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ON AN ABSTRACT RADIATION CONDITION

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INTRODUCTION

We shall present an abstract radiation condition in terms of the Mourre theory of conjugate operator method.

Let $\mathcal{H}$ be a Hilbert space and A be a self-adjoint operator in $\mathcal{H}$. For $s \geq 0$ consider the Hilbert space $\mathcal{A}^s = D(\langle A\rangle^s)$ with the graph norm, and if $s < 0$, $\mathcal{A}^s = (\mathcal{A}^{-s})^*$. Then, if $s \geq 0$, $\mathcal{A}^s \subseteq \mathcal{H} \subseteq \mathcal{A}^{-s}$ continuously and densely, and the scalar product of $\mathcal{H}$ extends to a natural duality $(\cdot,\cdot)_{s,-s} : \mathcal{A}^s \times \mathcal{A}^{-s} \to \mathbf{C}$ for all $s \in \mathbf{R}$. We denote by $P_{\pm}$ the spectral projectors of A associated to the half-lines $[0,+\infty)$ and $(-\infty, 0]$, respectively.

We recall now some (Besov) spaces of operators (see [ABG]). Let S be a bounded operator on $\mathcal{H}$. We say that $S \in C^k(A)$, k positive integer, if the application $\mathbf{R} \ni \tau \to \mathcal{W}(\tau)[S] = e^{i\tau A}Se^{-i\tau A} \in \mathbf{B}(\mathcal{H})$ is strongly C^k; in this case $ad_A^k S$ can be extended as a bounded operator on $\mathcal{H}$. Consider $\theta \in (0,1]$,$p \in [1,\infty]$; we say that $S \in \mathcal{C}^{\theta,p}(A)$ if $(\tau \to (\mathcal{W}(\tau) - I)^m[S]\|/|\tau|^{\theta+1/p}) \in L^p((0,\infty))$, where $m = 1$ if $\theta < 1$, and $m = 2$ if $\theta = 1$. (If $p = \infty$, this condition should be read as $\sup_{\tau>0} \|(\mathcal{W}(\tau) - I)^m[S]\|/|\tau|^\theta < \infty$.) For general $\theta > 0$, we say that $S \in \mathcal{C}^{\theta,p}(A)$ if $S \in C^l(A)$ and $ad_A^l S \in \mathcal{C}^{\theta-l,p}(A)$, where l is the largest integer $l < \theta$.

Let L be a self-adjoint operator in $\mathcal{H}$. Then $L \in \mathcal{C}^{\theta,p}(A)$ (or $C^k(A)$) if $(L-z)^{-1} \in \mathcal{C}^{\theta,p}(A)$ (or $C^k(A)$) for some (and hence all) $z \in \mathbf{C} \setminus \sigma(L)$.

If L is a self-adjoint operator of class $C^1(A)$, then the commutator $i[L,A]$ is defined as a continuous form on the domain of L. Then one can define the strict Mourre set $\mu^A(L)$ of L with respect to A as the set of $\lambda \in \mathbf{R}$ with the property that there exists $J = (\lambda - \delta, \lambda + \delta) \neq \emptyset$ and $d > 0$ such that

$$E_L(J)\, i[L,A]\, E_L(J) \geq d\, E_L(J).$$

We recall that if L has a spectral gap and $L \in \mathcal{C}^{1,1}(A)$, then there exist $R_L(\lambda \pm i0) = \lim_{\epsilon\to 0}(L-\lambda\mp i\epsilon)^{-1}$ uniformly in $\mathbf{B}(\mathcal{A}^s, \mathcal{A}^{-s})$, whenever $s > 1/2$.

The following theorem was given in [BGS1] (for the proof see [BGS2]; see also [J] for some earlier results).

THEOREM 1. *Let $s > 1/2$ be a real number and L be a self-adjoint operator with a spectral gap and of class $\mathcal{C}^{s+1/2,1}(A)$. Then we have $P_\mp R_L(\lambda \pm i0)\mathcal{A}^s \subseteq \mathcal{A}^{s-1}$* for each $\lambda \in \mu^A(L)$.

It turns out that in some stronger hypotheses this condition characterizes $R_L(\lambda \pm i0)$. Namely, we prove the following theorem, extending some results of [B2], [M].

THEOREM 2. *Let $1 \geq \theta > 1/2$ be a real number, $L \geq -M$ be a bounded from below self-adjoint operator of class $\mathcal{C}^{1+\theta,\infty}(A)$ such that $i[L,A] \in \mathbf{B}(\mathcal{G},\mathcal{G}^*)$, where $\mathcal{G}$ is the form domain of L, and $\lambda \in \mu^A(L)$. Suppose $u \in \mathcal{A}^{-s}$, $s \in (1/2,\theta)$ satisfies:*

a) $(u,(L-\lambda)\varphi)_{-s,s} = 0$ for all $\varphi \in (L+M)^{-1}\mathcal{A}^s$,

b) there exists $\alpha < \theta/2$ such that $\langle A\rangle^{-\alpha}P_-(A)u \in \mathcal{H}$ (or $\langle A\rangle^{-\alpha}P_+(A)u \in \mathcal{H}$).

Then $u = 0$.

The proof follows Isozaki's proof of some type of radiation conditions which are strongly related to those presented here. (See [I1], [I2], [I3].) We only remark here that Theorem 2 provides some useful results in the study of the layered media.

One of the tools needed here is the functional calculus using almost analytic extensions of symbols. Let $m \in \mathbf{R}$. We denote by S^m the set of symbols $f \in C^\infty(\mathbf{R})$ that satisfy

$$p_k(f) = \sup_{x\in\mathbf{R}} \langle x\rangle^{m-k} |f^{(k)}(x)| < \infty.$$

Then S^m, endowed with the seminorms p_k is a Fréchet space. The following result can be found in [B2], [M] (see also [DG] for the main idea).

