ON AN ABSTRACT RADIATION CONDITION

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INTRODUCTION

We shall present an abstract radiation condition in terms of the Mourre theory of conjugate operator method.

Let $\mathcal{H}$ be a Hilbert space and $A$ be a self-adjoint operator in $\mathcal{H}$. For $s \geq 0$ consider the Hilbert space $A^s = D((A^*)^s)$ with the graph norm, and if $s < 0$, $A^s = (A^{-s})^*$. Then, if $s \geq 0$, $A^s \subseteq \mathcal{H} \subseteq A^{-s}$ continuously and densely, and the scalar product of $\mathcal{H}$ extends to a natural duality $(\cdot, \cdot)_{s,-s} : A^s \times A^{-s} \to \mathbb{C}$ for all $s \in \mathbb{R}$. We denote by $P_\pm$ the spectral projectors of $A$ associated to the half-lines $[0, +\infty)$ and $(-\infty, 0]$, respectively.

We recall now some (Besov) spaces of operators (see [ABG]). Let $S$ be a bounded operator on $\mathcal{H}$. We say that $S \in C_k^s(A)$, $k$ positive integer, if the application $\mathbb{R} \ni \tau \to \mathcal{W}(\tau)[S] = e^{i\tau A} S e^{-i\tau A} \in \mathcal{B}(\mathcal{H})$ is strongly $C^k$; in this case $\text{ad}^s S$ can be extended as a bounded operator on $\mathcal{H}$. Consider $\theta \in (0,1], p \in [1, \infty]$; we say that $S \in C^\theta,p(A)$ if $(\tau \to (\mathcal{W}(\tau) - I)^m[S]||/|\tau|^{\theta + 1/p}) \in L^p((0, \infty))$, where $m = 1$ if $\theta < 1$, and $m = 2$ if $\theta = 1$. (If $p = \infty$, this condition should be read as $\sup_{\tau > 0} ||(\mathcal{W}(\tau) - I)^m[S]||/|\tau|^\theta < \infty$.) For general $\theta > 0$, we say that $S \in C^\theta,p(A)$ if $S \in C^l(A)$ and $\text{ad}^s S \in C^{\theta-l,p}(A)$, where $l$ is the largest integer $l < \theta$.

Let $L$ be a self-adjoint operator in $\mathcal{H}$. Then $L \in C^\theta,p(A)$ (or $C^k(A)$) if $(L - z)^{-1} \in C^\theta,p(A)$ (or $C^k(A)$) for some (and hence all) $z \in \mathbb{C} \setminus \sigma(L)$.

If $L$ is a self-adjoint operator of class $C^1(A)$, then the commutator $i[L,A]$ is defined as a continuous form on the domain of $L$. Then one can define the strict Mourre set $\mu^A(L)$ of $L$ with respect to $A$ as the set of $\lambda \in \mathbb{R}$ with the property that there exists $J = (\lambda - \delta, \lambda + \delta) \neq \emptyset$ and $d > 0$ such that

$$E_L(J) i[L,A] E_L(J) \geq d E_L(J).$$

We recall that if $L$ has a spectral gap and $L \in C^{1,1}(A)$, then there exist $R_L(\lambda \pm i0) = \lim_{\epsilon \to 0} (L - \lambda \mp i\epsilon)^{-1}$ uniformly in $\mathcal{B}(A^s, A^{-s})$, whenever $s > 1/2$.

The following theorem was given in [BGS1] (for the proof see [BGS2]; see also [J] for some earlier results).

THEOREM 1. Let $s > 1/2$ be a real number and $L$ be a self-adjoint operator with a spectral gap and of class $C^{s+1/2,1}(A)$. Then we have $P_\mp R_L(\lambda \pm i0)A^s \subseteq A^{s-1}$ for each $\lambda \in \mu^A(L)$.

It turns out that in some stronger hypotheses this condition characterizes $R_L(\lambda \pm i0)$. Namely, we prove the following theorem, extending some results of [B2], [M].

THEOREM 2. Let $1 \geq \theta > 1/2$ be a real number, $L \geq -M$ be a bounded from below self-adjoint operator of class $C^{1+\theta,\infty}(A)$ such that $i[L,A] \in \mathcal{B}(G,G^*)$, where $G$ is the form domain of $L$, and $\lambda \in \mu^A(L)$. Suppose $u \in A^s$, $s \in (1/2, \theta)$ satisfies:

a) $(u,(L - \lambda)\varphi)_{s,-s} = 0$ for all $\varphi \in (L + M)^{-1}A^s$,

b) there exists $\alpha < \theta/2$ such that $\langle A^{-\alpha}P_-(A)u \in \mathcal{H}$ (or $\langle A^{-\alpha}P_+(A)u \in \mathcal{H}$).

Then $u = 0$. 
The proof follows Isozaki's proof of some type of radiation conditions which are strongly related to those presented here. (See [I1], [I2], [I3].) We only remark here that Theorem 2 provides some useful results in the study of the layered media.

One of the tools needed here is the functional calculus using almost analytic extensions of symbols. Let \( m \in \mathbb{R} \). We denote by \( S^m \) the set of symbols \( f \in C^\infty(\mathbb{R}) \) that satisfy

\[
p_k(f) = \sup_{x \in \mathbb{R}} \langle x \rangle^{m-k} |f^{(k)}(x)| < \infty.
\]

Then \( S^m \), endowed with the seminorms \( p_k \) is a Fréchet space. The following result can be found in [B2], [M] (see also [DG] for the main idea).

