数理解析研究所講究録 1210 巻 2001 年 99-114

Asymptotic shape of a free-boundary arising in elliptic boundary value problem with non-homogeneous linearity

東京都立大理学研究科 柴田 将敬 (Masataka Shibata)
Depertment of Mathematics, Tokyo Metropolitan Univ.

1 Introduction and Main Results

In this paper, we consider the following partial differential equation with Dirichlet boundary condition

$$\begin{cases} \mu^2\Delta u + K(x)(u-1)_+^p = 0, u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a $C^{1,1}$ bounded domain in $\mathbf{R}^n$ $(n \geq 3)$, $\mu > 0$ and $1 < p < (n+2)/(n-2)$. We assume K is a nonnegative α-Hölder continuous function on $\overline{\Omega}$ for some $\alpha \in (0,1)$ and $K \not\equiv 0$. For a positive solution u, the set $A := \{x \in \Omega \mid u(x) > 1\}$ is called its core and its boundary is a free-boundary which is important in this problem.

This problem is a variant of a plasma confinement problem (see [12] for its physical background), which was studied by many authors. In [12, 13], the existence of solutions to (1.1) for the case $p = 1$ has been established by using an another equivalent formulation (see, e.g. [12, p54] for its equivalence between two problems). Actually, in [13] Temam obtained solutions as minimizers to a certain minimization problem (see [13, Section 1]). When $n = 2$, $p = 1$ and $K(x) \equiv 1$, for solutions obtained by [13], Caffarelli and Friedman studied in [2] a precise asymptotic location and a shape of the free-boundary as $\mu \to 0$. Especially, they showed that if μ is sufficiently small, then the core is approximated by a ball with the center x_μ, which converges to a harmonic center by passing to a subsequence if necessary, and its radius is comparable to μ. In [5], Flucher and Wei studied the problem (1.1) for the case $K \equiv 1$, $n \geq 3$, $1 < p < (n+2)/(n-2)$ and showed that if μ is sufficiently small a mountain pass solution u_μ and its core A_μ has a similar asymptotic behavior as the one obtained by [2]. The purpose of this paper is to study mountain pass solution u_μ to the problem (1.1) for general $K(x)$ and investigate the effect of K and the geometry of Ω on a concentration phenomenon of u_μ and an asymptotic location and a shape of its core.

Throughout this paper, we assume that Ω is a bounded $C^{1,1}$ domain in $\mathbf{R}^n$ with $n \geq 3$, $1 < p < (n+2)/(n-2)$ and K is a nonnegative α-Hölder continuous function in Ω with $K \not\equiv 0$. We use the notation:

$$\begin{aligned} g(t) &:= (t-1)_+^p, & f(t) &:= (t-1)_+^{p+1}/(p+1), \\ K_{\max} &:= \max_\Omega K, & \Omega_K &:= \{x \in \overline{\Omega} | K(x) = K_{\max}\} \\ B(x,r) &:= \{y \in \mathbf{R}^n | \, |y-x| < r\}, & B_r &:= B(0,r). \end{aligned}$$

We define the energy functional I_μ on $W_0^{1,2}(\Omega)$ by

$$I_\mu[u] := \frac{\mu^2}{2}\int_\Omega |\nabla u|^2\, dx - \frac{1}{p+1}\int_\Omega K(x)(u-1)_+^{p+1}\, dx.$$

Then a critical point u of I_μ is a solution to (1.1). We say that the critical point u is a least energy solution if u has the least energy among all nontrivial critical points, i.e. $I_\mu[u] \le I_\mu[v]$ and $I'_\mu[u] = 0$ for all $v \in W_0^{1,2}(\Omega)$ with $v \ne 0$ and $I'_\mu[v] = 0$. Actually, we can obtain a least energy solution u_μ to (1.1) for each $\mu > 0$ as a mountain pass solution (see Lemma 2.1). We denote the core of u_μ by A_μ, namely

$$A_\mu = \{x \in \Omega | u_\mu(x) > 1\}.$$

From now on, for each $\mu > 0$, we denote by u_μ a least energy solution to (1.1) obtained Lemma 2.1. We study the asymptotic shape of u_μ and its core A_μ. We state our first result in this paper,

Theorem A. *A least energy solution u_μ to* (1.1) *has the following properties:*

(i) If μ is sufficiently small, u_μ have only one local maximal point x_μ.

(ii) $\lim_{\mu\to 0} \mathrm{dist}(x_\mu, \Omega_K) = 0$.

(iii) There exist the unique constant R_1 (see Proposition 3.5) such that for all r, R with $r < R_1 < R$, $B(x_\mu, \mu r) \subset A_\mu = \{x \in \Omega | u_\mu(x) > 1\} \subset B(x_\mu, \mu R)$ holds and A_μ is convex, if μ is sufficiently small.

(iv) As $\mu \to 0$, the energy E_μ is asymptotically given by

$$E_\mu = I_\mu[u_\mu] = \mu^n \left\{ K_{\max}^{\frac{2-n}{2}} E_{0,1} + o(1) \right\},$$

where $E_{0,1}$ is a constant defined in Definition 2.1.

Theorem A says that if μ is sufficiently small, u_μ concentrate on a point x_μ which converges to $x_0 \in \Omega_K$ by taking subsequence if necessary, the core of u_μ is approximately a ball with the center x_μ and the radius μR_1.