PROPOSITION 3. *Consider a bounded family of symbols $\{f_\epsilon\} \subset S^m$. Then there exists a family of functions (the almost analytic extensions) $\{\tilde{f}_\epsilon\} \subset C^\infty(\mathbf{C})$ such that:*

i. $|Imz| \le \langle Rez\rangle$ *on* $\operatorname{supp}\tilde{f}_\epsilon$,

ii. $|\bar\partial \tilde{f}_\epsilon(z)| \le C_N \langle z\rangle^{m-N-1} |Imz|^{N-1}$ *for all* $N \ge 0$ *and all* $z \in \mathbf{C}$, *where the constants* C_N *do not depend on* z *and* ϵ.

This construction provides an useful representation for the functional calculus of a self-adjoint operator, due to Helffer- Sjöstrand ([HS]): Let A be a self-adjoint operator on $\mathcal{H}$ and $f \in S^{-\delta}$, $\delta > 0$. Then

$$f(A) = \frac{1}{\pi} \int_{\mathbf{C}} \bar\partial \tilde{f}(z)(A-z)^{-1}\, dx\, dy,$$

where $z = x + i\,y$ and $\tilde{f}$ is an almost analytic extension of f. If B is a bounded operator with $ad_A n$ is a bounded form on the domain of A, and $ad_A k f^{(k)}(A)$ (respectively $f^{(k)}(A) ad_A k$) $k = 1, \ldots, n-1$, are bounded operators, then

$$\begin{aligned}
[B, f(A)] &= \sum_{k=1}^{n-1} \frac{(-1)^{k+1}}{k!} ad_A^k(B) f^{(k)}(A) + R_n^r(A,B) \\
&= \sum_{k=1}^{n-1} \frac{(-1)^{k+1}}{k!} f^{(k)}(A) ad_A^k(B) + R_n^l(A,B),
\end{aligned}$$

where

$$\begin{aligned}
R_n^r(A,B) &= -\frac{1}{\pi} \int_{\mathbf{C}} \bar\partial \tilde{f}(z)(A-z)^{-1} ad_A^n(B)(A-z)^{-n}\, dx\, dy, \\
R_n^l(A,B) &= \frac{(-1)^n}{\pi} \int_{\mathbf{C}} \bar\partial \tilde{f}(z)(A-z)^{-n} ad_A^n(B)(A-z)^{-1}\, dx\, dy.
\end{aligned}$$

For a proof, see for instance [M].

1. COMMUTATORS

LEMMA 1.1. *Let* $B \in C^{\theta,\infty}(A)$, $0 < \theta < 1$, *be a bounded operator and* α_1, α_2 *positive numbers such that* $0 < \alpha_1 + \alpha_2 < \theta$. *Then*

$$\|\langle A\rangle^{\alpha_1} [B, (A-z)^{-1}] \langle A\rangle^{\alpha_2}\| \le C(|Imz|^{-\theta-1} + |Imz|^{-1} + \langle z\rangle |Imz|^{-2} + \langle z\rangle^2 |Imz|^{-3}) \tag{1.1}$$

whenever $|Imz| \neq 0$.

Proof. Consider $0 < \alpha < \theta$. We consider first the operator $\langle A\rangle^\alpha[B,(A-z)^{-1}]$. Suppose $Imz > 0$; the case $Imz < 0$ is similar.

(i) We have (weakly)

$$[B,(A-z)^{-1}] = \int_{-\infty}^{0} e^{\mu t} e^{i\lambda t}[B, e^{itA}]\, dt,$$

where $z = \lambda + i\mu$. Using that $B \in \mathcal{C}^{\theta,\infty}(A)$, we get

$$\|[B,(A-z)^{-1}]\| \leq \int_{-\infty}^{0} e^{\mu t}|t|^\theta\, dt \leq C\mu^{-\theta-1}\int_{-\infty}^{0} e^t|t|^\theta\, dt,$$

hence

(1.2)
$$\|[B,(A-z)^{-1}]\| \leq C\mu^{-\theta-1}.$$

(ii) Denote $\nu(\lambda) = \langle\lambda\rangle^\alpha$. Helffer-Sjöstrand formula gives (first as bounded operators between $\mathcal{A}^\alpha$ and $\mathcal{A}^{-\alpha}$)

(1.3)
$$[B,\langle A\rangle^\alpha] = \frac{1}{\pi}\int_{\mathbf{C}} \overline{\partial}\tilde{\nu}(z)[B,(A-z)^{-1}]\, dx\, dy.$$

The norm of the integrand in (3) can be bounded by

$$\|\overline{\partial}\tilde{\nu}(z)[B,(A-z)^{-1}]\| \leq C\langle z\rangle^{\alpha-1-N}|Imz|^{N-\theta-1}.$$

If one takes $N = \theta + 1$ to avoid the singularities, we get

$$\|\overline{\partial}\tilde{\nu}(z)[B,(A-z)^{-1}]\| \leq C\langle z\rangle^{\alpha-2-\theta},$$

which is integrable if $\alpha < \theta$. Hence

$$[B,\langle A\rangle^\alpha] \in \mathbf{B}(\mathcal{H}).$$

(iii) We can write then

(1.4)
$$\langle A\rangle^\alpha[B,(A-z)^{-1}] = [B,\langle A\rangle^\alpha(A-z)^{-1}] - [\langle A\rangle^\alpha, B](A-z)^{-1}.$$

The norm of the second hand in the rhs of (4) is bounded by $C|Imz|^{-1}$.