**PROPOSITION 3.** Consider a bounded family of symbols \( \{f_\epsilon\} \subset S^m \). Then there exists a family of functions (the almost analytic extensions) \( \{\tilde{f}_\epsilon\} \subset C^\infty(\mathbb{C}) \) such that:

1. \(|Imz| \leq |Rez|\) on \( \text{supp} \tilde{f}_\epsilon \),
2. \( |\overline{\partial}\tilde{f}_\epsilon(z)| \leq C_N(z)^{m-N-1}|Imz|^{N-1} \) for all \( N \geq 0 \) and all \( z \in \mathbb{C} \), where the constants \( C_N \) do not depend on \( z \) and \( \epsilon \).

This construction provides an useful representation for the functional calculus of a self-adjoint operator, due to Helffer- Sjöstrand ([HS]): Let \( A \) be a self-adjoint operator on \( \mathcal{H} \) and \( f \in S^{-\delta}, \delta > 0 \). Then

\[
f(A) = \frac{1}{\pi} \int_{\mathbb{C}} \overline{\partial}\tilde{f}(z)(A - z)^{-1} \, dx \, dy,
\]

where \( z = x + iy \) and \( \tilde{f} \) is an almost analytic extension of \( f \). If \( B \) is a bounded operator with \( ad_A^n \) is a bounded form on the domain of \( A \), and \( ad_A k f^{(k)}(A) \) (respectively \( f^{(k)}(A) ad_A k \)) \( k = 1, \ldots, n - 1 \), are bounded operators, then

\[
[B, f(A)] = \sum_{k=1}^{n-1} \frac{(-1)^{k+1}}{k!} ad_A^k(B) f^{(k)}(A) + R_0^n(A, B)
\]

where

\[
R_0^n(A, B) = \frac{1}{\pi} \int_{\mathbb{C}} \overline{\partial}\tilde{f}(z)(A - z)^{-1} ad_A^n(B)(A - z)^{-n} \, dx \, dy,
\]

\[
R_0^n(A, B) = \frac{(-1)^n}{\pi} \int_{\mathbb{C}} \overline{\partial}\tilde{f}(z)(A - z)^{-n} ad_A^n(B)(A - z)^{-1} \, dx \, dy.
\]

For a proof, see for instance [M].

1. **COMMUTATORS**

**LEMMA 1.1.** Let \( B \in C^{\theta,\infty}(A) \), \( 0 < \theta < 1 \), be a bounded operator and \( \alpha_1, \alpha_2 \) positive numbers such that \( 0 < \alpha_1 + \alpha_2 < \theta \). Then

\[
\|(A)^{\alpha_1}[B, (A - z)^{-1}](A)^{\alpha_2}\| \leq C(|Imz|^{-\theta-1} + |Imz|^{-1} + \langle z \rangle |Imz|^{-2} + \langle z \rangle^2 |Imz|^{-3})
\]
whenever $|Imz| \neq 0$.

Proof. Consider $0 < \alpha < \theta$. We consider first the operator $\langle A \rangle^\alpha [B, (A - z)^{-1}]$. Suppose $Imz > 0$; the case $Imz < 0$ is similar.

(i) We have (weakly)

$$[B, (A - z)^{-1}] = \int_{-\infty}^{0} e^{\mu t} [B, e^{i\lambda t} A] e^{itA} dt,$$

where $z = \lambda + i\mu$. Using that $B \in C^{\theta, \infty}(A)$, we get

$$||[B, (A - z)^{-1}]|| \leq \int_{-\infty}^{0} e^{\mu t} |t|^\theta dt \leq C \mu^{-\theta-1} \int_{-\infty}^{0} e^{t} |t|^\theta dt,$$

hence

$$\tag{1.2} \quad \quad ||[B, (A - z)^{-1}]|| \leq C \mu^{-\theta-1}.$$

(ii) Denote $\nu(\lambda) = \langle \lambda \rangle^\alpha$. Helffer-Sjöstrand formula gives (first as bounded operators between $A^\alpha$ and $A^{-\alpha}$)

$$\tag{1.3} [B, \langle A \rangle^\alpha] = \frac{1}{\pi} \int_{C} \overline{\nu}(z) [B, (A - z)^{-1}] dx dy.$$

The norm of the integrand in (3) can be bounded by

$$\|\overline{\nu}(z) [B, (A - z)^{-1}]\| \leq C(z)^{\alpha-1-N} |Imz|^{N-\theta-1}.$$

If one takes $N = \theta + 1$ to avoid the singularities, we get

$$\|\overline{\nu}(z) [B, (A - z)^{-1}]\| \leq C(z)^{\alpha-2-\theta},$$

which is integrable if $\alpha < \theta$. Hence

$$[B, \langle A \rangle^\alpha] \in \mathcal{B}(\mathcal{H}).$$

(iii) We can write then

$$\tag{1.4} \quad \quad \langle A \rangle^\alpha [B, (A - z)^{-1}] = [B, \langle A \rangle^\alpha (A - z)^{-1}] - [(A)^\alpha, B](A - z)^{-1}.$$

The norm of the second hand in the rhs of (4) is bounded by $C|Imz|^{-1}$.

