

Scattering theory for N -body quantum systems in a time-periodic electric field

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1 Introduction

In this article, we study the scattering theory for N -body quantum systems in a time-periodic electric field. First we give the notations in the N -body scattering theory for describing our results. We consider a system of N particles moving in a given time-periodic electric field $\mathcal{E}(t) \in \mathbf{R}^d$, $\mathcal{E}(t) \neq 0$. Let m_j , e_j and $r_j \in \mathbf{R}^d$, $1 \leq j \leq N$, denote the mass, charge and position vector of the j -th particle, respectively. We suppose that the particles under consideration interact with one another through the pair potentials $V_{jk}(r_j - r_k)$, $1 \leq j < k \leq N$. We assume that these pair potentials are independent of time t . Then the total Hamiltonian for the system is given by

$$\tilde{H}(t) = \sum_{1 \leq j \leq N} \left\{ -\frac{1}{2m_j} \Delta_{r_j} - e_j \langle \mathcal{E}(t), r_j \rangle \right\} + V,$$

where $\langle \xi, \eta \rangle = \sum_{j=1}^d \xi_j \eta_j$ for $\xi, \eta \in \mathbf{R}^d$ and

$$V = \sum_{1 \leq j < k \leq N} V_{jk}(r_j - r_k).$$

We consider the Hamiltonian $\tilde{H}(t)$ in the center-of-mass frame. We equip $\mathbf{R}^{d \times N}$ with the metric $r \cdot \tilde{r} = \sum_{j=1}^N m_j \langle r_j, \tilde{r}_j \rangle$ for $r = (r_1, \dots, r_N)$ and $\tilde{r} = (\tilde{r}_1, \dots, \tilde{r}_N) \in \mathbf{R}^{d \times N}$ and denote $|r| = \sqrt{r \cdot \tilde{r}}$. Let X and X_{cm} be the configuration spaces for the inner motion of the particles and the center of mass motion, respectively:

$$X = \left\{ r \in \mathbf{R}^{d \times N} \mid \sum_{1 \leq j \leq N} m_j r_j = 0 \right\},$$

$$X_{\text{cm}} = \left\{ r \in \mathbf{R}^{d \times N} \mid r_j = r_k \text{ for } 1 \leq j < k \leq N \right\}.$$

X and X_{cm} are mutually orthogonal and $\mathbf{R}^{d \times N} = X \oplus X_{\text{cm}}$. We denote by $\pi : \mathbf{R}^{d \times N} \rightarrow X$ and $\pi_{\text{cm}} : \mathbf{R}^{d \times N} \rightarrow X_{\text{cm}}$ the orthogonal projections onto X and X_{cm} , respectively, and write $x = \pi r$ and $x_{\text{cm}} = \pi_{\text{cm}} r$ for $r \in \mathbf{R}^{d \times N}$, and

$$E(t) = \pi \left(\frac{e_1}{m_1} \mathcal{E}(t), \dots, \frac{e_N}{m_N} \mathcal{E}(t) \right), \quad E_{\text{cm}}(t) = \pi_{\text{cm}} \left(\frac{e_1}{m_1} \mathcal{E}(t), \dots, \frac{e_N}{m_N} \mathcal{E}(t) \right).$$

Then $\tilde{H}(t)$ is decomposed into

$$\tilde{H}(t) = H(t) \otimes Id + Id \otimes T_{\text{cm}}(t) \quad \text{on } L^2(X) \otimes L^2(X_{\text{cm}}),$$

where Id are the identity operators,

$$\begin{aligned} H(t) &= -\frac{1}{2}\Delta - E(t) \cdot x + V && \text{on } L^2(X), \\ T_{\text{cm}}(t) &= -\frac{1}{2}\Delta_{\text{cm}} - E_{\text{cm}}(t) \cdot x_{\text{cm}} && \text{on } L^2(X_{\text{cm}}), \end{aligned}$$

and Δ (resp. Δ_{cm}) is the Laplace-Beltrami operator on X (resp. X_{cm}). Throughout this article, we assume that there exists at least one pair (j, k) such that $e_j/m_j \neq e_k/m_k$. Under this assumption, when $\mathcal{E}(t) \neq 0$, we have $|E(t)| \neq 0$. We consider the Hamiltonian $H(t)$ which satisfies this assumption in this article.

A non-empty subset of the set $\{1, \dots, N\}$ is called a cluster. Let C_j , $1 \leq j \leq m$, be clusters. If $\cup_{1 \leq j \leq m} C_j = \{1, \dots, N\}$ and $C_j \cap C_k = \emptyset$ for $1 \leq j < k \leq m$, $a = \{C_1, \dots, C_m\}$ is called a cluster decomposition. We denote by $\#(a)$ the number of clusters in a . Let \mathcal{A} be the set of all cluster decompositions. Suppose $a, b \in \mathcal{A}$. If b is obtained as a refinement of a , that is, if each cluster in b is a subset of a cluster in a , we say $b \subset a$, and its negation is denoted by $b \not\subset a$. Any a is regarded as a refinement of itself. We identify the pair $\alpha = (j, k)$ with the $(N-1)$ -cluster decomposition $\{(j, k), (1), \dots, (\hat{j}), \dots, (\hat{k}), \dots, (N)\}$.

Next, for $a \in \mathcal{A}$, the two subspaces X^a and X_a of X are defined by

$$\begin{aligned} X^a &= \left\{ r \in X \mid \sum_{j \in C} m_j r_j = 0 \text{ for each cluster } C \text{ in } a \right\}, \\ X_a &= \left\{ r \in X \mid r_j = r_k \text{ for each pair } \alpha = (j, k) \subset a \right\}. \end{aligned}$$

In particular, for $\alpha = (j, k)$, X^α is the configuration space for the relative position of j -th and k -th particles. Hence we can write $V_\alpha(x^\alpha) = V_{jk}(r_j - r_k)$. These spaces are mutually orthogonal and span the total space $X = X^a \oplus X_a$, so that $L^2(X)$ is decomposed into the tensor product $L^2(X) = L^2(X^a) \otimes L^2(X_a)$. We also denote by $\pi^a : X \rightarrow X^a$ and $\pi_a : X \rightarrow X_a$ the orthogonal projections onto X^a and X_a , respectively, and write $x^a = \pi^a x$ and $x_a = \pi_a x$ for $x \in X$. The intercluster potential I_a is defined by

$$I_a(x) = \sum_{\alpha \not\subset a} V_\alpha(x^\alpha),$$

and the cluster Hamiltonian

$$H_a(t) = -\frac{1}{2}\Delta - E(t) \cdot x + V^a, \quad V^a(x) = V^a(x^a) = \sum_{\alpha \subset a} V_\alpha(x^\alpha),$$

governs the motion of the system broken into non-interacting clusters of particles. Let $E^a(t) = \pi^a E(t)$ and $E_a(t) = \pi_a E(t)$. Then the cluster Hamiltonian $H_a(t)$ acting on $L^2(X)$ is decomposed into

$$H_a(t) = H^a(t) \otimes Id + Id \otimes T_a(t) \quad \text{on } L^2(X^a) \otimes L^2(X_a),$$

where $H^a(t)$ is the subsystem Hamiltonian defined by

$$H^a(t) = -\frac{1}{2}\Delta^a - E^a(t) \cdot x^a + V^a \quad \text{on } L^2(X^a),$$

$T_a(t)$ is the free Hamiltonian defined by

$$T_a = -\frac{1}{2}\Delta_a - E_a(t) \cdot x_a \quad \text{on } L^2(X_a),$$

and Δ^a (resp. Δ_a) is the Laplace-Beltrami operator on X^a (resp. X_a).