(iv) We estimate now the first term in the rhs of (4). Let g be a smooth function on $\mathbf{R}$, $g(t) = 1$ if $|t| \geq 1$ and $g(t) = 0$ if $|t| < 1/2$. Then

(1.5)
$$[B,\langle A\rangle^\alpha(A-z)^{-1}] = [B, g(A)\langle A\rangle^\alpha(A-z)^{-1}] + [B,(1-g(A))\langle A\rangle^\alpha(A-z)^{-1}].$$

The second term of the rhs of (1.5) equals

$$[B,\langle A\rangle^\alpha](A-z)^{-1} + [B,(A-z)^{-1}](1-g(A))\langle A\rangle^\alpha,$$

and has the norm less then (using (2))

$$C(|Imz|^{-1} + |Imz|^{\theta+1}). \tag{1.6}$$

We denote $g_z(\lambda) = g(\lambda)\langle\lambda\rangle^\alpha(\lambda - z)^{-1}$. We shall use the following form of the Helffer–Sjöstrand formı (see [BGS2], section 4):

$$\begin{aligned}(1.7)\quad [B, g_z(A)] = &\frac{1}{\pi}\int_{\mathbf{R}} ((g_z(\lambda) - \lambda g_z'(\lambda))[B, ImR_A(\lambda + i\lambda)] - \partial_\lambda(\lambda g_z(\lambda))[B, ImiR_A(\lambda + i\lambda)])\, d\lambda \\ &- \frac{1}{\pi}\int_{\mathbf{R}}\int_0^\lambda g_z^{(2)}(\lambda))[B, ImR_A(\lambda + i\mu)]\mu\, d\mu\, d\lambda.\end{aligned}$$

The norm of the integrand in the first term of (1.7) can be estimated by (using (2) and on supp$g$)

$$C\left(\frac{\langle\lambda\rangle^\alpha}{|\lambda - z|} + \frac{\langle\lambda\rangle^{\alpha+1}}{|\lambda - z|^2}\right)\langle\lambda\rangle^{-\theta-1} \le C\langle\lambda\rangle^{-\theta-1}(|Imz|^{-1} + \langle z\rangle|Imz|^{-2}).$$

Hence the first integral in (7) can be bounded as follows

$$(1.8)\quad \|\int_{\mathbf{R}} (g_z(\lambda) - \lambda g_z'(\lambda) + 2(i+1)^{-1}\partial_\lambda(\lambda g_z(\lambda))[B, R_A(\lambda + i\lambda)]\, d\lambda\| \le C(|Imz|^{-1} + \langle z\rangle|Imx|^{-2}).$$

To estimate the second integral we note first that

$$\|\int_0^\lambda g_z^{(2)}(\lambda))[B, R_A(\lambda + i\mu)]\mu\, d\mu\, d\lambda\| \le C\langle\lambda\rangle^{1-\theta} \tag{1.9}$$

on supp$g$. Then

$$\begin{aligned}(1.10)\quad \|g_z^{(2)}(\lambda)\int_0^\lambda g_z^{(2)}(\lambda))[B, R_A(\lambda + i\mu)]\mu\, d\mu d\lambda\| &\le C\langle\lambda\rangle^{1-\theta}\left(\frac{\langle\lambda\rangle^\alpha}{|\lambda - z|^3} + \frac{\langle\lambda\rangle^{\alpha-1}}{|\lambda - z|^2} + \frac{\langle\lambda\rangle^{\alpha-2}}{|\lambda - z|}\right) \\ &\le C\langle\lambda\rangle^{-1+\alpha-\theta}(|Imz|^{-1} + \langle z\rangle|Imz|^{-2} + \langle z\rangle^2|Imz|^{-3})\end{aligned}$$

Summing up:

$$\|[B, g_z(A)]\| \le C(\langle z\rangle^2|Imz|^{-3}) + \langle z\rangle|Imz|^{-2} + \lambda\rangle^{-1+\alpha-\theta}|Imz|^{-1}).$$

Then one gets

$$\|\langle A\rangle^\alpha[B, (A - z)^{-1}]\| \le C(\langle z\rangle^2|Imz|^{-3}) + \langle z\rangle|Imz|^{-2} + \lambda\rangle^{-1+\alpha-\theta}|Imz|^{-1} + |Imz|^{-\theta-1}). \tag{1.11}$$

In the same way

$$\|[B, (A - z)^{-1}\langle A\rangle^\alpha]\| \le C(\langle z\rangle^2|Imz|^{-3}) + \langle z\rangle|Imz|^{-2} + \lambda\rangle^{-1+\alpha-\theta}|Imz|^{-1} + |Imz|^{-\theta-1}). \tag{1.12}$$

The general result follows by interpolation. ∎

LEMMA 1.2. *Let* $\{\chi_t\} \in S^a$, $a < 1$ *be a bounded family of symbols, and* $B \in C^{1+\theta,\infty}(A)$ *a bounded operator. Then*

$$i[B, \chi_t(A)] = i[B, A]\chi_t'(A) + R_{1,t},$$
$$i[B, \chi_t(A)] = \chi_t'(A)i[B, A] + R_{2,t},$$

where

$$\langle A\rangle^{\alpha_1} R_{1,t}\langle A\rangle^{\alpha_2} \in \mathbf{B}(\mathcal{H}) \quad \textit{and} \quad \|\langle A\rangle^{\alpha_1} R_{1,t}\langle A\rangle^{\alpha_2}\| \le C$$
$$\langle A\rangle^{\alpha_2} R_{2,t}\langle A\rangle^{\alpha_1} \in \mathbf{B}(\mathcal{H}) \quad \textit{and} \quad \|\langle A\rangle^{\alpha_2} R_{2,t}\langle A\rangle^{\alpha_1}\| \le C,$$

whenever $\alpha_1 + \alpha_2 + a < 1 + \theta$, $\alpha_1 + \alpha_2 < 1 + \theta$, $\alpha_1 < \theta$. *Here* C *stands for constants not depending on* t.

Proof. We have $i[B, \chi_t(A)] = i[B, A]\chi_t'(A) + R_{1,t}$ where

$$R_{1,t} = \frac{1}{\pi}\int_{\mathbf{C}} \overline{\partial}\tilde{\chi}_t i[D, (A - z)^{-1}](A - z)^{-1}\, dx\, dy,$$

with $D = i[B, A] \in C^{\theta,\infty}(A)$, bounded.