However, if Ω_K contains more than one point the property (ii) above does not give us precise information on the behavior of the maximal point x_μ. For example, consider the case that K has exactly two maximal points x_1 and x_2, i.e. $\Omega_K = \{x_1, x_2\}$. To which point x_μ converges as $\mu \to 0$? Theorem A does not answer to this question. To answer this question, we need to compute a higher order asymptotic of the energy E_μ of u_μ as $\mu \to 0$. The geometry of Ω, namely the Robin function for Ω, plays an important role in the higher order asymptotic. The Robin function is defined by

$$t(x) := H_x(x),$$

where $H_x(y)$ is a solution of

$$\begin{cases} \Delta_y H_x(y) = 0 & \text{in } \Omega, \\ H_x(y) = (n-2)^{-1} |\partial B_1|^{-1} |x-y|^{2-n} & \text{on } \partial\Omega. \end{cases}$$

Here B_1 is a ball with radius 1. It is well-known that the Robin function $t(x)$ is a positive continuous function $t(x) \to \infty$ as $x \to \partial\Omega$. A minimal point of $t(x)$ is called a harmonic center. So there exists at least one harmonic center for any bounded domain Ω. For the details of the harmonic center, see e.g. [1]. Now we can answer to the question above. Under the situation $n = 3$, $K \in C^2(\Omega)$ and $t(x_1) < t(x_2)$ with $\Omega_K = \{x_1, x_2\}$, we can say x_μ converges to x_1 as $\mu \to 0$. This is a consequence of the following our main theorem. To state our main theorem, we need additional assumptions on $K(x)$:

$$\text{(K1)} \quad K \in C^2(\overline{\Omega}), \qquad \text{(K2)} \quad \Omega_K \cap \Omega \neq \emptyset.$$

We also use the notation: $\tilde{\Omega}_K = \{x \in \Omega_K | t(x) = \min_{\Omega_K} t\}$.

Theorem B (Main Theorem). *Suppose $n = 3$ and K satisfies the additional assumptions (K1), (K2). Then a least energy solution u_μ of* (1.1) *obtained by Lemma 2.1 has the following properties:*

(i) $\lim_{\mu \to 0} \operatorname{dist}(x_\mu, \tilde{\Omega}_K) = 0$.

(ii) The energy E_μ has the following precise asymptotic as $\mu \to 0$.

$$E_\mu = \mu^3 \left\{ K_{\max}^{-\frac{1}{2}} E_{0,1} + c_1 \mu \min_{\Omega_K} t + o(\mu) \right\}$$

Here $E_{0,1}$ is the same constant as in Theorem A and c_1 is a positive constant defined by $c_1 = \{|\partial B_1| R_{0,K_{\max}}\}^2/2$.

Theorem B is an extension of the result of [5]. In [5], they treated the case $K(x) \equiv 1$ and showed that a maximal point x_μ of u_μ converges to a harmonic center by passing to a subsequence if necessary. Furthermore, Theorem B has an application. Consider the case that $K \equiv 1$ and the Robin function $t(x)$ has exactly two local minimal points $x_[1]$ and x_2 with $t(x_1) < t(x_2)$. In this situation, the result of [5] implies that the maximal point x_μ of u_μ converges to x_1 as $\mu \to 0$ and u_μ concentrates near the point x_1. Can we construct a solution u of (1.1) which concentrates near the point x_2? We can answer to this question affirmatively by using Theorem B. We state this result in somewhat generalized form as Theorem C.

To state Theorem C, we define the local maximal (minimal) point and the local maximal (minimal) set. Assume g be a continuous function on $\overline{\Omega}$. We call x the local maximal point for g if there is a open neighborhood $U \subset \mathbf{R}^n$ of x such that x is maximal point of g on $U \cap \overline{\Omega}$ and $g(x) > g(y)$ for all $y \in \partial U \cap \overline{\Omega}$. Define the local minimal set V by $V := \{x \in \overline{\Omega} | x \text{is a local maximal point for} g\}$. Here, Similarly, we define the local minimal point and the local minimal set. we can find K is constant on each component of the local maximal (minimal) set. Now we state Theorem C.

Theorem C. *Suppose that $n = 3$, M is a component of the local minimal set for Robin function $t(x)$ and K is constant on some neighborhood of M. Then for any $\mu > 0$, there is a solution u_μ to*

$$\begin{cases} \mu^2 \Delta u + (u-1)_+^p = 0, u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

which satisfying following properties.

(i) If μ is sufficiently small, u_μ has only one maximal point x_μ.

(ii) $\lim_{\mu\to 0} \operatorname{dist}(x_\mu, M) = 0$.

(iii) There exists the constant R_1 such that for all r, R with $r < R_1 < R$, $B(x_\mu, \mu r) \subset A_\mu = \{x \in \Omega | u_\mu(x) > 1\} \subset B(x_\mu, \mu R)$ and A_μ is convex if μ is sufficiently small.

(iv) The energy E_μ have following asymptotic as $\mu \to 0$.

$$E_\mu = \quad \mu^n \left\{ K_M^{\frac{2-n}{2}} E_{0,1} + c_1 \mu^{n-2} \min_{\Omega_K} t + o(\mu^{n-2}) \right\}$$

Here $E_{0,1}$ is same constant as in Theorem A and c_1 is a positive constant defined by $c_1 = \{|\partial B_1|\, R_{0,K_M}\}^2 / 2$.

By using a similar strategy as in the proof of Theorem C, we can construct a solution of (1.1) which concentrates near a local maximal point of K.