Now we state the assumptions on the time-periodic electric field $\mathcal{E}(t)$ and the pair potentials. We suppose that $\mathcal{E}(t)$ is a \mathbf{R}^d -valued continuous function on \mathbf{R} , has its period $T > 0$, that is, $\mathcal{E}(t+T) = \mathcal{E}(t)$ for any $t \in \mathbf{R}$, and its average \mathcal{E} in time is non-zero, i.e.

$$\mathcal{E} = \frac{1}{T} \int_0^T \mathcal{E}(t) dt \neq 0.$$

We denote

$$E = \pi \left(\frac{e_1}{m_1} \mathcal{E}, \dots, \frac{e_N}{m_N} \mathcal{E} \right).$$

By the assumption that there exists at least one pair (j, k) such that $e_j/m_j \neq e_k/m_k$, we see that $E \neq 0$. We let c be a maximal element of the set $\{a \in \mathcal{A} \mid E^a = 0\}$ with respect to the relation \subset , where $E^a = \pi^a E$. Such a cluster decomposition uniquely exists and it follows that $E^a = 0$ if $\alpha \subset c$, and $E^a \neq 0$ if $\alpha \not\subset c$. Thus the potential V_α with $\alpha \not\subset c$ (resp. $\alpha \subset c$) describes the pair interaction between two particles with $e_j/m_j \neq e_k/m_k$ (resp. $e_j/m_j = e_k/m_k$). If, in particular, $e_j/m_j \neq e_k/m_k$ for any $j \neq k$, then c becomes the N -cluster decomposition. We put different assumptions on V_α according as $\alpha \not\subset c$ or $\alpha \subset c$. We consider the following conditions on the pair potentials:

(V)_{c,S} $V_\alpha(x^\alpha) \in C^\infty(X^\alpha)$, $\alpha \subset c$, is a real-valued function and has the decay property

$$|\partial_{x^\alpha}^\beta V_\alpha(x^\alpha)| \leq C_\beta \langle x^\alpha \rangle^{-(\rho' + |\beta|)},$$

with $\rho' > 1$.

(V)_{c,L} $V_\alpha(x^\alpha) \in C^\infty(X^\alpha)$, $\alpha \subset c$, is a real-valued function and has the decay property

$$|\partial_{x^\alpha}^\beta V_\alpha(x^\alpha)| \leq C_\beta \langle x^\alpha \rangle^{-(\rho' + |\beta|)},$$

with $\sqrt{3} - 1 < \rho' \leq 1$.

(V)_c $V_\alpha(x^\alpha) \in C^\infty(X^\alpha)$, $\alpha \not\subset c$, is a real-valued function and has the decay property

$$|\partial_{x^\alpha}^\beta V_\alpha(x^\alpha)| \leq C_\beta \langle x^\alpha \rangle^{-(\rho + |\beta|)/2},$$

with $\rho > 1$.

Under these assumptions, all the Hamiltonians defined above are essentially self-adjoint on C_0^∞ . We denote their closures by the same notations. If V_α , $\alpha \subset c$, satisfies the condition (V)_{c,S}, then V_α is called a short-range potential. And, if V_α , $\alpha \subset c$, satisfies the condition (V)_{c,L}, then V_α is called a long-range potential. We note that if V_α , $\alpha \not\subset c$, satisfies the condition (V)_c, then V_α should be called a ‘‘Stark short-range’’ potential.

To formulate the obtained results precisely, we define the usual and the modified wave operators. Let $U(t, s)$ and $\bar{U}_a(t, s)$, $a \subset c$, be the unitary propagators generated by the time-dependent Hamiltonians

$H(t)$ and $T_a(t)$, respectively, whose existence and uniqueness are guaranteed by virtue of the result of Yajima [Ya2] and the Avron-Herbst formula [CFKS]. Here the unitary propagator $U(t, s)$ generated by the time-dependent Hamiltonian $H(t)$ means the family of unitary operators $\{U(t, s)\}_{t, s \in \mathbf{R}}$ on $L^2(X)$ with the following properties:

- (1) $(t, s) \mapsto U(t, s)$ is strongly continuous.
- (2) $U(t, s) = U(t, r)U(r, s)$ holds for any $r, s, t \in \mathbf{R}$.
- (3) $U(t + T, s + T) = U(t, s)$ holds for any $s, t \in \mathbf{R}$.
- (4) For $\psi \in \mathcal{D}$,

$$\frac{d}{dt}U(t, s)\psi = -iH(t)U(t, s)\psi, \quad \frac{d}{ds}U(t, s)\psi = iU(t, s)H(s)\psi$$

hold, where \mathcal{D} is the common domain of $H(t)$.

Here we note that for $a \subset c$, $H^a(t)$ is independent of time t because of $E^a(t) \equiv 0$. Then we write it as H^a , and we put

$$U_a(t, s) = e^{-i(t-s)H^a} \otimes \bar{U}_a(t, s). \quad (1.1)$$

Under the assumptions $(V)_{c,S}$ and $(V)_{\bar{c}}$, we define the usual wave operators $W_a^\pm(s)$, $a \subset c$ and $s \in \mathbf{R}$, by

$$W_a^\pm(s) = s - \lim_{t \rightarrow \pm\infty} U(t, s)^* U_a(t, s) (P^a \otimes Id), \quad (1.2)$$

where $P^a : L^2(X^a) \rightarrow L^2(X^a)$ is the eigenprojection associated with H^a . On the other hand, we suppose that the assumptions $(V)_{c,L}$ and $(V)_{\bar{c}}$ are satisfied. We put

$$U_{a,D}(t, s) = U_a(t, s) e^{-i \int_s^t I_a^c(p_a u) du} \quad (1.3)$$

for $a \subset c$. Here $I_a^c = I_a - I_c$ and $p_a = -i\nabla_a$ is the velocity operator on $L^2(X_a)$. Now we define the modified wave operators $W_{a,D}^\pm(s)$, $a \subset c$, by

$$W_{a,D}^\pm(s) = s - \lim_{t \rightarrow \pm\infty} U(t, s)^* U_{a,D}(t, s) (P^a \otimes Id). \quad (1.4)$$