We take $\delta = \theta - \alpha_1 - \epsilon$ with ϵ sufficiently small such that $\alpha_2 - \delta < 1$ and $a + \alpha_2 - \delta < 1$. (This is possible by hypothesis.) Then, by Lemma 1,

$$\begin{aligned}&\|\partial\tilde{\chi}_t\langle A\rangle^{\alpha_1} i[D, (A - z)^1]\langle A\rangle^{\delta}(A - z)^{-1}\langle A\rangle^{\alpha_2-\delta}\| \\ &\le C_N\, \langle z\rangle^{a-1-N}|Imz|^N(|Imz|^{-1} + |Imz|^{-\theta-1})\langle z\rangle^{\alpha_2-\delta}|Imz|^{-1}\end{aligned}$$

on supp$\partial\tilde{\chi}_t$. We take $N = \theta + 2$ and thus obtain that

$$\|\partial\tilde{\chi}_t\langle A\rangle^{\alpha_1} i[D, (A - z)^1](A - z)^{-1}\langle A\rangle^{\alpha_2}\| \le C\langle z\rangle^{a-3+\alpha-\delta}$$

which is integrable and C does not depend on t. Hence $\langle A\rangle^{\alpha_1} R_{1,t}\langle A\rangle^{\alpha_2}$ extends to a bounded operator and the estimate in the statement holds. One proceed similarly to get the second assertion. ∎

LEMMA 1.3. *Let* B *be a bounded operator of class* $C^{\theta,\infty}(A)$, $0 < \theta \le 1$ *and* α_1, α_2 *positive numbers such that* $\alpha_1 + \alpha_2 < \theta$. *Then* $\langle A\rangle^{\alpha_1}[B, \langle A\rangle^{\alpha_2}]$ *extends to a bounded operator on* $\mathcal{H}$.

Proof. Recall that in the proof of Lemma 1 we proved that $[B, \langle A\rangle^{\delta}] \in \mathbf{B}(\mathcal{H})$ whenever $\delta = \alpha_1 + \alpha_2 + \epsilon < \theta$. We denote $\theta_i = \alpha_i/\delta$, $i = 1, 2$ and set $A_\delta = \langle A\rangle^{\delta}$; this is a self-adjoint operator $A_\delta \ge 1$. We have than to control $A_\delta^{\theta_1}[B, h(A_\delta)]$, where $h \in S^{\theta_2}$, $h(s) = s^{\theta_2}$ if $s \ge 1/2$ and $h(s) = 0$ if $s \le 1/4$. We have (first in form sense)

$$A_\delta^{\theta_1} i[B, h(A_\delta)] = -\frac{1}{\pi}\int_{\mathbf{C}} \overline{\partial}\tilde{h}(z) A_\delta^{\theta_1}(A_\delta - z)^{-1} i[B, A_\delta](A_\delta - z)^{-1}\, dx\, dy.$$

On the support of $\overline{\partial}\tilde{h}$ the norm of the integrand can be estimated as

$$\|\overline{\partial}\tilde{h}(z) A_\delta^{\theta_1}(A_\delta - z)^{-1} i[B, A_\delta](A_\delta - z)^{-1}\| \le C\langle z\rangle^{\theta_1+\theta_2-1-2}.$$

The rhs is an integrable function, since $\theta_1 + \theta_2 < 1$. Therefore $A_\delta^{\theta_1} i[B, h(A_\delta)]$ extends to a bounded operator on $\mathcal{H}$. ∎

LEMMA 1.4. *Let B be a bounded operator of class $C^{\theta,\infty}(A)$, $0 < \theta \leq 1$ and α_1, α_2 positive numbers such that $\alpha_1 + \alpha_2 < \theta$, and $\{g_t\} \subset S^a$, $a \leq 0$, a bounded family of symbols. Then:*

$$\|\langle A\rangle^{\alpha_1} i[B, g_t(A)]\langle A\rangle^{\alpha_2}\| \leq C,$$

where C does not depend on t.

Proof. (i) Consider first the case where $a < 0$. Then

$$\langle A\rangle^{\alpha_1} i[B, g_t(A)]\langle A\rangle^{\alpha_2} = \frac{1}{\pi}\int_{\mathbf{C}} \bar{\partial}\tilde{g}_t(z)\langle A\rangle^{\alpha_1} i[B,(A-z)^{-1}]\langle A\rangle^{\alpha_2}\, dx\, dy.$$

Using Lemma 1 the norm of the integrand can be majorized by $C\langle z\rangle^{a-2}$.
(ii) If $a = 0$, let $\epsilon > 0$ be such that $\alpha_1 + \alpha_2 + \epsilon < \theta$ and write

$$\langle A\rangle^{\alpha_1+\epsilon}\langle A\rangle^{-\epsilon} i[B, g_t(A)]\langle A\rangle^{\alpha_2} = \langle A\rangle^{\alpha_1+\epsilon} i[\langle A\rangle^{-\epsilon}, B]\langle A\rangle^{\alpha_2} g_t(A) + \langle A\rangle^{\alpha_1+\epsilon}[B, g_t(A)\langle A\rangle^{-\epsilon}]\langle A\rangle^{\alpha_2}.$$

We use the proof of the previous lemma to show that the first term is a bounded operator and its norm can be bounded by a constant not depending on t. For the second term we use (i). ∎

2. THE PROOF OF THEOREM 2

We can suppose, without restricting the generality, that in Theorem 2 we have $M = 1$ and $\lambda = 0$.