(iv) We estimate now the first term in the rhs of (4). Let $g$ be a smooth function on $\mathbb{R}$, $g(t) = 1$ if $|t| \geq 1$ and $g(t) = 0$ if $|t| < 1/2$. Then

$$\tag{1.5} \quad \quad [B, \langle A \rangle^\alpha (A - z)^{-1}] = [B, g(A)\langle A \rangle^\alpha (A - z)^{-1}] + [B, (1 - g(A))\langle A \rangle^\alpha (A - z)^{-1}].$$

The second term of the rhs of (1.5) equals

$$[B, \langle A \rangle^\alpha (A - z)^{-1} + [B, (A - z)^{-1}] (1 - g(A))\langle A \rangle^\alpha,$$
and has the norm less than (using (2))

\[(1.6)\quad C(|\text{Im}z|^{-1} + |\text{Im}z|^{\theta+1}).\]

We denote \(g_z(\lambda) = g(\lambda)(\lambda)^{\alpha}(\lambda - z)^{-1}\). We shall use the following form of the Helffer–Sjöstrand form (see [BGS2], section 4):

\[(1.7)\quad [B, g_z(A)] = \frac{1}{\pi} \int_{\mathbb{R}} \left( (g_z(\lambda) - \lambda g_z'(\lambda))[B, \text{Im} R_A(\lambda + i\lambda)] - \partial_\lambda(\lambda g_z(\lambda))[B, \text{Im}i R_A(\lambda + i\lambda)] \right) d\lambda \]

\[- \frac{1}{\pi} \int_{\mathbb{R}} \int_0^\lambda g_z^{(2)}(\lambda)) [B, \text{Im} R_A(\lambda + i\mu)] \mu d\mu d\lambda.\]

The norm of the integrand in the first term of (1.7) can be estimated by (using (2) and on \(\text{supp} g\))

\[C\left(\frac{\langle \lambda \rangle^\alpha}{|\lambda-z|} + \frac{\langle \lambda \rangle^{\alpha+1}}{|\lambda-z|^2}\right) \langle \lambda \rangle^{-\theta-1} \leq C \langle \lambda \rangle^{-\theta-1} (|\text{Im}z|^{-1} + \langle z \rangle |\text{Im}z|^{-2}).\]

Hence the first integral in (7) can be bounded as follows

\[(1.8) \quad \| \int_{\mathbb{R}} (g_z(\lambda) - \lambda g_z'(\lambda) + 2(i+1)^{-1}\partial_\lambda(\lambda g_z(\lambda))[B, R_A(\lambda + i\lambda)] d\lambda \| \leq C(|\text{Im}z|^{-1} + \langle z \rangle |\text{Im}z|^{-2}).\]

To estimate the second integral we note first that

\[(1.9) \quad \| \int_0^\lambda g_z^{(2)}(\lambda)) [B, R_A(\lambda + i\mu)] \mu d\mu \| \leq C\langle \lambda \rangle^{1-\theta}\]

on \(\text{supp} g\). Then

\[\|g_z^{(2)}(\lambda) \int_{0}^{\lambda} g_z^{(2)}(\lambda)) [B, R_A(\lambda + i\mu)] \mu d\mu d\lambda \| \leq C\langle \lambda \rangle^{1-\theta} \left( \frac{\langle \lambda \rangle^\alpha}{|\lambda-z|^{\theta}} + \frac{\langle \lambda \rangle^{\alpha-1}}{|\lambda-z|^2} + \frac{\langle \lambda \rangle^{\alpha-2}}{|\lambda-z|} \right)\]

\[\leq C\langle \lambda \rangle^{1+\alpha-\theta} (|\text{Im}z|^{-1} + \langle z \rangle |\text{Im}z|^{-2} + \langle z \rangle^2 |\text{Im}z|^{-3})\]

Summing up:

\[\|[B, g_z(A)]\| \leq C((z)^2|\text{Im}z|^{-3}) + \langle z \rangle |\text{Im}z|^{-2} + \lambda^{\alpha+\theta} |\text{Im}z|^{-1}).\]

Then one gets

\[(1.11) \quad ||A|^\alpha [B, (A - z)^{-1}]|| \leq C((z)^2|\text{Im}z|^{-3}) + \langle z \rangle |\text{Im}z|^{-2} + \lambda^{1+\alpha-\theta} |\text{Im}z|^{-1} + |\text{Im}z|^{-\theta-1}).\]

In the same way

\[(1.12) \quad \|[B, (A - z)^{-1} A]^\alpha\| \leq C((z)^2|\text{Im}z|^{-3}) + \langle z \rangle |\text{Im}z|^{-2} + \lambda^{1+\alpha-\theta} |\text{Im}z|^{-1} + |\text{Im}z|^{-\theta-1}).\]

The general result follows by interpolation.
1.2. Let \( \{ \chi_t \} \in S^a \), \( a < 1 \) be a bounded family of symbols, and \( B \in C^{1+\theta, \infty}(A) \) a bounded operator. Then

\[
i[B, \chi_t(A)] = i[B, A] \chi'_t(A) + R_{1,t},
i[B, \chi_t(A)] = \chi'_t(A)i[B, A] + R_{2,t},
\]

where

\[
\langle A \rangle^{\alpha_1} R_{1,t}(A)^{\alpha_2} \in \mathcal{B}(\mathcal{H}) \quad \text{and} \quad \| \langle A \rangle^{\alpha_1} R_{1,t}(A)^{\alpha_2} \| \leq C
\]

\[
\langle A \rangle^{\alpha_2} R_{2,t}(A)^{\alpha_1} \in \mathcal{B}(\mathcal{H}) \quad \text{and} \quad \| \langle A \rangle^{\alpha_2} R_{2,t}(A)^{\alpha_1} \| \leq C,
\]

whenever \( \alpha_1 + \alpha_2 + a < 1 + \theta \), \( \alpha_1 + \alpha_2 < 1 + \theta \), \( \alpha_1 < \theta \). Here \( C \) stands for constants not depending on \( t \).