The main results of this article are the following two theorems:

Theorem 1.1. *Assume that $(V)_{c,S}$ and $(V)_{\bar{c}}$ are fulfilled. Let c be as above. Then the usual wave operators $W_a^\pm(s)$, $a \subset c$ and $s \in \mathbf{R}$, exist, and are asymptotically complete*

$$L^2(X) = \sum_{a \subset c} \oplus \text{Ran } W_a^\pm(s).$$

Theorem 1.2. *Assume that $(V)_{c,L}$ and $(V)_{\bar{c}}$ are fulfilled. Let c be as above. Then the modified wave operators $W_{a,D}^\pm(s)$, $a \subset c$ and $s \in \mathbf{R}$, exist, and are asymptotically complete*

$$L^2(X) = \sum_{a \subset c} \oplus \text{Ran } W_{a,D}^\pm(s).$$

Remark. As it follows from the proof below, one can be allowed to include the time-periodicity with the same period T as the one of the electric field $\mathcal{E}(t)$ in the pair potentials V_α with $\alpha \notin c$. But we do not consider such cases here.

The problem of the asymptotic completeness for N -body quantum systems has been studied by many mathematicians and they have achieved a great success. For N -body Schrödinger operators, this problem was first solved by Sigal-Soffer [SS] for a large class of short-range potentials, and some alternative proofs appeared (e.g. Graf [Gr2] and Yafaev [Y]). On the other hand, for the long-range case, Dereziński [D] solved this problem with arbitrary N for the class of potentials decaying like $O(|x^\alpha|^{-\rho})$ with some $\rho > \sqrt{3} - 1$ (see also Zielinski [Z]). Also the case of potentials decaying more slowly has been dealt with (see the references in [DG]). Also for N -body Stark Hamiltonians, satisfactory results of this problem have been obtained (see e.g. [AT1,AT2] and [HMS1,HMS2]). For other systems, see [DG]. These results are concerned with time-independent Hamiltonians.

On the other hand, for time-dependent Hamiltonians, the lack of energy conservation is a barrier in studying this problem. For instance, in [Gr1], the time-boundedness of the kinetic energy was the key fact for studying the charge transfer model. Howland [Ho1] proposed the stationary scattering theory for time-dependent Hamiltonians, whose formulation was the quantum analogue to the procedure in the classical mechanics in order to ‘recover’ the conservation of energy. Yajima [Ya1] applied this Howland method to the two-body quantum systems with time-periodic potentials and studied the problem of the asymptotic completeness for the systems under short-range assumptions (see also [Ho2]). His result was extended to the three-body case by Nakamura [N] later. As for the scattering theory in a time-periodic electric field, for instance, Kitada-Yajima [KY] dealt with the so-called AC Stark case for two-body quantum systems with long-range interactions. Recently Møller [Mø] studied the scattering theory for two-body quantum systems with short-range interactions in a time-periodic electric field whose average in time is non-zero, by using the so-called Howland-Yajima method. In his work, it seems to be important that he used the so-called Avron-Herbst formula (see [CFKS]) in order to remove the oscillating part of the electric field, and reduced the problem to the scattering problem for two-body Stark Hamiltonians with time-periodic potentials. This point of view motivates partly us to study the present problem.

The plan of this article is as follows: In §2, we reduce the present problem to the one which it is easier to deal with. The Howland-Yajima method plays an important role, combining the notion of the asymptotic clustering developed by ourselves and Tamura [AT1,AT2] for N -body Stark Hamiltonians. In §3, we state results on the spectral theory and propagation estimates for the Floquet Hamiltonian associated with this problem, which are obtained in [A3]. Finally, in §4, we prove Theorems 1.1 and 1.2.

2 Reduction of the problem

In this section, we reduce the problem under consideration to the one which it is easier to deal with.

Following the idea of Møller [Mø], we remove the oscillating part of the electric field, and reduce the present problem to the scattering problem for so-called N -body Stark Hamiltonians with time-periodic potentials. In removing the oscillating part of the field, we will use a version of the Avron-Herbst formula:

We define C^1 periodic functions on \mathbf{R}

$$\begin{aligned} b(t) &= \int_0^t (E(s) - E) ds - b_0, & b_0 &= \frac{1}{T} \int_0^T \int_0^t (E(s) - E) ds dt, \\ c(t) &= \int_0^t b(s) ds - c_0, & c_0 &= \frac{1}{T} \int_0^T \left(-\frac{1}{2}|b(t)|^2 + \int_0^t E \cdot b(s) ds \right) dt \frac{E}{|E|^2}, \\ a(t) &= \int_0^t \left(\frac{1}{2}|b(s)|^2 - E \cdot c(s) \right) ds, \end{aligned} \quad (2.1)$$

where $b(t), c(t) \in X$ and $a(t) \in \mathbf{R}$, and a strongly continuous periodic family of unitary operators on $L^2(X)$ by

$$\mathcal{T}(t) = e^{-ia(t)} e^{ib(t) \cdot x} e^{-ic(t) \cdot p}, \quad (2.2)$$

where $p = -i\nabla$ is the velocity operator on $L^2(X)$. Moreover we define the time-dependent Hamiltonian $H^S(t)$ by

$$H^S(t) = -\frac{1}{2}\Delta - E \cdot x + V(x + c(t)), \quad (2.3)$$

and define the time-independent Hamiltonian H_c^S by

$$H_c^S = -\frac{1}{2}\Delta - E \cdot x + V^c(x). \quad (2.4)$$

We note that the time-periodic potential $V(x + c(t))$ in the definition of the Hamiltonian $H^S(t)$ are written as

$$V(x + c(t)) = V^c(x) + I_c(x + c(t)), \quad (2.5)$$

because $c(t) \in X_c$ by definition and $V^c(x) = V^c(x^c)$ is independent of $x_c \in X_c$ also by definition. Let $\tilde{U}(t, s)$ be the unitary propagator generated by the Hamiltonian $H^S(t)$, whose existence and uniqueness are guaranteed by the result of Yajima [Ya2] and the Avron-Herbst formula [CFKS]. Then the Avron-Herbst formula which we use here is that

$$U(t, s) = \mathcal{T}(t) \tilde{U}(t, s) \mathcal{T}(s)^*, \quad U_c(t, s) = \mathcal{T}(t) e^{-i(t-s)H_c^S} \mathcal{T}(s)^*, \quad (2.6)$$

where we used the relationships (2.1) and (2.2).