LEMMA 2.1. *If $\Phi \in C_0^\infty(\mathbf{R})$ is a real function, $\Phi = 1$ on a neighborhood of 0, then*

(2.1) $$(u, \Phi(L)\varphi)_{-s,s} = (u,\varphi)_{-s,s}, \quad \textit{for all} \quad \varphi \in \mathcal{A}^s.$$

Proof. We have, for $\varphi \in \mathcal{A}^1$,

(2.2) $$(u,(1-\Phi(L))\varphi)_{-s,s} = (u, L\Psi(L)\varphi)_{-s,s},$$

where $\Psi(t) = (1-\Phi(t))t^{-1}$. Therefore, to have (2) for $\varphi \in \mathcal{A}^1$ it suffices to prove that $\Psi(L) = (L+1)^{-1}\varphi_1$ with $\varphi_1 \in \mathcal{A}^1$. We can write $(L+1)\Psi(L) = (1-\Phi(L)) + \Psi(L)$. Thus, since $(1\Phi(L))\phi \in \mathcal{A}^1$, it remains to show that $\Psi(L)\varphi \in \mathcal{A}^1$ if $\varphi \in \mathcal{A}^1$. We have

$$\begin{aligned} i[\Psi(L), A] &= \frac{1}{\pi}\int_{\mathbf{C}} \bar{\partial}\tilde{\Psi} i[(L-z)^{-1}, A]\, dx\, dy \\ &= -\frac{1}{\pi}\int_{\mathbf{C}} \bar{\partial}\tilde{\Psi}(L-z)^{-1}(L+1)^{1/2}(L+1)^{-1/2} i[L,A](L+1)^{-1/2}(L+1)^{1/2}(L-z)^{-1}\, dx\, dy \end{aligned}$$

The norm of the integrand can be bounded by $C\langle z\rangle^{-2-2}|Imz|^2\langle z\rangle|Imz|^{-2} = C\langle z\rangle^{-3}$. We get that $i[\Psi(L), A]$ is a bounded operator and we obtain easily that $\Psi(L)\varphi \in \mathcal{A}^1$ if $\varphi \in \mathcal{A}^1$. Thus equation (1) holds for $\varphi \in \mathcal{A}^1$; the general result follows by density using the fact that $\Phi(L) \in \mathbf{B}(\mathcal{A}^s)$. ∎

Remark. In fact the previous Lemma says that $\Phi(L)u = u$ for all $\Phi \in C_0^\infty(\mathbf{R})$, $\Phi = 1$ on a neighborhood of 0; this fact can be easily seen using that $\Phi(L) \in \mathbf{B}(\mathcal{A}^s) \cap \mathbf{B}(\mathcal{A}^{-s})$ and it is symmetric with respect to the duality $(.,.)_{s,-s}$.

LEMMA 2.2. *Let $\chi \in C_0^\infty(\mathbf{R})$ such that $0 \le \chi(s) \le 1$, $\chi(s) = 1$ for $|s| \le 1$, $\chi(s) = 0$ for $|s| \ge 2$. We consider the $C_0^\infty(\mathbf{R})$ function*

$$\chi_t(y) = \int_{\langle y\rangle}^{\infty} s^{-2\beta}\chi^2(s/t)\,ds,$$

where $\beta > \max(\alpha, s/2)$, $\beta < \theta/2$. Then

$$(L\phi^2(L)\chi_t(A)u, u)_{s,-s} = 0. \tag{2.3}$$

Proof. The Lemma follows by hypothesis as $\Phi^2(L)\chi_t(A)u \in (L+1)^{-1}\mathcal{A}^s$. ∎

We shall set T for different bounded operators with norm independent on t.
Remark. We have

$$2\,Re\, i\,((L\phi^2(L)\chi_t(A)u, u)_{s,-s} = 0.$$

We shall give to this relation the form and the meaning

$$i([L\Phi^2(L), \chi_t(A)]u, u)_{s,-s} = 0.$$

Set $L_1 = \Phi^2(L)L$. Then L_1 is a bounded operator of class $C^{\theta+1,\infty}(A)$ (Thm. 6.2.5 [ABG]).

LEMMA 2.3. *We have*

$$i[L_1, \chi_t(A)] = i[L_1, A]A\langle A\rangle^{-2\beta-1}\chi^2(\langle A\rangle/t) + \langle A\rangle^{-s}T\langle A\rangle^{-s}.$$

Proof. One applies Lemma 1.2 for $R_{1,t}$, $\alpha_1 = \alpha_2 = s$, $a = 1-2\beta$. (Here $\alpha_1+\alpha_2+a = 2s+1-2\beta < \theta+1$ since $\beta > \theta/2$, and $s < \theta$.) ∎

As a direct consequence we get the next Lemma.

LEMMA 2.3'. $\sup_{t\ge1} |(i[L_1, A]A\langle A\rangle^{-2\beta-1}\chi^2(\langle A\rangle/t)u, u)_{s,-s}| < \infty$.

LEMMA 2.4. *If $\tilde\Phi \in C_0^\infty(\mathbf{R})$, $\tilde\Phi = 1$ on a small enough neighborhood of 0, then*

$$\sup_{t\ge1} |(\tilde\Phi(L)i[L, A]\tilde\Phi(L)A\langle A\rangle^{-2\beta-1}\chi^2(\langle A\rangle/t)u, u)_{s,-s}| < \infty. \tag{2.4}$$

Proof. We know that $\tilde\Phi(L)u = u$. We have

$$A\langle A\rangle^{-2\beta-1}\chi^2(\langle A\rangle/t)\tilde\Phi(L) = \tilde\Phi(L)A\langle A\rangle^{-2\beta-1}\chi^2(\langle A\rangle/t) + [\tilde\Phi(L), A\langle A\rangle^{-2\beta-1}\chi^2(\langle A\rangle/t)].$$