Theorem D. *Suppose that M is a component of the local maximal set for K. Then for any $\mu > 0$, there is a solution u_μ which satisfies properties (i) - (iv) in Theorem A with changing from Ω_K to M.*

The paper is organized as follows. In Section 2, we prove the existence of a least energy solution and study the ground state to the corresponding problem on $\mathbf{R}^n$ and on a ball. In Section 3, we give the proof of Theorem A. In Section 4, we give the proof of Theorem B. In Section 5, we give the proof of Theorem C and D.

2 Preliminaries

In this section, we establish the existence of a least energy solution to (1.1) and prepare several facts about the ground state. First, we note the existence of a critical point of the mountain pass type for I_μ.

Lemma 2.1. *There exist $\eta_\mu \in W_0^{1,2}(\Omega)$ which satisfies $I_\mu[\eta_\mu] < 0$, and a mountain-pass solution u_μ such that*

$$I_\mu[u_\mu] = E_\mu > 0, I'_\mu[u_\mu] = 0.$$

Here

$$E_\mu := \inf_{\gamma \in \Gamma_\mu} \sup_{t \in [0,1]} I_\mu[\gamma(t)], \quad \Gamma_\mu := \left\{ \gamma \in C\big([0,1]; W_0^{1,2}(\Omega)\big) \;\middle|\; \gamma(0) = 0, \gamma(1) = \eta_\mu \right\}.$$

Note that the mountain-pass solution u_μ obtained above is non-trivial since $I_\mu[u_\mu] > 0$. By the following elementary lemma, the core A_μ of u_μ is non-empty.

Lemma 2.2. *If $u \in W_0^{1,2}(\Omega) \setminus \{0\}$ satisfies $I'[u] = 0$, then*

(i) $u \in C^{2,\alpha}(\Omega) \cap C^{1,\beta}(\overline{\Omega})$ for any $\beta \in (0,1)$.

(ii) $u > 0$ in Ω, $(Ku)_+ \not\equiv 0$.

(iii) $A = \{x \in \Omega; u(x) > 1\} \neq \emptyset$.

By using standard elliptic regularity theorems and the maximal principle, we can prove this lemma. We omit the proof of this lemma.

Next, we comment the mountain-pass solution u_μ is a least energy solution. To see this, we define M_μ by

$$M_\mu[v] := \sup_{t>0} I_\mu[tv]$$

for $v \in W_0^{1,2}(\Omega)$ with $(K(x)v)_+ \neq 0$. Note that $I_\mu[tv]$ has only one critical point in $(0,\infty)$, hence it is unique maximal point.

Lemma 2.3. *The energy E_μ of u_μ has following property:*

$$E_\mu = \inf\left\{M_\mu[v] \mid v \in W_0^{1,2}(\Omega), (K(x)v)_+ \neq 0\right\}.$$

Note that $I_\mu[v] = M_\mu[v]$, if $I'_\mu[v] = 0$. This lemma asserts the mountain-pass solution u_μ is a least energy solution since

$$E_\mu = I_\mu[u_\mu] = M_\mu[u_\mu] \leq M_\mu[v] = I_\mu[v]$$

for any critical point v of I_μ. Because of this, we call E_μ the least energy. We prepare several facts for the ground states on $\mathbf{R}^n$ and on a ball B_R. In the following Lemma, we denote by $u \in C^2(\overline{B_1})$ the unique positive solution of $\Delta u + u^p = 0$ in B_1 with Dirichlet zero boundary condition (see [6]).

Lemma 2.4. *Let $c > 0$, $R \in (0,\infty]$ be fixed constants. Then a function $v \in C^2(\overline{B_R})$ satisfying*

$$\begin{cases} \Delta v + c(v-1)_+^p = 0, v > 0 & \text{in } B_R, \\ v = 0 & \text{on } \partial B_R, \\ \left(\lim_{|x|\to\infty} v(x) = 0, \nabla v(0) = 0 \right. & \left. \text{if } R = \infty \right) \end{cases} \tag{2.3}$$

is radially symmetric about the origin, $v'(r) < 0$ for $r > 0$, exists uniquely, and can be expressed as

$$v(x) = \begin{cases} \left(\dfrac{2-n}{u'(1)}\right) \dfrac{R_c^{2-n}}{R_c^{2-n} - R^{2-n}} u\left(\dfrac{x}{R_c}\right) + 1 & (0 \leq |x| \leq R_c), \\ \dfrac{1}{R_c^{2-n} - R^{2-n}} (|x|^{2-n} - R^{2-n}) & (R_c < |x| < R). \end{cases} \tag{2.4}$$

Here R_c is the constant uniquely determined by

$$\left(\frac{1}{cR_c^2}\right)^{\frac{1}{p-1}} = \frac{2-n}{u'(1)} \frac{R_c^{2-n}}{R_c^{2-n} - R^{2-n}}.$$

The case $R = \infty$ of Lemma 2.4 was shown in Flucher-Wei [5]. Similarly, we can compute the case $R \in (0,\infty)$. we omit the proof of this lemma.