Proof. We have \( i[B, \chi_t(A)] = i[B, A] \chi'_t(A) + R_{1,t} \) where

\[
R_{1,t} = \frac{1}{\pi} \int_{\mathcal{C}} \overline{\partial} \chi_t i[D, (A - z)^{-1}] (A - z)^{-1} \, dz \, dy,
\]

with \( D = i[B, A] \in C^{\theta, \infty}(A) \), bounded.

We take \( \delta = \theta - \alpha_1 - \epsilon \) with \( \epsilon \) sufficiently small such that \( \alpha_2 - \epsilon < 1 \) and \( a + \alpha_2 - \delta < 1 \). (This is possible by hypothesis.) Then, by Lemma 1,

\[
\| \partial \chi_t(A)^{\alpha_1} i[D, (A - z)^{-1}] (A)^{\delta} (A - z)^{-1} (A)^{\alpha_2 - \delta} \|
\leq C_N (z)^{a-1-N} |Imz|^N (|Imz|^{-\theta} + |Imz|^{-\theta-1}) (z)^{\alpha_2 - \delta} |Imz|^{-1}
\]

on \( \text{supp} \overline{\partial} \chi_t \). We take \( N = \theta + 2 \) and thus obtain that

\[
\| \partial \chi_t(A)^{\alpha_1} i[D, (A - z)^{-1}] (A - z)^{-1} (A)^{\alpha_2} \| \leq C (z)^{a-3+\alpha-\delta}
\]

which is integrable and \( C \) does not depend on \( t \). Hence \( \langle A \rangle^{\alpha_1} R_{1,t}(A)^{\alpha_2} \) extends to a bounded operator and the estimate in the statement holds. One proceed similarly to get the second assertion.

1.3. Let \( B \) be a bounded operator of class \( C^{\theta, \infty}(A) \), \( 0 < \theta \leq 1 \) and \( \alpha_1, \alpha_2 \) positive numbers such that \( \alpha_1 + \alpha_2 < \theta \). Then \( \langle A \rangle^{\alpha_1} [B, \langle A \rangle^{\alpha_2}] \) extends to a bounded operator on \( \mathcal{H} \).

Proof. Recall that in the proof of Lemma 1 we proved that \( [B, \langle A \rangle^{\delta}] \in \mathcal{B}(\mathcal{H}) \) whenever \( \delta = \alpha_1 + \alpha_2 + \epsilon < \theta \). We denote \( \delta_i = \alpha_i / \delta \), \( i = 1, 2 \) and set \( A_\delta = \langle A \rangle^{\delta} \); this is a self-adjoint operator \( A_\delta \geq 1 \). We have to control \( A_\delta^k [B, h(A_\delta)] \), where \( h \in S^{\theta_2}, h(s) = s^\theta \) if \( s \geq 1/2 \) and \( h(s) = 0 \) if \( s \leq 1/4 \). We have (first in form sense)

\[
A_\delta^k i[B, h(A_\delta)] = \frac{1}{\pi} \int_{\mathcal{C}} \overline{\partial} h(z) A_\delta^k (A_\delta - z)^{-1} i[B, A_\delta] (A_\delta - z)^{-1} \, dz \, dy.
\]

On the support of \( \overline{\partial} h \) the norm of the integrand can be estimated as

\[
\| \overline{\partial} h(z) A_\delta^k (A_\delta - z)^{-1} i[B, A_\delta] (A_\delta - z)^{-1} \| \leq C (z)^{\theta_1 + \theta_2 - 1/2}
\]

The rhs is an integrable function, since \( \theta_1 + \theta_2 < 1 \). Therefore \( A_\delta^k i[B, h(A_\delta)] \) extends to a bounded operator on \( \mathcal{H} \).
LEMMA 1.4. Let $B$ be a bounded operator of class $C^{0,\infty}(A)$, $0 < \theta \leq 1$ and $\alpha_1$, $\alpha_2$ positive numbers such that $\alpha_1 + \alpha_2 < \theta$, and $\{g_t\} \subset S^a$, $a \leq 0$, a bounded family of symbols. Then:

\[ \|\langle A \rangle^{\alpha_1}i[B, g_t(A)]\langle A \rangle^{\alpha_2}\| \leq C, \]

where $C$ does not depend on $t$.

Proof. (i) Consider first the case where $a < 0$. Then

\[ \langle A \rangle^{\alpha_1}i[B, g_t(A)]\langle A \rangle^{\alpha_2} = \frac{1}{\pi} \int_{C} \overline{\partial} \tilde{g}_t(z)\langle A \rangle^{\alpha_1}i[B, (A-z)^{-1}]\langle A \rangle^{\alpha_2} dx dy. \]

Using Lemma 1 the norm of the integrand can be majorized by $C\langle z \rangle^{-2-2\epsilon}$. (ii) If $a = 0$, let $\epsilon > 0$ be such that $\alpha_1 + \alpha_2 + \epsilon < \theta$ and write

\[ \langle A \rangle^{\alpha_1+\epsilon}\langle A \rangle^{-\epsilon}i[B, g_t(A)]\langle A \rangle^{\alpha_2} = \langle A \rangle^{\alpha_1+\epsilon}i[B, g_t(A)]\langle A \rangle^{-\epsilon}\].

We use the proof of the previous lemma to show that the first term is a bounded operator and its norm can be bounded by a constant not depending on $t$. For the second term we use (i). $\blacksquare$

2. THE PROOF OF THEOREM 2

We can suppose, without restricting the generality, that in Theorem 2 we have $M = 1$ and $\lambda = 0$.