Set $g_t(A) = A\langle A\rangle^{-2\beta-1}\chi^2(\langle A\rangle/t)$. Here $\{g_t\} \in S_{-2\beta}$ is a bounded family of symbols. One applies Lemma 1.2 to get

$$[\tilde\Phi(L), g_t(A)] = [\tilde\Phi(L), A]g_t'(A) + R_{1,t}.$$

(In this case $\alpha_1 = \alpha_2 = s$, $a = -\beta$.) Thus, since $2\beta + 1 > s + 1 > 2s$,

$$A\langle A\rangle^{-2\beta-1}\chi^2(\langle A\rangle/t)\tilde{\Phi}(L) = \tilde{\Phi}(L)A\langle A\rangle^{-2\beta-1}\chi^2(\langle A\rangle/t) + \langle A\rangle^{-s}T\langle A\rangle^{-s}. \tag{2.5}$$

Hence

$$\begin{aligned}(i[L_1, A]A\langle A\rangle^{-2\beta-1}\chi^2(\langle A\rangle/t)\tilde{\Phi}(L)u, \tilde{\Phi}(L)u)_{s,-s} \\ = (i[L_1, A]\tilde{\Phi}(L)A\langle A\rangle^{-2\beta-1}\chi^2(\langle A\rangle/t)u, \tilde{\Phi}(L)u)_{s,-s} + (i[L_1, A]\langle A\rangle^{-s}T\langle A\rangle^{-s}u.u)_{s,-s}.\end{aligned} \tag{2.6}$$

Since $i[L_1, A] \in \mathbf{B}(\mathcal{H}) \cap \mathcal{C}^{\theta,\infty}(A)$ (Prop. 5.2.2 [ABG]), $i[L_1, A]$ is a bounded operator on $\mathcal{A}^s$, $s < \theta$ (Thm 5.3.3, Lemma 5.3.2 [ABG]). Therefore the second term in (2.6) is bounded by a constant independent on t. But, in form sense,

$$\tilde{\Phi}(L)i[L_1, A]\tilde{\Phi}(L) = \tilde{\Phi}(L)\Phi(L)i[L, A]\Phi(L)\tilde{\Phi}(L) + L\tilde{\Phi}(L)i[A, 1 - \Phi(L)]\tilde{\Phi}(L) + \tilde{\Phi}(L)i[A, 1 - \Phi(L)]L\tilde{\Phi}(L).$$

We take supp$\tilde{\Phi}$ to be in the set where $\Phi = 1$; then

$$\tilde{\Phi}(L)i[A, 1 - \Phi(L)]\tilde{\Phi}(L) = 0 \quad \text{on} \quad \mathcal{A}^1 \times \mathcal{A}^1.$$

Therefore the bounded operator given by this form on $\mathcal{H}$ ($L \in C^1(A)$) is zero. Similarly we get that $\tilde{\Phi}(L)i[A, 1 - \Phi(L)]L\tilde{\Phi}(L) = 0$. Summing up

$$\tilde{\Phi}(L)i[L_1, A]\tilde{\Phi}(L) = \tilde{\Phi}(L)i[L, A]\tilde{\Phi}(L).$$

The lemma follows by (2.6), (2.5) and the previous relation. ∎

We can denote $\tilde{\Phi}$ also by Φ.

LEMMA 2.5. $\sup_{t\geq 1} |(\Phi(L)i[L, A]\Phi(L)A\langle A\rangle^{-\beta}\chi(\langle A\rangle/t)u, \langle A\rangle^{-\beta}\chi(\langle A\rangle/t)u)| \leq \infty$.

Proof. Set $B = \Phi(L)i[L, A]\Phi(L)$. Then B is a bounded operator of class $\mathcal{C}^{\theta,\infty}(A)$. Denote $f_t(x) = \langle x\rangle^{-\beta}\chi(\langle x\rangle/t)$, $x \in \mathbf{R}$. We take $\beta_0 < \beta$, but still $\beta_0 > s/2$, $\beta_0 < \theta/2$. We write

$$\begin{aligned}\langle A\rangle^s[B, f_t(A)]\langle A\rangle^{s-\beta} &= \langle A\rangle^{s-\beta_0}\langle A\rangle^{\beta_0}[B, f_t(A)]\langle A\rangle^{s-\beta} \\ &= \langle A\rangle^{s-\beta_0}[B, f_t(A)\langle A\rangle^{\beta_0}]\langle A\rangle^{s-\beta} - \langle A\rangle^{s-\beta_0}[B, \langle A\rangle^{\beta_0}]\langle A\rangle^{s-2\beta}\chi(\langle A\rangle/t).\end{aligned}$$

But $2s - 2\beta_0 < 2s - \theta < \theta$, so the first term is a bounded operator and its norm does not depend on t (Lemma 1.4). The second term is bounded since $s - 2\beta < 0$ and $\langle A\rangle^{s-\beta_0}[B, \langle A\rangle^{\beta_0}]$ is bounded by Lemma 1.3. Now the lemma follows easily. ∎

LEMMA 2.6. *For all $\beta > \alpha$ we have $\langle A\rangle^{-\beta}u \in \mathcal{H}$.*

Proof. (i) Consider first $\beta > \max(\alpha, s/2)$, $\beta < \theta/2$. Let F_+ be a smooth bounded real function, $F_+ = 1$ on $[1, \infty)$, $F_+ = 0$ on $(-\infty, 1/2]$. We shall show first that

$$\sup_{t\geq 1} |(\Phi(L)i[L, A]\Phi(L)A\langle A\rangle^{-\beta-1}F_+(A)\chi(\langle A\rangle/t))u, \langle A\rangle^{-\beta}F_+(A)\chi(\langle A\rangle/t)u)| < \infty. \tag{2.7}$$