Definition 2.1 (ground state $w_{0,c}$). For a positive constant c, we denote by $w_{0,c}$ the unique solution of

$$\begin{cases} \Delta v + c(v-1)_+^p = 0, v > 0 & \text{in } \mathbf{R}^n , \\ \lim\limits_{|x|\to\infty} v(x) = 0, \nabla v(0) = 0. \end{cases}$$

And we define the energy $E_{0,c}$ of the ground state $w_{0,c}$ by

$$E_{0,c} := \frac{1}{2}\int_{\mathbf{R}^n} |\nabla w_{0,c}(x)|^2 \, dx - \int_{\mathbf{R}^n} cf\left(w_{0,c}(x)\right) dx.$$

It is easy to see expression $E_{0,c}$ by using $E_{0,1}$.

Lemma 2.5. *$w_{0,c}(x) = w_{0,1}(\sqrt{c}\,x)$ and $E_{0,c} = c^{(2-n)/2}E_{0,1}$ for all $c > 0$.*

Proof. Put $v(x) = w_{0,1}(\sqrt{c}\,x)$. Then v is a solution of $\Delta v + cg(v) = 0$ in $\mathbf{R}^n$, $v > 0$ in $\mathbf{R}^n$, $\lim_{|x|\to\infty} v(x) = 0$ and $\nabla v(0) = 0$. By Lemma 2.4, we obtain $v \equiv w_{0,c}$ in $\mathbf{R}^n$. It yields this lemma immediately. □

3 Proof of Theorem A

In this section, we give the proof of Theorem A. The following asymptotic formula is a key to the proof.

Proposition 3.1. *The least energy E_μ has the following asymptotic formula as $\mu \to 0$.*

$$E_\mu = \mu^n \left\{ [\max_{x\in\overline{\Omega}} K(x)]^{\frac{2-n}{2}} E_{0,1} + o(1) \right\}.$$

The idea to estimate the least energy E_μ is similar to [11].

Proof. Fix any $x_0 \in \Omega$ such that $K(x_0) > 0$. Take r_1, $r_2 > 0$ so that $U_1 = B(x_0, r_1) \subset \Omega \subset \overline{\Omega} \subset U_2 = B(x_0, r_2)$ and $\min_{x\in U_1} K(x) > 0$. Define $\underline{I}_\mu$ and $\overline{I}_\mu$ by

$$\underline{I}_\mu[u] := \frac{\mu^2}{2}\int_{U_1} |\nabla u|^2 \, dx - \int_{U_1} [\min_{U_1} K] f(u)\, dx \qquad (u \in W_0^{1,2}(U_1)),$$

$$\overline{I}_\mu[u] := \frac{\mu^2}{2}\int_{U_2} |\nabla u|^2 \, dx - \int_{U_2} K_{\max} f(u)\, dx \qquad (u \in W_0^{1,2}(U_2)).$$

Through a trivial extension, we may write $W_0^{1,2}(U_1) \subset W_0^{1,2}(\Omega) \subset W_0^{1,2}(U_2)$. Let $\underline{u}_\mu$, $\overline{u}_\mu$ be the mountain-pass solution of $\underline{I}_\mu$, $\overline{I}_\mu$ respectively. By the definition of $\overline{I}_\mu$ and Lemma 2.3, and $u_\mu \in W_0^{1,2}(\Omega) \subset W_0^{1,2}(U_2)$, we have $\max_{t>0} I_\mu[tu_\mu] \geq \max_{t>0} \overline{I}_\mu[tu_\mu]$. Applying Lemma 2.3 for $\overline{I}_\mu$, we find $\max_{t>0} \overline{I}_\mu[tu_\mu] \geq \overline{I}_\mu[\overline{u}_\mu]$. Consequently, we obtain $I_\mu[u_\mu] \geq \overline{I}_\mu[\overline{u}_\mu]$. Similarly we have $\max_{t>0} \underline{I}_\mu[\underline{u}_\mu] \geq I_\mu[u_\mu]$ and hence

$$\overline{I}_\mu[\overline{u}_\mu] \leq I_\mu[u_\mu] \leq \underline{I}_\mu[\underline{u}_\mu]. \tag{3.5}$$

We define $\underline{w}_\mu$ by $\underline{w}_\mu(y) := \underline{u}_\mu(x_0 + \mu y)$ in $U_{1,\mu} := B(0, r_1/\mu)$. Then $\underline{w}_\mu$ is a solution of

$$\begin{cases} \Delta \underline{w}_\mu + [\min_{x\in U_1} K(x)] g(\underline{w}_\mu) = 0, \underline{w}_\mu > 0 & \text{in } U_{1,\mu}, \\ \underline{w}_\mu = 0 & \text{on } U_{1,\mu}. \end{cases}$$

By using Lemma 2.4 and 2.5, we have

$$\frac{1}{2}\int_{U_{1,\mu}} |\nabla \underline{w}_\mu|^2 \, dy - \int_{U_{1,\mu}} \min_{x\in U_1} K(x) f(\underline{w}_\mu) \, dy = [\min_{x\in U_1} K(x)]^{\frac{2-n}{2}} E_{0,1} + o(1).$$

So, we obtain

$$\underline{I}_\mu[\underline{u}_\mu] = \mu^n \{[\min_{x\in U_1} K(x)]^{\frac{2-n}{2}} E_{0,1} + o(1)\}$$

Since the argument above is valid for all $B(x_0, r_1) \subset \Omega$ with $\min_{B(x_0,r_1)} K > 0$, we have

$$\overline{\lim_{\mu\to 0}}\, \mu^{-n} I_\mu[u_\mu] \le \inf_{B(x_0,r_1)\subset\Omega} [\min_{U_1} K]^{\frac{2-n}{2}} E_{0,1} = [K_{\max}]^{\frac{2-n}{2}} E_{0,1}.$$