In order to prove Theorems 1.1 and 1.2, we claim that the following theorem holds:

Theorem 2.1. (The Asymptotic Clustering) *Assume that $(V)_{c,S}$ or $(V)_{c,L}$, and $(V)_{\bar{c}}$ are fulfilled. Let $s \in \mathbf{R}$. Then the strong limits*

$$\hat{W}_c^\pm(s) = s - \lim_{t \rightarrow \pm\infty} U(t, s)^* U_c(t, s) \quad (2.7)$$

exist and are unitary on $L^2(X)$.

This property played an important role to prove the asymptotic completeness for N -body Stark Hamiltonians in the works of ourselves and Tamura [AT1,AT2] (see also [A1] and [HMS2]). As for the present problem, since the propagator $U_c(t, s)$ can be decomposed into

$$U_c(t, s) = e^{-i(t-s)H^c} \otimes \bar{U}_c(t, s), \quad (2.8)$$

we have only to study the scattering theory for the many body Schrödinger operator H^c by virtue of Theorem 2.1. Thus Theorems 1.1 and 1.2 can be proved, if we see that Theorem 2.1 holds. Now, using the above Avron-Herbst formula (2.6), Theorem 2.1 can be translated into the following theorem:

Theorem 2.2. (The Asymptotic Clustering) *Assume that $(V)_{c,S}$ or $(V)_{c,L}$, and $(V)_\varepsilon$ are fulfilled. Let $s \in \mathbf{R}$. Then the strong limits*

$$\tilde{W}_c^\pm(s) = s - \lim_{t \rightarrow \pm\infty} \tilde{U}(t, s)^* e^{-i(t-s)H_c^S} \quad (2.9)$$

exist and are unitary on $L^2(X)$.

Therefore the end of this article is to show that Theorem 2.2 holds. In order to prove Theorem 2.2, we follow the argument of Yajima [Ya1] (see also Howland [Ho1,Ho2]). We let $\mathbf{T} = \mathbf{R}/(\mathbf{T}\mathbf{Z})$ be the torus and introduce $\mathcal{H} = L^2(\mathbf{T}; L^2(X)) \cong L^2(\mathbf{T}) \otimes L^2(X)$. We define two families of operators $\{\hat{U}(\sigma)\}_{\sigma \in \mathbf{R}}$ and $\{\hat{U}_c(\sigma)\}_{\sigma \in \mathbf{R}}$ on \mathcal{H} by

$$(\hat{U}(\sigma)f)(t) = U(t, t - \sigma)f(t - \sigma), \quad (2.10)$$

$$(\hat{U}_c(\sigma)f)(t) = e^{-i\sigma H_c^S} f(t - \sigma), \quad (2.11)$$

for $f \in \mathcal{H}$. Then $\{\hat{U}(\sigma)\}_{\sigma \in \mathbf{R}}$ and $\{\hat{U}_c(\sigma)\}_{\sigma \in \mathbf{R}}$ form strongly continuous unitary groups on \mathcal{H} . Now one can write

$$\hat{U}(\sigma) = e^{-i\sigma K}, \quad \hat{U}_c(\sigma) = e^{-i\sigma K_c}, \quad (2.12)$$

where K and K_c are self-adjoint operators on \mathcal{H} . We call these self-adjoint operators K and K_c the Floquet Hamiltonians associated with the Hamiltonians $H^S(t)$ and H_c^S , respectively. From now on we denote the norm and scalar product in \mathcal{H} by $\|\cdot\|$ and (\cdot, \cdot) , respectively. We also denote the operator norm on \mathcal{H} by $\|\cdot\|$.

Proving Theorem 2.2 is equivalent to showing the following theorem, by virtue of the argument of Yajima [Ya1]:

Theorem 2.3. (The Asymptotic Clustering) *Assume that $(V)_{c,S}$ or $(V)_{c,L}$, and $(V)_\varepsilon$ are fulfilled. Then the strong limits*

$$\mathcal{W}_c^\pm = s - \lim_{\sigma \rightarrow \pm\infty} e^{i\sigma K} e^{-i\sigma K_c} \quad (2.13)$$

exist and are unitary on \mathcal{H} .

Now by assuming that Theorem 2.3 holds and the wave operators $\tilde{W}_c^\pm(s)$, $s \in \mathbf{R}$, exist, we prove the unitarity of them in Theorem 2.2. The existence of $\tilde{W}_c^\pm(s)$ is guaranteed by the argument similar to the ones of ourselves and Tamura [AT1,AT2], Herbst-Møller-Skibsted [HMS2] and ourselves [A2].

Proof under the assumption mentioned above.

First we note that the wave operators \mathcal{W}_c^\pm are the multiplication operators by $\tilde{W}_c^\pm(t)$. Let \mathcal{V} and \mathcal{V}_c be unitary operators on \mathcal{H} defined by

$$(\mathcal{V}f)(t) = \tilde{U}(t, s)f(t), \quad (\mathcal{V}_c f)(t) = e^{-i(t-s)H_c^S} f(t)$$

for $f \in \mathcal{H}$. By the unitarity of \mathcal{V} , we have

$$\mathcal{H} = \mathcal{V}\mathcal{H} = \mathcal{V}L^2(\mathbf{T}; L^2(X)). \quad (2.14)$$

On the other hand, letting $\hat{\mathcal{W}}_c^\pm$ be the multiplication operator by $\tilde{W}_c^\pm(s)$, we see that

$$\begin{aligned} \text{Ran } \mathcal{W}_c^\pm &= \text{Ran } \mathcal{V}\hat{\mathcal{W}}_c^\pm\mathcal{V}_c^* \\ &= \mathcal{V}\text{Ran } \hat{\mathcal{W}}_c^\pm \\ &= \mathcal{V}L^2(\mathbf{T}; \text{Ran } \tilde{W}_c^\pm(s)). \end{aligned} \quad (2.15)$$

By virtue of Theorem 2.3, comparing (2.14) with (2.15), we have

$$\text{Ran } \tilde{W}_c^\pm(s) = L^2(X),$$

which implies the unitarity of $\tilde{W}_c^\pm(s)$. □

Therefore we have only to study the scattering theory for the pair of the Floquet Hamiltonians K and K_c .

3 Mourre estimate and propagation estimates for K

In this section, we state results on the spectral theory and propagation estimates for the Floquet Hamiltonian K . Since the pages of this article are limited, we omit the proofs. As for the proofs, see [A3].

First of all, we claim the absence of bound states of the Floquet Hamiltonian K , which is a key fact on the spectral theory for K :

Theorem 3.1. (The Absence of Bound States) *Suppose that $(V)_{c,S}$ or $(V)_{c,L}$, and $(V)_{\bar{c}}$ are fulfilled. Then the pure point spectrum $\sigma_{pp}(K)$ of the Floquet Hamiltonian K is empty.*

Moreover, we obtain the following Mourre estimate for K .