LEMMA 2.1. If $\Phi \in C_0^\infty(\mathbb{R})$ is a real function, $\Phi = 1$ on a neighborhood of 0, then

\[ (u, \Phi(L)\varphi)_{-s,s} = (u, \varphi)_{-s,s}, \text{ for all } \varphi \in \mathcal{A}^s. \]

Proof. We have, for $\varphi \in \mathcal{A}^1$,

\[ (u, (1 - \Phi(L))\varphi)_{-s,s} = (u, L\Psi(L)\varphi)_{-s,s}, \]

where $\Psi(t) = (1-\Phi(t))t^{-1}$. Therefore, to have (2) for $\varphi \in \mathcal{A}^1$ it suffices to prove that $\Psi(L) = (L+1)^{-1}\varphi_1$ with $\varphi_1 \in \mathcal{A}^1$. We can write $(L + 1)\Psi(L) = (1 - \Phi(L)) + \Psi(L)$. Thus, since $(1\Phi(L))\varphi \in \mathcal{A}^1$, it remains to show that $\Psi(L)\varphi \in \mathcal{A}^1$ if $\varphi \in \mathcal{A}^1$. We have

\[ i[\Psi(L), A] = \frac{1}{\pi} \int_{C} \overline{\partial} \Psi i[(L-z)^{-1}, A] dx dy = \frac{1}{\pi} \int_{C} \overline{\partial} \Psi (L-z)^{-1}((L+1)^{-1/2}(L+1)^{-1/2}i[L, A](L+1)^{-1/2}((L+1)^{1/2}(L-z)^{-1} dx dy \]

The norm of the integrand can be bounded by $C\langle z \rangle^{-2-2\epsilon}|\text{Im}z|^2(\langle z \rangle^{|\text{Im}z|^2} = C\langle z \rangle^{-3}$. We get that $i[\Psi(L), A]$ is a bounded operator and we obtain easily that $\Psi(L)\varphi \in \mathcal{A}^1$ if $\varphi \in \mathcal{A}^1$. Thus equation (1) holds for $\varphi \in \mathcal{A}^1$; the general result follows by density using the fact that $\Phi(L) \in \mathcal{B}(\mathcal{A}^s)$. $\blacksquare$
Remark. In fact the previous Lemma says that $\Phi(L)u = u$ for all $\Phi \in \mathcal{C}_0^\infty(\mathbb{R})$, $\Phi = 1$ on a neighborhood of 0; this fact can be easily seen using that $\Phi(L) \in \mathcal{B}(A^t) \cap \mathcal{B}(A^{-t})$ and it is symmetric with respect to the duality $(.,.)_{\iota,-\iota}$.

**Lemma 2.2.** Let $\chi \in \mathcal{C}_0^\infty(\mathbb{R})$ such that $0 \leq \chi(s) \leq 1$, $\chi(s) = 1$ for $|s| \leq 1$, $\chi(s) = 0$ for $|s| \geq 2$. We consider the $\mathcal{C}_0^\infty(\mathbb{R})$ function

$$
x_t(y) = \int_{(y)}^\infty s^{-2\beta} \chi^2(s/t) ds,
$$

where $\beta > \max(\alpha, s/2)$, $\beta < \theta/2$. Then

$$
(\mathcal{L}\Phi^2(L)\chi_t(A)u, u)_{\iota,-\iota} = 0.
$$

**Proof.** The Lemma follows by hypothesis as $\Phi^2(L)\chi_t(A)u \in (L+1)^{-1}A^\iota$.

We shall set $T$ for different bounded operators with norm independent on $t$.

**Remark.** We have

$$2 \text{Re } i ((\mathcal{L}\Phi^2(L)\chi_t(A)u, u)_{\iota,-\iota} = 0.$$

We shall give to this relation the form and the meaning

$$i([L\Phi^2(L), \chi_t(A)]u, u)_{\iota,-\iota} = 0.$$

Set $L_1 = \Phi^2(L)L$. Then $L_1$ is a bounded operator of class $\mathcal{C}^{\theta+1,\infty}(A)$ (Thm. 6.2.5 [ABG]).

**Lemma 2.3.** We have

$$i[L_1, \chi_t(A)] = i[L_1, A]A(A^{-2\beta-1}\chi^2(\langle A \rangle/t) + \langle A \rangle^{-s}T\langle A \rangle^{-s}.$$

**Proof.** One applies Lemma 1.2 for $R_{1,t}$, $\alpha_1 = \alpha_2 = s$, $a = 1 - 2\beta$. (Here $\alpha_1 + \alpha_2 + a = 2s + 1 - 2\beta < \theta + 1$ since $\beta > \theta/2$, and $s < \theta$.)

As a direct consequence we get the next Lemma.

**Lemma 2.3'.** $\sup_{t \geq 1} |(i[L_1, A]A(A^{-2\beta-1}\chi^2(\langle A \rangle/t)u, u)_{\iota,-\iota}| < \infty$.

**Lemma 2.4.** If $\tilde{\Phi} \in \mathcal{C}_0^\infty(\mathbb{R})$, $\tilde{\Phi} = 1$ on a small enough neighborhood of 0, then

$$\sup_{t \geq 1} |\Phi(L)[i[L, A]\Phi(L)A(A^{-2\beta-1}\chi^2(\langle A \rangle/t)u, u)_{\iota,-\iota}| < \infty.$$

**Proof.** We know that $\Phi(L)u = u$. We have

$$A(A^{-2\beta-1}\chi^2(\langle A \rangle/t)\tilde{\Phi}(L) = \Phi(L)A(A^{-2\beta-1}\chi^2(\langle A \rangle/t) + [\Phi(L), A(A^{-2\beta-1}\chi^2(\langle A \rangle/t)].$$