Similarly, by using $\overline{u}_\mu$, we obtain

$$\underline{\lim}_{\mu\to 0}\, \mu^{-n} I_\mu[u_\mu] \ge \underline{\lim}_{\mu\to 0}\, \mu^{-n} \overline{I}_\mu[\overline{u}_\mu] = [K_{\max}]^{\frac{2-n}{2}} E_{0,1}.$$

Combining these estimates, we conclude this Proposition. $\square$

Let x_μ be the maximal point of u_μ and put $w_\mu(y) := u_\mu(x_\mu + \mu y)$, $\Omega_\mu := (\Omega - x_\mu)/\mu$, $K_\mu(y) := K(x_\mu + \mu y)$. Then w_μ is a solution of

$$\begin{cases} \Delta w_\mu + g(w_\mu) = 0, \quad w_\mu > 0 & \text{in } \Omega_\mu. \\ w_\mu = 0 & \text{on } \partial\Omega_\mu. \end{cases} \tag{3.6}$$

We consider where w_μ converges to as $\mu \to 0$. Define the energy functional J_μ by

$$J_\mu[v] := \frac{1}{2}\int_{\Omega_\mu} |\nabla w|^2 \, dy - \int_{\Omega_\mu} K_\mu f(w) \, dy.$$

By using (3.6), we have

$$\mu^{-n} I_\mu[u_\mu] = J_\mu[w_\mu] \ge \{\frac{1}{2} - \frac{1}{p+1}\} \int_{\Omega_\mu} |\nabla w_\mu|^2 \, dy.$$

So Proposition 3.1 asserts $\|\nabla w_\mu\|^2_{L^2(\Omega)}$ is uniformly bounded with respect to μ. By using Moser's iteration, we can find $\|w_\mu\|_{L^\infty(\Omega_\mu)}$ is uniformly bounded (see e.g. [9]). By using [7, Theorem 8.33], we obtain $\|\nabla w_\mu\|_{L^\infty(\Omega_\mu)}$ is uniformly bounded. So the Schauder estimate and Ascoli-Arzela's theorem assert that, by passing to a subsequence if necessary,

$$w_{\mu_j} \to W_0 \quad \text{in } C^2_{\text{loc}}(\underline{\lim}_{j\to\infty} \Omega_{\mu_j}) \text{ as } j \to \infty. \tag{3.7}$$

and $\Delta W_0 + K(\lim_{j\to\infty} x_{\mu_j}) g(W_0) = 0$ in $\underline{\lim}_{j\to\infty} \Omega_{\mu_j}$. By using these facts, we can show the following lemma (see e.g. [5],[11]).

Lemma 3.2. *Let x_μ be a local maximal point of the least energy solution u_μ. Then we have* $\lim_{\mu\to 0}\operatorname{dist}(x_\mu, \partial\Omega)/\mu = +\infty$.

This Lemma implies $\underline{\lim}_{\mu\to 0}\Omega_\mu = \mathbf{R}^n$.

Lemma 3.3. $\operatorname{dist}(x_\mu, \Omega_K) = 0$.

Proof. For any subsequence satisfying $\lim_{j\to\infty} x_{\mu_j} = x_0$, by using (3.7), we have $w_{\mu_j} \to w_{0,K(x_0)}$ in $C^2_{\mathrm{loc}}(\mathbf{R}^n)$ as $j \to \infty$. Hence, $\underline{\lim}_{j\to\infty} J_{\mu_j}[w_{\mu_j}] \geq E_{0,K(x_0)}$. Proposition 3.1 asserts $K(x_0) = K_{\max}$. So we can obtain this Lemma. □

Proposition 3.4. *u_μ has only one local maximal point if μ is sufficiently small.*

For proof this Proposition, see e.g. [5],[11]. The following proposition almost completes the proof of Theorem A.

Proposition 3.5. *Let x_μ be the unique maximal point of u_μ for sufficiently small μ. For any subsequence $\{\mu_j\}_{j=1}^\infty$ of $\mu \to 0$ with $\lim_{j\to\infty} x_{\mu_j} = x_0$, $x_0 \in \Omega_K$ holds. And for any constant r,R with $r < R_0 < R$, we have*

$$B(x_{\mu_j}, \mu_j r) \subset A_{\mu_j} = \{x \in \Omega; u_{\mu_j}(x) > 1\} \subset B(x_{\mu_j}, \mu_j R),$$

for sufficiently large j. Furthermore the free-boundary ∂A_{μ_j} is of class C^2 and the core A_{μ_j} is strictly convex. Here R_0 is the radius of $A_0 = \{x \in \mathbf{R}^n | W_0(x) > 1\}$.