We use again the notation $B = \Phi(L)i[L, A]\Phi(L)$. If $F_- = 1 - F_+$ then

$$\begin{aligned}&(BA\langle A\rangle^{-\beta-1}\chi(\langle A\rangle/t)(A)u, \langle A\rangle^{-\beta}\chi(\langle A\rangle/t)u) \\ =&(BA\langle A\rangle^{-\beta-1}\chi(\langle A\rangle/t)(F_+ + F_-)(A)u, \langle A\rangle^{-\beta}(F_+ + F_-)(A)\chi(\langle A\rangle/t)u).\end{aligned}$$

Here $\langle A\rangle^{-\beta}u \in \mathcal{H}$. Moreover, by Thm 3.10 [BGS2], the fact that B is of class $C^{s,2}$ for all $s < \theta$ ensure that $F_+(A)BF_-(A) \in \mathbf{B}(\mathcal{A}^{\beta-s}, \mathcal{A}^{\beta-s})$. Hence $\langle A\rangle^{\beta}F_+(A)BF_-(A)\langle A\rangle^{\beta} = T \in \mathbf{B}(\mathcal{H})$, and this gives

$$\begin{aligned}&(BA\langle A\rangle^{-\beta-1}\chi(\langle A\rangle/t)F_+(A)u, \langle A\rangle^{-\beta}\chi(\langle A\rangle/t)F_-(A))\\ =&(T\langle A\rangle^{\beta-s}\langle A\rangle^{-\beta-1}\chi(\langle A\rangle/t)Au, \langle A\rangle^{-2\beta}\chi(\langle A\rangle/t)F_-(A)u).\end{aligned}$$

Therefore

$$\sup_{t\geq 1}|(BA\langle A\rangle^{-\beta-1}\chi(\langle A\rangle/t)F_+(A)u, \langle A\rangle^{-\beta}\chi(\langle A\rangle/t)F_-(A))| < \infty. \tag{2.8}$$

Similarly one gets

$$\sup_{t\geq 1}|(BA\langle A\rangle^{-\beta-1}\chi(\langle A\rangle/t)F_-(A)u, \langle A\rangle^{-\beta}\chi(\langle A\rangle/t)F_+(A))| < \infty. \tag{2.9}$$

Now (2.7) follows by (2.8), (2.9) and the previous lemma.

We can write $A\langle A\rangle^{-1}F_+(A) = g^2(A)F_+(A)$ with $g \in S^0$. But $\langle A\rangle^{s-\beta}[B, g(A)]\langle A\rangle^{s-\beta}$ is bounded by Lemma 1.4 ($2s - 2\beta < 2s - \theta < 2\theta - \theta = \theta$). Hence

$$\sup_{t\geq 1}|(B\langle A\rangle^{-\beta}\chi(\langle A\rangle/t)g(A)F_+(A)u, \langle A\rangle^{-\beta}\chi(\langle A\rangle/t)g(A)F_+(A))| < \infty.$$

Using now the Mourre estimate we get

$$\sup_{t\geq 1}\|\Phi(L)\langle A\rangle^{-\beta}\chi(\langle A\rangle/t)g(A)F_+(A)u\| \leq \infty.$$

As $[\Phi(L), \langle A\rangle^{-\beta}\chi(\langle A\rangle/t)g(A)F_+(A)]\langle A\rangle^{s}$ is a bounded operator with norm independent on t (by Lemma 1.4) it follows

$$\sup_{t\geq 1}\|\langle A\rangle^{-\beta}\chi(\langle A\rangle/t)g(A)F_+(A)u\| \leq \infty.$$

This provide, using Beppo-Levi Theorem,

$$\langle A\rangle^{-\beta}g(A)F_+(A)u \in \mathcal{H}. \tag{2.10}$$

If we take $\tilde{F}_+$ to be a smooth bounded real function on $\mathbf{R}$, $\tilde{F}_+ = 1$ on $[2, \infty)$ and $\operatorname{supp}\tilde{F}_+ \subset [1, \infty)$, we can write

$$\tilde{F}_+(A)\langle A\rangle^{-\beta} = (\tilde{F}_+/gF_+)(A)(gF_+)(A)\langle A\rangle^{-\beta},$$

and $(\tilde{F}_+/gF_+)(A)(gF_+)(A)$ is a bounded operator. Then (2.10) gives that $\tilde{F}_+(A)\langle A\rangle^{-\beta}u \in \mathcal{H}$. Thus the lemma follows in this case since $(1 - \tilde{F}_+(A)\langle A\rangle^{-\beta}u \in \mathcal{H}$ by hypothesis (b) of Thm 2.

(ii) Now we can repeat the argument with s replaced by 2α and see that $\langle A\rangle^{-\beta}u \in \mathcal{H}$ for all $\beta < \theta/2$, $\beta > \alpha$. ∎

LEMMA 2.7. *In the conditions of Thm. 2,* $u \equiv 0$.

Proof. Denote $u_\epsilon = \langle \epsilon A\rangle^{-\beta}u$. We shall show that $\|u_\epsilon\| \leq C$, where C does not depend on ϵ. This imlpies that $u \in \mathcal{H}$. Since $u = \Phi(L)u$, u is in the domain of L; and, as $Lu = 0$, it follows that either 0 is a eigenvalue of L, or $u \equiv 0$. The first case is imposible due to the Mourre estimate.