Set $g_t(A) = A(A^{-2\beta-1}\chi^2(\langle A \rangle/t)$. Here $\{g_t\} \in S_{-2\beta}$ is a bounded family of symbols. One applies Lemma 1.2 to get

$$[\Phi(L), g_t(A)] = [\Phi(L), A]\Phi(L)\Phi_t(A) + R_{1,t}.$$


In this case $\alpha_1 = \alpha_2 = s$, $a = -\beta$. Thus, since $2\beta + 1 > s + 1 > 2s$,

\[(2.5) \quad A(A)^{-2\beta-1}\chi^2(\langle A \rangle/t)\tilde{\Phi}(L) = \tilde{\Phi}(L)A(A)^{-2\beta-1}\chi^2(\langle A \rangle/t) + \langle A \rangle^{-s}T\langle A \rangle^{-s}.
\]

Hence

\[(2.6) \quad (i[L_1, A]A(A)^{-2\beta-1}\chi^2(\langle A \rangle/t)\tilde{\Phi}(L)u, \tilde{\Phi}(L)u)_{s,-s} = (i[L_1, A]\tilde{\Phi}(L)A(A)^{-2\beta-1}\chi^2(\langle A \rangle/t)u, \tilde{\Phi}(L)u)_{s,-s} + (i[L_1, A]\langle A \rangle^{-s}T\langle A \rangle^{-s}u.u)_{s,-s}.
\]

Since $i[L_1, A] \in B(\mathcal{H}) \cap \mathcal{C}^{\theta,\infty}(A)$ (Prop. 5.2.2 [ABG]), $[L_1, A]$ is a bounded operator on $A^*$, $s < \theta$ (Thm 5.3.3, Lemma 5.3.2 [ABG]). Therefore the second term in (2.6) is bounded by a constant independent on $t$. But, in form sense,

\[
\tilde{\Phi}(L)i[L_1, A]\tilde{\Phi}(L) = \tilde{\Phi}(L)\Phi(L)[L, A]\Phi(L) + \tilde{\Phi}(L)i[A, 1 - \Phi(L)]\tilde{\Phi}(L) + \tilde{\Phi}(L)i[A, 1 - \Phi(L)]L\tilde{\Phi}(L).
\]

We take supp$\tilde{\Phi}$ to be in the set where $\Phi = 1$; then

\[
\tilde{\Phi}(L)i[A, 1 - \Phi(L)]\tilde{\Phi}(L) = 0 \quad \text{on} \quad A^1 \times A^1.
\]

Therefore the bounded operator given by this form on $\mathcal{H}$ ($L \in C^1(A)$) is zero. Similarly we get that $\tilde{\Phi}(L)i[A, 1 - \Phi(L)]L\tilde{\Phi}(L) = 0$. Summing up

\[
\tilde{\Phi}(L)i[L_1, A]\tilde{\Phi}(L) = \tilde{\Phi}(L)i[L, A]\tilde{\Phi}(L).
\]

The lemma follows by (2.6), (2.5) and the previous relation. \[\square\]

We can denote $\tilde{\Phi}$ also by $\Phi$.

**Lemma 2.5.** $\sup_{t \geq 1} |(\Phi(L)i[L, A]\Phi(L)A(A)^{-\beta}\chi(\langle A \rangle/t)u, \langle A \rangle^{-\beta}\chi(\langle A \rangle/t)u)| \leq \infty$.

**Proof.** Set $B = \Phi(L)i[L, A]\Phi(L)$. Then $B$ is a bounded operator of class $\mathcal{C}^{\theta,\infty}(A)$. Denote $f_t(x) = \langle x \rangle^{-\beta}(\langle x \rangle/t)$, $x \in \mathbb{R}$. We take $\beta_0 < \beta$, but still $\beta_0 > s/2$, $\beta_0 < \theta/2$. We write

\[
\langle A \rangle^{-s}\beta_0[B, f_t(A)]\langle A \rangle^{-\beta} = \langle A \rangle^{-s}\beta_0[B, f_t(A)]\langle A \rangle^{-\beta} - \langle A \rangle^{-s}\beta_0[B, \langle A \rangle^{\beta_0}]\langle A \rangle^{-s-\beta}(\langle A \rangle/t).
\]

But $2s - 2\beta_0 < 2s - \theta < \theta$, so the first term is a bounded operator and its norm does not depend on $t$ (Lemma 1.4). The second term is bounded since $s - 2\beta < 0$ and $\langle A \rangle^{-s-\beta}[B, \langle A \rangle^{\beta_0}]$ is bounded by Lemma 1.3. Now the lemma follows easily. \[\square\]

**Lemma 2.6.** For all $\beta > \alpha$ we have $\langle A \rangle^{-\beta}u \in \mathcal{H}$.

**Proof.** (i) Consider first $\beta > \max(\alpha, s/2)$, $\beta < \theta/2$. Let $F_+$ be a smooth bounded real function, $F_+ = 1$ on $[1, \infty)$, $F_+ = 0$ on $(\infty, 1/2]$. We shall show first that

\[(2.7) \quad \sup_{t \geq 1} |(\Phi(L)i[L, A]\Phi(L)A(A)^{-\beta-1}F_+(A)\chi(\langle A \rangle/t))u, \langle A \rangle^{-\beta}F_+(A)\chi(\langle A \rangle/t)u)| < \infty.
\]