Proof. A_{μ_j} has only one component if μ is sufficiently small, because each component has a maximal point and u_{μ_j} has only one maximal point if μ is small. By Proposition 3.4, $W_0(y)$ is radially symmetric and strictly decreasing, and hence there are unique s and t such that $s > 1 > t$ and

$$B_r = \{y \in \mathbf{R}^n | W_0(y) > s\} \subset A_0 \subset \{y \in \mathbf{R}^n | W_0(y) > t\} = B_R.$$

By $B_R \subset \Omega_{\mu_j}$ if μ is small, we have

$$w_{\mu_j} \to W_0 \qquad \text{in } C^2(\overline{B_R}) \text{ as } j \to \infty. \tag{3.8}$$

So, if μ_j is small, then $|w_{\mu_j} - W_0| \leq \min\{s-1, 1-t\}/2$ and

$$w_{\mu_j} > \frac{s+1}{2} > 1 \quad \text{in } B_r, \qquad w_{\mu_j} < \frac{t+1}{2} < 1 \quad \text{in } B_R^c.$$

Since A_{μ_j} has only one component,

$$B_r \subset \{y \in \Omega_{\mu_j} | w_{\mu_j}(y) > 1\} \subset B_R.$$

Hence $B(x_{\mu_j}, \mu_j r) \subset A_{\mu_j} \subset B(x_{\mu_j}, \mu_j R)$.

Next, we show that ∂A_{μ_j} is of class C^2 if μ is small. Since $W_0'(s) < 0$ on $(0,\infty)$, there exists $a > 0$ such that

$$|\nabla W_0(y)| = |W_0'(|y|)| > a \qquad \text{in } \overline{B_R} \setminus B_r.$$

As (3.8), $\left||\nabla W_0| - |\nabla w_{\mu_j}|\right| < a/2$ in $\overline{B_R}$ if μ_j is small. So we have $|\nabla w_{\mu_j}| > a/2$ in $\overline{B_R} \setminus B_r$. Especially $\nabla w_{\mu_j} \neq 0$ on ∂A_{μ_j}. Since w_{μ_j} is of class C^2, the implicit function theorem asserts that ∂A_{μ_j} is of class C^2 if μ_j is sufficiently small.

Finally, we show that A_{μ_j} is strictly convex if μ_j is sufficiently small. It follows from $A_{\mu_j} \subset B_R$ for all small μ_j, the principal curvature of ∂A_{μ_j} is determined by $D^2 w_{\mu_j}$ and (3.8) that A_{μ_j} is strictly convex for sufficiently small μ because of strict positivity of $D^2 W_0$. □

Proof of Theorem A. If Theorem A (ii) is not, i.e. there exist a subsequence $\{\mu_j\}_{j=1}^{\infty}$ of $\mu \to 0$ with $\lim \operatorname{dist}\{x_{\mu_j}, \partial\Omega_K\} = c > 0$. Then, by passing to a subsequence if necessary, we can assume $x_{\mu_j} \to x_0$ as $j \to \infty$. By using Proposition 3.5, we have $x_0 \in \Omega_K$. Hence $\lim_{j\to\infty} \operatorname{dist}\{x_{\mu_j}, \Omega_K\} = 0$ and it is contradiction. So Theorem A (ii) holds.

Similarly, we can prove Theorem A. □

4 Proof of Theorem B

To prove Theorem B, we need more precise asymptotic formula for E_μ as $\mu \to 0$. Throughout this section, we assume the assumptions (K1) and (K2) for $K(x)$.

4.1 Upper energy bound

First, we establish the following upper energy bound.

Proposition 4.1. *The least energy E_μ has the following estimate as $\mu \to 0$:*

$$E_\mu \le \mu^3 \left(K_{\max}^{-\frac{1}{2}} E_{0,1} + c_1 \mu \min_{s \in \Omega_K} t(x) + o(\mu) \right). \tag{4.9}$$

Here c_1 is a positive constant determined by $c_1 := \left\{|\partial B_1|\, R_0\right\}^2/2$.

Fix $x \in \Omega \cap \Omega_K$, $R > 0$ and put $K_\mu(y) := K(x + \mu y)$. Since (K2) and Lemma 3.2, we have

$$K_{\max} - K_\mu(y) = \nabla K(x) \cdot y + o(\mu) = o(\mu) \text{ in } B_{R/\mu} \text{ as } \mu \to 0.$$

By using this formula and similar strategy as in [5], we can prove Proposition 4.1. In this paper, we omit the proof of Proposition 4.1.

4.2 Lower energy bound

Next, we establish the following lower energy bound. Throughout this section 4.2, we use following notations:

$$\Omega_\mu := (\Omega - x_\mu)/\mu, \quad w_\mu(y) := u_\mu(x_\mu + y), \quad K_\mu(y) := K_{\mu, x_\mu}(y) = K(x_\mu + \mu y).$$

Remark. Here, we note that $\nabla K(x_\mu) \to 0$ in B_{2R_0} as $\mu \to 0$. Indeed, if not, there exist a subsequence $\{\mu_j\}_{j=1}^{\infty}$ such that $x_{\mu_j} \to x \in \Omega_K$ and it is contradiction. Hence,

$$K_\mu - K(x_\mu) = \mu y \cdot \nabla K(x_\mu) + o(\mu) = o(\mu) \text{ in } B_{2R_0} \text{ as } \mu \to 0. \tag{4.10}$$

Proposition 4.2. *Let $x_\mu \in \Omega$ be the unique maximal point of u_μ. Then we have the following asymptotic lower bound as $\mu \to 0$.*

$$E_\mu \geq \mu^3 \left\{ K_{\max}^{-\frac{1}{2}} E_{0,1} + c_1 t(x_\mu)\mu + o(t(x_\mu)\mu) \right\}.$$

Remark. In Proposition 4.2, $\mu t(x_\mu) \to 0$ as $\mu \to 0$ holds because of $t(x) \leq c/\operatorname{dist}(x, \partial\Omega)$ (see e.g. [1, p196]) and $\operatorname{dist}(x_\mu, \partial\Omega)/\mu \to \infty$ (see Lemma 3.2).