Theorem 3.2. (The Mourre Estimate) *(1) Let $0 < \nu < |E| < \nu'$. Then one can take $\epsilon > 0$ so small uniformly in $\lambda \in \mathbf{R}$ that*

$$\eta_\epsilon(K - \lambda)i[K, A]\eta_\epsilon(K - \lambda) \geq \nu\eta_\epsilon(K - \lambda)^2, \quad (3.1)$$

$$\eta_\epsilon(K - \lambda)i[K, -A]\eta_\epsilon(K - \lambda) \geq -\nu'\eta_\epsilon(K - \lambda)^2 \quad (3.2)$$

hold.

(2) The spectrum of K is purely absolutely continuous.

These two results are closely related to ones due to Herbst-Møller-Skibsted [HMS1] for N -body Stark Hamiltonians.

Next we state some useful propagation estimates for K . Before stating them, we introduce the following smooth cut-off functions F with $0 \leq F \leq 1$: For sufficiently small $\delta > 0$, we define

$$\begin{aligned} F(s \leq d) &= 1 \quad \text{for } s \leq d - \delta, \quad = 0 \quad \text{for } s \geq d, \\ F(s \geq d) &= 1 \quad \text{for } s \geq d + \delta, \quad = 0 \quad \text{for } s \leq d, \end{aligned}$$

and $F(d_1 \leq s \leq d_2) = F(s \geq d_1) F(s \leq d_2)$. The choice of $\delta > 0$ does not matter to the argument below.

By virtue of the estimates (3.1) and (3.2), we obtain the following propagation estimates.

Theorem 3.3. *Let $f \in C_0^\infty(\mathbf{R})$. Then the following estimate holds as $\sigma \rightarrow \infty$:*

$$\left\| F\left(\left|\frac{p}{\sigma} - E\right| \geq \epsilon\right) e^{-i\sigma K} f(K) \langle z \rangle^{-1/2} \langle p \rangle^{-1} \langle D_t \rangle^{-1} \right\| = O(\sigma^{-1/2}), \quad (3.3)$$

$$\left\| F\left(\left|\frac{x}{\sigma^2} - \frac{E}{2}\right| \geq \epsilon\right) e^{-i\sigma K} f(K) \langle x \rangle^{-1} \langle p \rangle^{-2} \langle D_t \rangle^{-1} \right\| = O(\sigma^{-1/2}). \quad (3.4)$$

Theorem 3.4. *Let $0 < \epsilon < \min_{\alpha \in \mathcal{Q}_c} |E^\alpha|/2$. Put*

$$Z(\sigma) = F\left(\left|\frac{x}{\sigma^2} - \frac{E}{2}\right| \leq \epsilon\right) F\left(\left|\frac{p}{\sigma} - E\right| \leq \epsilon\right) e^{-i\sigma K} f(K) \langle x \rangle^{-1} \langle p \rangle^{-2} \langle D_t \rangle^{-1}.$$

Then we have as $\sigma \rightarrow \infty$

$$\| |p - E\sigma| Z(\sigma) \| = O(\sigma^{1/2}), \quad (3.5)$$

$$\left\| \left| x - \frac{E}{2}\sigma^2 \right| Z(\sigma) \right\| = O(\sigma^{3/2}). \quad (3.6)$$

These propagation estimates should be compared to ones due to ourselves [A2] for N -body Stark Hamiltonians. But, in the proofs, it is crucial that $\langle z \rangle^{-1/2} p(K + i)^{-1}$ is *not* bounded on \mathcal{H} . Here we note that $\langle z \rangle^{-1/2} p(H_0^S + i)^{-1}$ is bounded on $L^2(X)$, where $H_0^S = -\Delta/2 - E \cdot x$ is the free Stark Hamiltonian.

4 Proof of the asymptotic completeness

In this section, we prove Theorems 1.1 and 1.2.

First we prove Theorem 2.3.

Proof of Theorem 2.3. We have only to prove the existence of the adjoint of \mathcal{W}_c^\pm , that is,

$$s\text{-}\lim_{\sigma \rightarrow \pm\infty} e^{i\sigma K_c} e^{-i\sigma K}, \quad (4.1)$$

because one can prove the existence of the wave operators \hat{W}_c^\pm similarly, and this fact implies the unitarity of \hat{W}_c^\pm by a standard argument in the scattering theory. We consider the case $\sigma \rightarrow \infty$. Since the set

$\mathcal{D} = \{\psi \in \mathcal{H} | f(K)\psi = \psi \text{ for some } f \in C_0^\infty(\mathbf{R}) \text{ and } \langle D_t \rangle \langle p \rangle^2 \langle x \rangle \psi \in \mathcal{H}\}$ is dense in \mathcal{H} , it suffices to show the existence of the limit

$$\lim_{\sigma \rightarrow \infty} e^{i\sigma K_c} e^{-i\sigma K} \psi \quad (4.2)$$

for $\psi \in \mathcal{D}$. By virtue of (3.4), we see that

$$\lim_{\sigma \rightarrow \infty} e^{i\sigma K_c} \left\{ 1 - F \left(\left| \frac{x}{\sigma^2} - \frac{E}{2} \right| \leq \epsilon \right) \right\} e^{-i\sigma K} \psi = 0, \quad (4.3)$$

where we take $\epsilon > 0$ as $\epsilon < \min_{\alpha \in \mathcal{Z}_c} |E^\alpha|/2$. Moreover by (3.3), we have

$$\lim_{\sigma \rightarrow \infty} e^{i\sigma K_c} F \left(\left| \frac{x}{\sigma^2} - \frac{E}{2} \right| \leq \epsilon \right) \left\{ 1 - F \left(\left| \frac{p}{\sigma} - E \right| \leq \epsilon \right) \right\} e^{-i\sigma K} \psi = 0. \quad (4.4)$$

Thus we have only to show the existence of the limit

$$\lim_{\sigma \rightarrow \infty} e^{i\sigma K_c} F \left(\left| \frac{x}{\sigma^2} - \frac{E}{2} \right| \leq \epsilon \right) F \left(\left| \frac{p}{\sigma} - E \right| \leq \epsilon \right) e^{-i\sigma K} \psi. \quad (4.5)$$