We use again the notation $B = \Phi(L)i[L, A]\Phi(L)$. If $F_- = 1 - F_+$ then

\[
(\mathcal{B}A(A)^{-\beta-1}\chi(\langle A \rangle/t)(A)u, \langle A \rangle^{-\beta}\chi(\langle A \rangle/t)u) = (\mathcal{B}A(A)^{-\beta-1}\chi(\langle A \rangle/t)(F_+ + F_-)(A)u, \langle A \rangle^{-\beta}(F_+ + F_-)(A)\chi(\langle A \rangle/t)u).
\]
Here \( (A)^{-\beta} u \in \mathcal{H} \). Moreover, by Thm 3.10 [BGS2], the fact that \( B \) is of class \( C^{s,2} \) for all \( s < \theta \) ensure that \( F_+(A)B F_-(A) \in \mathcal{B}(A^{\beta-2}, A^{\beta-4}) \). Hence \( (A)^{-\beta} F_+(A)B F_-(A) (A)^{\theta} = T \in \mathcal{B}(\mathcal{H}) \), and this gives

\[
(BA(A)^{-\beta-1} \chi((A)/t) F_+(A)u, (A)^{-\beta} \chi((A)/t) F_-(A)) \\
= (T(A)^{-\beta-t}(A)^{-\beta-1} \chi((A)/t) Au, (A)^{-2\beta} \chi((A)/t) F_-(A)u).
\]

Therefore

\[(2.8) \sup_{t \geq 1} |(BA(A)^{-\beta-1} \chi((A)/t) F_+(A)u, (A)^{-\beta} \chi((A)/t) F_-(A))| < \infty.\]

Similarly one gets

\[(2.9) \sup_{t \geq 1} |(BA(A)^{-\beta-1} \chi((A)/t) F_-(A)u, (A)^{-\beta} \chi((A)/t) F_+(A))| < \infty.\]

Now (2.7) follows by (2.8), (2.9) and the previous lemma.

We can write \( A^{1-s} F_+(A) = g^2(A) F_+(A) \) with \( g \in S^0 \). But \( (A)^{-s-\beta}[B, g(A)](A)^{1-\beta} \) is bounded by Lemma 1.4 \((2s - 2s \beta < 2s \theta < 2\theta - \theta = \theta)\). Hence

\[
\sup_{t \geq 1} |(B(A)^{-\beta} \chi((A)/t) g(A) F_+(A)u, (A)^{-\beta} \chi((A)/t) g(A) F_+(A))| < \infty.
\]

Using now the Mourre estimate we get

\[
\sup_{t \geq 1} \| \Phi(L)(A)^{-\beta} \chi((A)/t) g(A) F_+(A)u \| \leq \infty.
\]

As \( [\Phi(L), (A)^{-\beta} \chi((A)/t) g(A) F_+(A)](A)^s \) is a bounded operator with norm independent on \( t \) (by Lemma 1.4) it follows

\[
\sup_{t \geq 1} \| (A)^{-\beta} \chi((A)/t) g(A) F_+(A)u \| \leq \infty.
\]

This provide, using Beppo-Levi Theorem,

\[(2.10) \langle A \rangle^{-\beta} g(A) F_+(A)u \in \mathcal{H}.\]

If we take \( \tilde{F}_+ \) to be a smooth bounded real function on \( \mathbb{R} \), \( \tilde{F}_+ = 1 \) on \([2, \infty)\) and \( \text{supp} \tilde{F}_+ \subset [1, \infty) \), we can write

\[
\tilde{F}_+(A)(A)^{-\beta} = (\tilde{F}_+/gF_+)(A)(gF_+)(A)(A)^{-\beta},
\]

and \( (\tilde{F}_+/gF_+)(A)(gF_+)(A) \) is a bounded operator. Then (2.10) gives that \( \tilde{F}_+(A)(A)^{-\beta} u \in \mathcal{H} \). Thus the lemma follows in this case since \( (1 - \tilde{F}_+(A)(A)^{-\beta} u \in \mathcal{H} \) by hypothesis (b) of Thm 2.

(ii) Now we can repeat the argument with \( s \) replaced by \( 2\alpha \) and see that \( (A)^{-\beta} u \in \mathcal{H} \) for all \( \beta < \theta/2, \beta > \alpha \). □

**Lemma 2.7. In the conditions of Thm. 2, \( u \equiv 0 \).**

**Proof.** Denote \( u_\epsilon = (A)^{-\beta} u \). We shall show that \( \| u_\epsilon \| \leq C \), where \( C \) does not depend on \( \epsilon \). This implies that \( u \in \mathcal{H} \). Since \( u = \Phi(L)u \), \( u \) is in the domain of \( L \); and, as \( Lu = 0 \), it follows that either 0 is a eigenvalue of \( L \), or \( u \equiv 0 \). The first case is impossible due to the Mourre estimate.
Recall that $L_1 = L\Phi^2(L)$. We shall denote by $T$ different bounded operators with norm independent on $t$ and $\epsilon$. We begin by computing

\[
(i[L_1,A]u_\epsilon, u_\epsilon) = \lim_{t \to \infty} (i[L_1,A(A+itA)^{-1}it]u_\epsilon, u_\epsilon)
\]

\[
= \lim_{t \to \infty} i(L_1A(A+itA)^{-1}it\langle\epsilon A\rangle^{-\beta}u_\epsilon, u_\epsilon)
\]

\[
- \lim_{t \to \infty} i(L_1A(A-itA)^{-1}it\langle\epsilon A\rangle^{-\beta}u_\epsilon, u_\epsilon)
\]

\[
= -i([\langle\epsilon A\rangle^{-\beta}, L_1]A(A+itA)^{-1}it\langle\epsilon A\rangle^{-\beta}u_\epsilon, u_\epsilon)
\]