For the proof of Proposition 4.2, we approximate w_μ by using the unique solution v_μ to

$$\begin{cases} \Delta v_\mu + K(x_\mu) g(W_{0,\mu}) = 0 & \text{in } \Omega_\mu, \\ v_\mu = 0 & \text{on } \partial\Omega_\mu. \end{cases}$$

Here, $W_{0,\mu} := w_{0,K(x_\mu)}$ is the ground state defined in Definition 2.1. We put

$$\phi_\mu := \frac{w_\mu - v_\mu}{\mu}, \qquad g_\mu(\phi_\mu) := \frac{K_\mu g(w_\mu) - K(x_\mu) g(W_{0,\mu})}{\mu} - K(x_\mu) g'(W_{0,\mu})\phi_\mu.$$

Then, it follows from w_μ and v_μ tends to W_0 as $\mu \to 0$ and R_0 is a core of W_0 that

$$g_\mu(\phi_\mu) = 0 \qquad \text{in } B_{2R_0}^c. \tag{4.11}$$

For ϕ_μ, we have the following.

Lemma 4.3. *If μ is sufficiently small, then we have the following formulas:*

$$\begin{aligned} & w_\mu = W_{0,\mu} - b\mu K(x_\mu)^{-\frac{1}{2}} \mu h_{x_\mu} + \mu\phi_\mu, \\ & g_\mu(\phi_\mu) + bK(x_\mu)^{\frac{1}{2}} g'(W_{0,\mu}) h_{x_\mu} \\ & = \mu^{-1} \left\{ K_\mu g(w_\mu) - K(x_\mu) g(W_{0,\mu}) - K(x_\mu) g'(W_{0,\mu})(w_\mu - W_{0,\mu}) \right\}, \\ & \begin{cases} L_\mu \phi_\mu + g_\mu(\phi_\mu) = 0 & \text{in } \Omega_\mu, \\ \phi_\mu = 0 & \text{on } \partial\Omega_\mu. \end{cases} \end{aligned}$$

where $L_\mu := \Delta + K(x_\mu) g'(W_{0,\mu})$, b is the constant satisfying $k_0 = bK_{\max}^{-\frac{1}{2}}$, $h_{x_\mu}(y) := H_{x_\mu}(x_\mu + \mu y)$.

Proof. By the definition of w_μ, v_μ, ϕ_μ, $g_\mu(\phi_\mu)$ and L_μ, we have $L_\mu \phi_\mu = -g_\mu(\phi_\mu)$ in Ω_μ. Let $R(\mu)$ be the radius of the core of $W_{0,\mu}$, namely $(K(x_\mu)R(\mu)^2)^{-1/(p-1)} = (-1)/(u'(1))$ (see Lemma 2.4). By this formula and $K(x_\mu) \to K_{\max}$, we have $R(\mu) \to R_0$ as $\mu \to 0$. So, it follows from Lemma 3.2 that $R(\mu) \leq \operatorname{dist}(x_\mu, \partial\Omega)/\mu$ for sufficiently small $\mu > 0$. Therefore, $W_{0,\mu} - v_\mu$ satisfies

$$\begin{cases} \Delta(W_{0,\mu} - v_\mu) = 0 & \text{in } \Omega_\mu, \\ W_{0,\mu}(y) - v_\mu(y) = R(\mu)/|y| & \text{on } \partial\Omega_\mu. \end{cases}$$

by the definition of h_{x_μ}, we can obtain

$$v_\mu(y) = W_{0,\mu}(y) - k(\mu)\mu h_{x_\mu}(y) \quad y \in \Omega_\mu,$$

where $k(\mu) = R(\mu)\,|\partial B_1|$. Define b by $b = |\partial B_1|\,((-1)/u'(1))^{(p-1)/2} = k_0 K_{\max}^{\frac{1}{2}}$ then $k(\mu) = bK(x_\mu)^{-\frac{1}{2}}$ and we arrive at

$$w_\mu = \mu\phi_\mu + v_\mu = \mu\phi_\mu + W_{0,\mu} - bK(x_\mu)^{-\frac{1}{2}}\mu h_{x_\mu}$$

This formula yields this Lemma easily. □

Lemma 4.4. *There exists a positive constant C such that*

$$\left| g_\mu(\phi_\mu) + bK(x_\mu)^{\frac{1}{2}} g'(W_{0,\mu}) h_{x_\mu} \right| \le C\mu^{-1} |w_\mu - W_{0,\mu}|^{1+\sigma} + o(1) \text{ as } \mu \to 0. \tag{4.12}$$

Proof of Lemma 4.4. By using the mean value theorem, we have

$$\begin{aligned} &|g(w_\mu) - g(W_{0,\mu}) - g'(W_{0,\mu})(w_\mu - W_{0,\mu})| \\ &= |g'(\theta w_\mu + (1-\theta)W_{0,\mu})(w_\mu - W_{0,\mu}) - g'(W_{0,\mu})(w_\mu - W_{0,\mu})| \\ &\le |g'(\theta w_\mu + (1-\theta)W_{0,\mu}) - g'(W_{0,\mu})|\, |w_\mu - W_{0,\mu}| \end{aligned}$$