We compute

$$\begin{aligned} & \frac{d}{d\sigma} \left(e^{i\sigma K_c} F \left(\left| \frac{x}{\sigma^2} - \frac{E}{2} \right| \leq \epsilon \right) F \left(\left| \frac{p}{\sigma} - E \right| \leq \epsilon \right) e^{-i\sigma K} \psi \right) \\ &= e^{i\sigma K_c} \left\{ \left(-\frac{2x}{\sigma^3} + \frac{p}{\sigma^2} \right) \cdot \left(\frac{x}{\sigma^2} - \frac{E}{2} \right) \left| \frac{x}{\sigma^2} - \frac{E}{2} \right|^{-1} \right. \\ & \quad F' \left(\left| \frac{x}{\sigma^2} - \frac{E}{2} \right| \leq \epsilon \right) F \left(\left| \frac{p}{\sigma} - E \right| \leq \epsilon \right) + O(\sigma^{-1}) F \left(\left| \frac{p}{\sigma} - E \right| \geq \epsilon \right) \\ & \quad \left. - F \left(\left| \frac{x}{\sigma^2} - \frac{E}{2} \right| \leq \epsilon \right) I_c(x + c(t)) F \left(\left| \frac{p}{\sigma} - E \right| \leq \epsilon \right) + O(\sigma^{-4}) \right\} e^{-i\sigma K} \psi, \end{aligned} \quad (4.6)$$

where we used $[V(x + c(t)), F(|p/\sigma - E| \leq \epsilon)] = O(\sigma^{-1})F(|p/\sigma - E| \geq \epsilon) + O(\sigma^{-\infty})$. Noting that $-2x/\sigma^3 + p/\sigma^2 = -2(x - E\sigma^2/2)/\sigma^3 + (p - E\sigma)/\sigma^2$, the property of ψ and

$$\left\| F \left(\left| \frac{x}{\sigma^2} - \frac{E}{2} \right| \leq \epsilon \right) I_c(x + c(t)) \right\| = O(\sigma^{-\rho})$$

with $\rho > 1$, by virtue of (3.3), (3.5) and (3.6), we have

$$\left\| \frac{d}{d\sigma} \left(e^{i\sigma K_c} F \left(\left| \frac{x}{\sigma^2} - \frac{E}{2} \right| \leq \epsilon \right) F \left(\left| \frac{p}{\sigma} - E \right| \leq \epsilon \right) e^{-i\sigma K} \psi \right) \right\| = O(\sigma^{-\min(3/2, \rho)}) \|\psi\|,$$

which implies the existence of the desired limit by Cook's method. Thus the theorem is proved. \square

Next we prove Theorem 2.2, which implies Theorem 2.1 as mentioned in §2. By the argument of §2, we have only to prove the existence of the wave operators $\tilde{W}_c^\pm(s)$, $s \in \mathbf{R}$ in (2.9). In order to prove their existence, we need some propagation properties of the evolution of the N -body Stark Hamiltonian H_c^S . Here we refer to the results in [A2], because one can compare Theorems 3.3 and 3.4 with them. One should also refer to [AT1, AT2] and [HMS2] about the propagation properties of $e^{-itH_c^S}$. We omit the proof of the following theorem (see [A2]).

Theorem 4.1. Suppose that $(V)_{c,S}$ or $(V)_{c,L}$, and $(V)_{\bar{c}}$ are fulfilled. Let $f \in C_0^\infty(\mathbf{R})$.

(1) Let $\epsilon > 0$ and $u > u' > 0$. Then the following estimates hold as $t \rightarrow \infty$:

$$\left\| F \left(\left| \frac{p}{t} - E \right| \geq \epsilon \right) e^{-itH_c^S} f(H_c^S) \langle x \rangle^{-u/2} \right\|_{B(L^2(X))} = O(t^{-u'}), \quad (4.7)$$

$$\left\| F \left(\left| \frac{x}{t^2} - \frac{E}{2} \right| \geq \epsilon \right) e^{-itH_c^S} f(H_c^S) \langle x \rangle^{-u/2} \right\|_{B(L^2(X))} = O(t^{-u'}). \quad (4.8)$$

(2) Let $0 < \epsilon < \min_{\alpha \in \mathcal{Q}_c} |E^\alpha|/2$. Then the following estimates hold as $t \rightarrow \infty$:

$$\left\| |p - Et| F \left(\left| \frac{x}{t^2} - \frac{E}{2} \right| \leq \epsilon \right) e^{-itH_c^S} f(H_c^S) \langle x \rangle^{-u/2} \right\|_{B(L^2(X))} = O(1), \quad u > 1, \quad (4.9)$$

$$\left\| \left| x - \frac{E}{2} t^2 \right| F \left(\left| \frac{x}{t^2} - \frac{E}{2} \right| \leq \epsilon \right) e^{-itH_c^S} f(H_c^S) \langle x \rangle^{-u/2} \right\|_{B(L^2(X))} = O(t), \quad u > 1. \quad (4.10)$$

Proof of Theorem 2.2. We prove the existence of $\tilde{W}_c^+(s)$ only. The existence of $\tilde{W}_c^-(s)$ can be proved similarly.

Since the set $\mathcal{D} = \{ \psi \in L^2(X) \mid f(H_c^S)\psi = \psi \text{ for some } f \in C_0^\infty(\mathbf{R}) \text{ and } \langle x \rangle^{u/2}\psi \in L^2(X) \text{ for some } u > 1 \}$ is dense in $L^2(X)$, it suffices to show the existence of the limit

$$\lim_{t \rightarrow \infty} \tilde{U}(t, s)^* e^{-i(t-s)H_c^S} \psi \quad (4.11)$$

for $\psi \in \mathcal{D}$. By virtue of (4.8), we see that

$$\lim_{t \rightarrow \infty} \tilde{U}(t, s)^* \left\{ 1 - F \left(\left| \frac{x}{(t-s)^2} - \frac{E}{2} \right| \leq \epsilon \right) \right\} e^{-i(t-s)H_c^S} \psi = 0. \quad (4.12)$$

Thus we have only to show the existence of the limit

$$\lim_{t \rightarrow \infty} \tilde{U}(t, s)^* F \left(\left| \frac{x}{(t-s)^2} - \frac{E}{2} \right| \leq \epsilon \right) e^{-i(t-s)H_c^S} \psi, \quad (4.13)$$

where we take ϵ as $0 < \epsilon < \min_{\alpha \in \mathcal{Q}_c} |E^\alpha|/2$. By virtue of (2.5), noting that $I_c(x + c(t))F(|x/(t-s)^2 - E/2| \leq \epsilon) = O((t-s)^{-\rho})$ with $\rho > 1$, we compute

$$\begin{aligned} & \frac{d}{dt} \left(\tilde{U}(t, s)^* F \left(\left| \frac{x}{(t-s)^2} - \frac{E}{2} \right| \leq \epsilon \right) e^{-i(t-s)H_c^S} \psi \right) \\ &= \tilde{U}(t, s)^* \left\{ \left(-\frac{2x}{(t-s)^3} + \frac{p}{(t-s)^2} \right) \cdot \left(\frac{x}{(t-s)^2} - \frac{E}{2} \right) \left| \frac{x}{(t-s)^2} - \frac{E}{2} \right|^{-1} \right. \\ & \quad \left. \times F' \left(\left| \frac{x}{(t-s)^2} - \frac{E}{2} \right| \leq \epsilon \right) + O((t-s)^{-\min(4, \rho)}) \right\} e^{-i(t-s)H_c^S} \psi. \end{aligned} \quad (4.14)$$