\[
- \lim_{t \to \infty} i(u_\epsilon, (A-itA)^{-1}it\langle\epsilon A\rangle^{-1}u_\epsilon)
\]

\[
+ \lim_{t \to \infty} i(L_1A(A-itA)^{-1}it\langle\epsilon A\rangle^{-\beta}u_\epsilon, u_\epsilon)
\]

\[
- \lim_{t \to \infty} i(u_\epsilon, \langle\epsilon A\rangle^{-\beta}(A-itA)^{-1}it\langle\epsilon A\rangle^{-\beta}u_\epsilon)
\]

\[
= -i([\langle\epsilon A\rangle^{-\beta}, L_1]A(A-itA)^{-1}it\langle\epsilon A\rangle^{-\beta}u_\epsilon, u_\epsilon)
\]

\[
- \lim_{t \to \infty} i(u_\epsilon, \langle\epsilon A\rangle^{-\beta}(A-itA)^{-1}it\langle\epsilon A\rangle^{-\beta}u_\epsilon)
\]

We have

\[
i[L_1, \langle\epsilon A\rangle^{-\beta}] = -\beta i[L_1, A]\epsilon^2\langle\epsilon A\rangle^{-\beta-2} + \langle A \rangle^{-\beta}T\langle A \rangle^{-\beta-1}
\]

(by Lemma 2.1, with $1 + 2\beta < 1 + \theta$, $\beta < \theta$, $\alpha = 1$) and also:

\[
i[L_1, \epsilon A\rangle^{-\beta}] = -\beta\epsilon^2\langle\epsilon A\rangle^{-\beta-1}[L_1, A] + \langle A \rangle^{-\beta-1}T\langle A \rangle^{-\beta}.
\]

It follows then

\[
-i([\langle\epsilon A\rangle^{-\beta}, L_1]itA(A+itA)^{-1}(\epsilon A)^{-\beta}u_\epsilon, u_\epsilon)
\]

\[
= (\beta i[L_1, A]\epsilon^2\langle\epsilon A\rangle^{-2-\beta}B_1u, u_\epsilon)
\]

\[
+ \lim_{t \to \infty} i(u_\epsilon, (A-itA)^{-1}it\langle\epsilon A\rangle^{-1}u_\epsilon)
\]

\[
= -\beta i[L_1, A]\epsilon^2\langle\epsilon A\rangle^{-2-\beta}B_1u, u_\epsilon)
\]

\[
+ \lim_{t \to \infty} i(u_\epsilon, (A-itA)^{-1}it\langle\epsilon A\rangle^{-1}u_\epsilon)
\]

Similarly for the second commutator. We get thus

\[
(i[L_1,A]u_\epsilon, u_\epsilon) = 2\beta(\epsilon^2\langle\epsilon A\rangle^{-2-\beta}B_1u, u_\epsilon) + (T\langle A \rangle^{-\beta}u, \langle A \rangle^{-\beta}u),
\]

where $B_1 = i[L_1, A]$. We write $\epsilon^2\langle\epsilon A\rangle^{-2-\beta} = \langle\epsilon A\rangle^\beta - \langle\epsilon A\rangle^{-1-\beta}$. Lemma 1.4 gives

\[
(\langle\epsilon A\rangle^\beta B_1u, u_\epsilon) = (B_1u, u_\epsilon) + (T\langle A \rangle^{-\beta}u, \langle A \rangle^{-\beta}u)
\]

and

\[
(\langle\epsilon A\rangle^{-2}B_1u, u_\epsilon) = (B_1\langle\epsilon A\rangle^{-1}u_\epsilon, \langle\epsilon A\rangle^{-1}u_\epsilon) + (T\langle A \rangle^{-\beta}u, \langle A \rangle^{-\beta}u).
\]

Hence

\[
(1 - 2\beta)(B_1u, u_\epsilon) = -2\beta(B_1\langle\epsilon A\rangle^{-1}u_\epsilon, \langle\epsilon A\rangle^{-1}u_\epsilon) + (T\langle A \rangle^{-\beta}u, \langle A \rangle^{-\beta}u).
\]

We use that $u = \Phi(L)u$ as in Lemma 2.4 to get

\[
(B_1\langle\epsilon A\rangle^{-1}u_\epsilon, \langle\epsilon A\rangle^{-1}u_\epsilon) = (\Phi(L)i[L, A]\Phi(L)\langle\epsilon A\rangle^{-1}u_\epsilon, \langle\epsilon A\rangle^{-1}u_\epsilon) + (T\langle A \rangle^{-\beta}u, \langle A \rangle^{-\beta}u).
\]
The Mourre inequality (suppose supp$\Phi$ small enough) provide

\[(1 - 2\beta)(B_1 u_{\epsilon}, u_{\epsilon}) \leq (T\langle A\rangle^{-\beta} u, \langle A\rangle^{-\beta} u)\]

Again:

\[(B_1 u_{\epsilon}, u_{\epsilon}) = (\Phi(L)i[L, A]\Phi(L)u_{\epsilon}, u_{\epsilon}) + (T\langle A\rangle^{-\beta} u, \langle A\rangle^{-\beta} u)\]

Then the Mourre inequality gives

\[||\Phi(L)u_{\epsilon}|| \leq C\]

Commuting $\Phi(L)$ and $\langle \epsilon A \rangle^{-\beta}$ (by Lemma 1.4), we get

\[||\langle \epsilon A \rangle^{-\beta} u|| \leq C\]

which gives $u \in \mathcal{H}$ and thus finishes the proof.

REFERENCES


