quad \frac{\partial v}{\partial t}, \frac{\partial^{\alpha_1+\cdots+\alpha_n} v}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}} \in C(\mathbb{R}^n \times (t_* - \rho, t_* + \rho)) \tag{2.17}$$

where $\alpha = (\alpha_1, \cdots, \alpha_n)$ is an arbitrary multi-index with $|\alpha| = \alpha_1 + \cdots + \alpha_n \leq 2$. Notice that v is also a weak solution. Then uniqueness result of Theorem 1.3 (i) yields

$$u(t) \equiv v(t) \quad \text{for } t \in [t_* - \rho, t_* + \rho).$$

Since $t_* \in (0,T)$ can be taken arbitrarily, we can conclude that u is regular as in (1.2). $\square$

To deal with the problem on regularity of weak solutions in $L^\infty(0,T; L^n)$, the above proof proposes us the following question.

Question. For every weak solution u in $L^\infty(0,T; L^n)$ is the set

$$K = \{u(t); 0 < t < T\}$$

precompact in L^n_σ ?

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