Recall that $L_1 = L\Phi^2(L)$. We shall denote by T different bounded operators with norm independe on t and ϵ. We begin by computing

$$
\begin{aligned}
(i[L_1, A]u_\epsilon, u_\epsilon) &= \lim_{t\to\infty} (i[L_1, A(A+itA)^{-1}it]u_\epsilon, u_\epsilon) \\
&= \lim_{t\to\infty} i(L_1 A(A+itA)^{-1}it\langle\epsilon A\rangle^{-\beta}u, \langle\epsilon A\rangle^{-\beta}u) \\
&- \lim_{t\to\infty} i(\langle\epsilon A\rangle^{-\beta}u, L_1 A(A-itA)^{-1}it\langle\epsilon A\rangle^{-\beta}u) \\
&= -\lim_{t\to\infty} i([\langle\epsilon A\rangle^{-\beta}, L_1]A(A+itA)^{-1}it\langle\epsilon A\rangle^{-\beta}u, u)_{\beta,-\beta} \\
&- \lim_{t\to\infty} i(u, L_1 A(A-itA)^{-1}it\langle\epsilon A\rangle^{-2\beta}u)_{-\beta,+\beta} \\
&+ \lim_{t\to\infty} i(L_1 A(A+itA)^{-1}it\langle\epsilon A\rangle^{-2\beta}u, u)_{\beta,-\beta} \\
&+ \lim_{t\to\infty} i(u, [\langle\epsilon A\rangle^{-\beta}, L_1]A(A-itA)^{-1}it\langle\epsilon A\rangle^{-\beta}u)_{-\beta,+\beta} \\
&= -\lim_{t\to\infty} i([\langle\epsilon A\rangle^{-\beta}, L_1]A(A+itA)^{-1}it\langle\epsilon A\rangle u, u)_{\beta,-\beta} \\
&+ \lim_{t\to\infty} i(u, [\langle\epsilon A\rangle^{-\beta}, L_1]A(A-itA)^{-1}it\langle\epsilon A\rangle u)_{-\beta,+\beta}
\end{aligned}
$$

We have

$$i[L_1, \langle\epsilon A\rangle^{-\beta}] = -\beta i[L_1, A]\epsilon^2 A\epsilon\langle\epsilon A\rangle^{-\beta-2} + \langle A\rangle^{-\beta}T\langle A\rangle^{-\beta-1}$$

(by Lemma 2.1, with $1+2\beta < 1+\theta$, $\beta < \theta$, $a = 1$) and also:

$$i[L_1, \epsilon A\rangle^{-\beta}] = -\beta\epsilon^2 A\epsilon A\rangle^{-\beta-1}i[L_1, A] + \langle A\rangle^{-\beta-1}T\langle A\rangle^{-\beta}.$$

It follows then

$$
\begin{aligned}
&-i([\langle\epsilon A\rangle^{-\beta}, L_1]itA(A+itA)^{-1}\langle\epsilon A\rangle^{-\beta}u, u)_{\beta,-\beta} \\
=&(-\beta i[L_1, A]\epsilon^2 A^2 it(A+itA)^{-1}it\langle\epsilon A\rangle^{-2\beta-2}u, u)_{\beta,-\beta} + (T\langle A\rangle^{-\beta-1}itA(A+itA)^{-1}u_\epsilon, \langle A\rangle^{-\beta}u_\epsilon \to \\
\to&(-\beta i[L_1, A]\epsilon^2 A^2\langle\epsilon A\rangle^{-2\beta-2}u, u)_{\beta,-\beta} + (T\langle A\rangle^{-\beta-1}u_\epsilon, \langle A\rangle^{-\beta}u_\epsilon
\end{aligned}
$$

Similarly for the second commutator. We get thus

$$(i[L_1, A]u_\epsilon, u_\epsilon) = 2\beta(\epsilon^2 A^2\langle\epsilon A\rangle^{-2-\beta}B_1 u, u_\epsilon) + (T\langle A\rangle^{-\beta}u, \langle A\rangle^{-\beta}u),$$

where $B_1 = i[L_1, A]$. We write $\epsilon^2 A^2\langle\epsilon A\rangle^{-2-\beta} = \langle\epsilon A\rangle^{\beta} - \langle\epsilon A\rangle^{-1-\beta}$. Lemma 1.4 gives

$$(\langle\epsilon A\rangle^{\beta}B_1 u, u_\epsilon) = (B_1 u_\epsilon, u_\epsilon) + (T\langle A\rangle^{-\beta}u, \langle A\rangle^{-\beta}u)$$

and

$$(\langle\epsilon A\rangle^{-2}B_1 u_\epsilon, u_\epsilon) = (B_1\langle\epsilon A\rangle^{-1}u_\epsilon, \langle\epsilon A\rangle^{-1}u_\epsilon) + (T\langle A\rangle^{-\beta}u, \langle A\rangle^{-\beta}u).$$

Hence

$$(1-2\beta)(B_1 u_\epsilon, u_\epsilon) = -2\beta(B_1\langle\epsilon A\rangle^{-1}u_\epsilon, \langle\epsilon A\rangle^{-1}u_\epsilon) + (T\langle A\rangle^{-\beta}u, \langle A\rangle^{-\beta}u).$$

We use that $u = \Phi(L)u$ as in Lemma 2.4 to get

$$(B_1\langle\epsilon A\rangle^{-1}u_\epsilon, \langle\epsilon A\rangle^{-1}u_\epsilon) = (\Phi(L)i[L, A]\Phi(L)\langle\epsilon A\rangle^{-1}u_\epsilon, \langle\epsilon A\rangle^{-1}u_\epsilon) + (T\langle A\rangle^{-\beta}u, \langle A\rangle^{-\beta}u).$$

The Mourre inequality (suppose suppΦ small enough) provide

$$(1-2\beta)(B_1 u_\epsilon, u_\epsilon) \leq (T\langle A\rangle^{-\beta}u, \langle A\rangle^{-\beta}u).$$

Again:

$$(B_1 u_\epsilon, u_\epsilon) = (\Phi(L)i[L,A]\Phi(L)u_\epsilon, u_\epsilon) + (T\langle A\rangle^{-\beta}u, \langle A\rangle^{-\beta}u).$$

Then the Mourre inequality gives

$$||\Phi(L)u_\epsilon|| \leq C.$$

Commuting $\Phi(L)$ and $\langle \epsilon A\rangle^{-\beta}$ (by Lemma 1.4), we get

$$||\langle \epsilon A\rangle^{-\beta}u|| \leq C,$$

which gives $u \in \mathcal{H}$ and thus finishes the proof. ∎

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