Noting that $\psi = f(H_c^S)\psi$ for some $f \in C_0^\infty(\mathbf{R})$, $\langle x \rangle^{u/2}\psi \in L^2(X)$ for some $u > 1$ and that $-2x/(t-s)^3 + p/(t-s)^2 = -2(x - E(t-s)^2/2)/(t-s)^3 + (p - E(t-s))/(t-s)^2$, by virtue of (4.9) and (4.10), we have

$$\left\| \frac{d}{dt} \left(\tilde{U}(t, s)^* F \left(\left| \frac{x}{(t-s)^2} - \frac{E}{2} \right| \leq \epsilon \right) e^{-i(t-s)H_c^S} \psi \right) \right\|_{L^2(X)} = O(t^{-\min(2, \rho)}) \|\langle x \rangle^{u/2}\psi\|_{L^2(X)},$$

which implies the existence of (4.13) by Cook's method, because of $\rho > 1$. Thus, combining this with the argument in §2, the proof of Theorem 2.2 is completed. \square

We have just obtained Theorem 2.1 as well as Theorem 2.2. Now we prove Theorems 1.1 and 1.2. Since their proofs are similar to each other, we prove Theorem 1.2 only. First we need the following theorem proved by Dereziński [D] (see also [DG] and [Z]), which is concerned with the asymptotic completeness for the subsystem Hamiltonian H^c . We omit the proof. We note that H^c is not a many body Stark Hamiltonian but an usual many body Schrödinger operator. Before mentioning its statement, we introduce some notations. Suppose $a \subset c$. We define the cluster Hamiltonian

$$H_a^c = -\frac{1}{2}\Delta^c + V^a$$

on $L^2(X^c)$ and put

$$U_{a,D}^c(t, s) = e^{-i(t-s)H_a^c} e^{-i \int_s^t I_a^c(p_a u) du},$$

which is acting on $L^2(X^c)$, where we noted the definition of I_a^c (see §1). We denote the orthogonal complement of X^a in X^c with respect to the metric \cdot by X_a^c . Then we have $X^c = X^a \oplus X_a^c$ and see that $L^2(X^c)$ is decomposed into the tensor product $L^2(X^a) \otimes L^2(X_a^c)$. Thus the cluster Hamiltonian H_a^c is decomposed into

$$H_a^c = H^a \otimes Id + Id \otimes T_a^c$$

on $L^2(X^c) = L^2(X^a) \otimes L^2(X_a^c)$, where $T_a^c = -\Delta_a^c/2$ and Δ_a^c is the Laplace-Beltrami operator on X_a^c . It follows from this that

$$U_{a,D}^c(t, s) = e^{-i(t-s)H^a} \otimes (e^{-i(t-s)T_a^c} e^{-i \int_s^t I_a^c(p_a u) du}) \quad (4.15)$$

on $L^2(X^c) = L^2(X^a) \otimes L^2(X_a^c)$.

Theorem 4.2. *Assume that $(V)_{c,L}$ and $(V)_\varepsilon$ are fulfilled. Then the modified wave operators*

$$\Omega_a^{c,\pm}(s) = s\text{-}\lim_{t \rightarrow \pm\infty} e^{i(t-s)H^c} U_{a,D}^c(t, s) (P^a \otimes Id)$$

acting on $L^2(X^c)$, $s \in \mathbf{R}$, exist for all $a \subset c$, and are asymptotically complete

$$L^2(X^c) = \sum_{a \subset c} \oplus \text{Ran } \Omega_a^{c,\pm}(s).$$

Proof of Theorem 1.2. We first prove the existence of the modified wave operators $W_{a,D}^\pm(s)$, $s \in \mathbf{R}$, in (1.4). Since we have seen the existence of $\hat{W}_c^\pm(s)$ in (2.7) by virtue of Theorem 2.1, by the chain rule, we have only to show the existence of the strong limits

$$s\text{-}\lim_{t \rightarrow \pm\infty} U_c(t, s) * U_{a,D}(t, s) (P^a \otimes Id) \quad (4.16)$$

for $a \subset c$ and $s \in \mathbf{R}$. By the definition of $T_a(t)$, we see that

$$T_a(t) = T_a^c \otimes Id + Id \otimes T_c(t)$$

on $L^2(X_a) = L^2(X_a^c) \otimes L^2(X_c)$. Thus $\bar{U}_a(t, s)$ in (1.1) is decomposed into

$$\bar{U}_a(t, s) = e^{-i(t-s)T_a^c} \otimes \bar{U}_c(t, s).$$

Combining this with (4.15), (4.16) is rewritten as

$$\begin{aligned} \text{s - } \lim_{t \rightarrow \pm\infty} U_c(t, s)^* U_{a,D}(t, s) (P^a \otimes Id) &= \text{s - } \lim_{t \rightarrow \pm\infty} e^{i(t-s)H^c} U_{a,D}^c(t, s) (P^a \otimes Id) \otimes Id \\ &= \Omega_a^{c,\pm}(s) \otimes Id \end{aligned}$$

on $L^2(X) = L^2(X^c) \otimes L^2(X_c)$. The existence of the right-hand side is guaranteed by Theorem 4.2. Thus the existence of the modified wave operators $W_{a,D}^\pm(s)$ is proved. The closedness and mutual orthogonality of their ranges can be easily seen. Finally we prove the asymptotic completeness. By Theorem 2.1, for any $\psi \in L^2(X)$, there exists $\psi_c^\pm \in L^2(X)$ such that

$$U(t, s)\psi = U_c(t, s)\psi_c^\pm + o(1), \quad t \rightarrow \pm\infty. \quad (4.17)$$

In fact, we have $\psi_c^\pm = \hat{W}_c^\pm(s)^*\psi$. On the other hand, $\psi_c^\pm \in L^2(X)$ is decomposed into

$$\psi_c^\pm = \sum_{j:\text{finite}} \psi_j^{c,\pm} \otimes \psi_c^{j,\pm} + O(\epsilon),$$

with $\psi_j^{c,\pm} \in L^2(X^c)$ and $\psi_c^{j,\pm} \in L^2(X_c)$. Then by virtue of Theorem 4.2, we have, by (4.17), as $t \rightarrow \pm\infty$

$$\begin{aligned} U(t, s)\psi &= \sum_{j:\text{finite}} e^{-i(t-s)H^c} \psi_j^{c,\pm} \otimes \bar{U}_c(t, s)\psi_c^{j,\pm} + O(\epsilon) + o(1) \\ &= \sum_{j:\text{finite}} \sum_{a \subset c} e^{-i(t-s)H^c} \Omega_a^{c,\pm}(s) \phi_{a,j}^{c,\pm} \otimes \bar{U}_c(t, s)\psi_c^{j,\pm} + O(\epsilon) + o(1) \\ &= \sum_{j:\text{finite}} \sum_{a \subset c} U_{a,D}^c(t, s) \phi_{a,j}^{c,\pm} \otimes \bar{U}_c(t, s)\psi_c^{j,\pm} + O(\epsilon) + o(1) \end{aligned}$$

for some $\phi_{a,j}^{c,\pm}$, whose existence is guaranteed by Theorem 4.2. This implies

$$\left\| \psi - \sum_{j:\text{finite}} \sum_{a \subset c} W_{a,D}^\pm(s) (\phi_{a,j}^{c,\pm} \otimes \psi_c^{j,\pm}) \right\|_{L^2(X)} = O(\epsilon).$$

Because $\epsilon > 0$ is arbitrary and $\sum_{a \subset c} \oplus \text{Ran } W_{a,D}^\pm(s)$ is closed, we see

$$\psi \in \sum_{a \subset c} \oplus \text{Ran } W_{a,D}^\pm(s).$$

This implies the asymptotic completeness. The proof of Theorem 1.2 is completed. \square

参考文献

- [A1] T. Adachi, Long-range scattering for three-body Stark Hamiltonians, *J. Math. Phys.* **35** (1994), 5547–5571.

- [A2] T. Adachi, Propagation estimates for N -body Stark Hamiltonians, *Ann. Inst. H. Poincaré Phys. Theor.* **62** (1995), 409–428.
- [A3] T. Adachi, Scattering theory for N -body quantum systems in a time-periodic electric field, to appear in *Funkcialaj Ekvacioj*.
- [AT1] T. Adachi and H. Tamura, Asymptotic Completeness for Long-Range Many-Particle Systems with Stark Effect, *J. Math. Sci., The Univ. of Tokyo* **2** (1995), 77–116.
- [AT2] T. Adachi and H. Tamura, Asymptotic Completeness for Long-Range Many-Particle Systems with Stark Effect, II, *Commun. Math. Phys.* **174** (1996), 537–559.
- [AH] J. E. Avron and I. W. Herbst, Spectral and scattering theory of Schrödinger operators related to the Stark effect, *Commun. Math. Phys.* **52** (1977), 239–254.
- [CFKS] H. Cycon, R. G. Froese, W. Kirsch and B. Simon, *Schrödinger Operators with Application to Quantum Mechanics and Global Geometry*, Springer-Verlag (1987).
- [D] J. Dereziński, Asymptotic completeness of long-range N -body quantum systems, *Ann. of Math.* **138** (1993), 427–476.
- [DG] J. Dereziński and C. Gérard, *Scattering theory of classical and quantum mechanical N -particle systems*, Springer-Verlag (1997).
- [FH] R. Froese and I. W. Herbst, Exponential bounds and absence of positive eigenvalues for N -body Schrödinger operators, *Commun. Math. Phys.* **87** (1982), 429–447.
- [Gr1] G. M. Graf, Phase space analysis of the charge transfer model, *Helv. Phys. Acta* **64** (1990), 107–138.
- [Gr2] G. M. Graf, Asymptotic completeness for N -body short-range quantum systems: a new proof, *Commun. Math. Phys.* **132** (1990), 73–101.
- [HeSj] B. Helffer and J. Sjöstrand, Equation de Schrödinger avec champ magnétique et équation de Harper, *Lecture Notes in Physics* **345** (1989), Springer-Verlag, 118–197.
- [H] I. W. Herbst, Temporal exponential decay for the Stark effect in atoms, *J. Funct. Anal.* **48** (1982), 224–251.
- [HMS1] I. W. Herbst, J. S. Møller and E. Skibsted, Spectral analysis of N -body Stark Hamiltonians, *Commun. Math. Phys.* **174** (1995), 261–294.
- [HMS2] I. W. Herbst, J. S. Møller and E. Skibsted, Asymptotic completeness for N -body Stark Hamiltonians, *Commun. Math. Phys.* **174** (1996), 509–535.
- [Ho1] J. S. Howland, Scattering theory for time-dependent Hamiltonians, *Math. Ann.* **207** (1974), 315–335.

- [Ho2] J. S. Howland, Scattering theory for Hamiltonians periodic in time, *Indiana Univ. Math. J.* **28** (1979), 471–494.
- [JY] A. Jensen and K. Yajima, On the long-range scattering for Stark Hamiltonians, *J. reine angew. Math.* **420** (1991), 179–193.
- [KY] H. Kitada and K. Yajima, Scattering theory for time-dependent long-range potentials, *Duke Math. J.* **49** (1982), 341–376.
- [Mø] J. S. Møller, Two-body short-range systems in a time-periodic electric field, *Duke Math. J.* **105** (2000), 135–166.
- [M] E. Mourre, Absence of singular continuous spectrum for certain self-adjoint operators, *Commun. Math. Phys.* **78** (1981), 391–408.
- [N] S. Nakamura, Asymptotic completeness for three-body Schrödinger equations with time-periodic potentials, *J. Fac. Sci. Univ. Tokyo Sect. IA, Math.* **33** (1986), 379–402.
- [P] P. A. Perry, *Scattering Theory by the Enss Method*, Math. Rep. Vol.1 (1983), Harwood Academic.
- [RS] M. Reed and B. Simon, *Methods of Modern Mathematical Physics I–IV*, Academic Press.
- [S] I. M. Sigal, Stark effect in multielectron systems: non-existence of bound states, *Commun. Math. Phys.* **122** (1989), 1–22.
- [SS] I. M. Sigal and A. Soffer, The N -particle scattering problem: asymptotic completeness for short-range systems, *Ann. of Math.* **125** (1987), 35–108.
- [Sk] E. Skibsted, Propagation estimates for N -body Schroedinger Operators, *Commun. Math. Phys.* **142** (1991), 67–98.
- [Y] D. Yafaev, Radiation conditions and scattering theory for N -particle Hamiltonians, *Commun. Math. Phys.* **154** (1993), 523–554.
- [Ya1] K. Yajima, Schrödinger equations with potentials periodic in time, *J. Math. Soc. Japan* **29** (1977), 729–743.
- [Ya2] K. Yajima, Existence of solutions for Schrödinger evolution equations, *Commun. Math. Phys.* **110** (1987), 415–426.
- [Yo] K. Yokoyama, Mourre theory for time-periodic systems, *Nagoya Math. J.* **149** (1998), 193–210.
- [Z] L. Zielinski, A proof of asymptotic completeness for N -body Schrödinger operators, *Commun. Partial Differ. Eqs.* **19** (1994), 455–522.