It is easy to see that $|g'(s) - g'(t)| < C\,|s-t|^\sigma$ on each bounded domain, where $\sigma = \min\{1, p-1\}$. Since w_μ and $W_{0,\mu}$ is uniformly bounded, we have

$$|g(w_\mu) - g(W_{0,\mu}) - g'(W_{0,\mu})(w_\mu - W_{0,\mu})| \le C\,|w_\mu - W_{0,\mu}|^{1+\sigma}.$$

By Lemma 4.3, we have $w_\mu - W_{0,\mu} = \mu(\phi_\mu - bK(x_\mu)^{-\frac{1}{2}} h_{x_\mu})$. By using (4.10), we obtain this Lemma. □

Lemma 4.5. *There exists a subsequence $\{\mu_j\}_{j=1}^\infty$ of $\mu \to 0$ such that*

$$\phi_{\mu_j} = bK(x_\mu)^{-\frac{1}{2}} t(x_{\mu_j})\,(\phi_0 + o(1)) \qquad (j \to \infty).$$

Here, the convergence is uniformly in $\mathbf{R}^3$ and ϕ_0 is the solution to

$$L_0\phi_0 = K_{\max} g'(w_{0,K_{\max}}) \text{ in } \mathbf{R}^3, \quad \phi_0 \in L^\infty(\mathbf{R}^3).$$

Here, $L_0 = \Delta + K_{\max} g'(w_{0,K_{\max}})$.

We will prove Lemma 4.5 at the end of this section. To prove Proposition 4.2, we use this Lemma.

Lemma 4.6.

$$\lim_{\mu\to 0} h_{x_\mu}/t(x_\mu) = 1 \qquad \text{in } C^0(\overline{B_{2R_0}}).$$

In particular, $\mu h_{x_\mu} = \mu t(x_\mu) + o(\mu t(x_\mu))$ as $\mu \to 0$ and $h_{x_\mu}/t(x_\mu)$ is uniformly bounded on B_{2R_0} for sufficiently small μ.

We can prove this Lemma by using similar way as in [1, p196]. Now we give the proof of Proposition 4.2.

Proof of Proposition 4.2. By Lemma 4.3, Lemma 4.6 and Lemma 4.5, we obtain

$$W_{0,\mu} - w_\mu = bK(x_\mu)^{-\frac{1}{2}}\mu t(x_\mu)(1-\phi_0) + o\left(\mu t(x_\mu)\right) = O\left(\mu t(x_\mu)\right) \text{ in } B_{2R_0}. \tag{4.13}$$

By using Lemma 4.4 and $K(x_\mu) - K(x) = o(\mu)$, we have

$$\begin{aligned} &K(x_\mu)\left|g(w_\mu) - g(W_{0,\mu}) - g'(W_{0,\mu})(w_\mu - W_{0,\mu})\right| \\ =&\left|K_\mu g(w_\mu) - K(x_\mu)g(W_{0,\mu}) - K(x_\mu)g'(W_{0,\mu})(w_\mu - W_{0,\mu})\right| + o(\mu) \\ \le& C\mu\left|w_\mu - W_{0,\mu}\right|^{1+\sigma} + o(\mu) = o\left(\mu t(x_\mu)\right). \end{aligned} \tag{4.14}$$

Note that $g(w_\mu) - g(W_{0,\mu}) = o(1)$ since both w_μ and $W_{0,\mu}$ tends to W_0 on B_{2R_0} as $\mu \to 0$. Then we have

$$\left(g(w_\mu) - g(W_{0,\mu})\right)\left(w_\mu - W_{0,\mu}\right) = o\left(\mu t(x_\mu)\right). \tag{4.15}$$

Since $K_\mu - K(x_\mu) = o(1)$ and $(w_\mu - 1)_+ = 0$ on $B^c_{2R_0}$, we obtain

$$\begin{aligned} \frac{E_\mu}{\mu^3} =& K(x_\mu)\int_{B_{2R_0}} \frac{1}{2}g(w_\mu)w_\mu - f(w_\mu)\,dy + o(\mu) \\ =&K(x_\mu)\int_{B_{2R_0}} \frac{1}{2}g(W_{0,\mu})W_{0,\mu} - f(W_{0,\mu})\,dy + \frac{K(x_\mu)}{2}\int_{B_{2R_0}} g(w_\mu)w_\mu - g(W_{0,\mu})W_{0,\mu}\,dy \\ &+ K(x_\mu)\int_{B_{2R_0}} f(W_{0,\mu}) - f(w_\mu)\,dy + o(\mu) =: (\mathrm{I}) + (\mathrm{II}) + (\mathrm{III}) + o(\mu). \end{aligned}$$

By Lemma 2.5, (I) $= K(x_\mu)^{-1/2}E_{0,1}$ holds. By (4.13), (4.14), we obtain

$$(\mathrm{II}) = \frac{bK(x_\mu)^{\frac{1}{2}}\mu t(x_\mu)}{2}\int_{B_{2R_0}} \left(g'(W_{0,\mu})W_{0,\mu} + g(W_{0,\mu})\right)(\phi_0 - 1)\,dy + o\left(\mu t(x_\mu)\right).$$

From the mean value theorem, (4.13) and (4.15), it follows that

$$(\mathrm{III}) = bK(x_\mu)^{\frac{1}{2}}\mu t(x_\mu)\int_{B_{2R_0}} g(W_{0,\mu})(1-\phi_0)\,dy + o\left(\mu t(x_\mu)\right).$$

Since $L_0\phi_0 = K_{\max}g'(w_{0,K_{\max}})$ by Lemma 4.5, it follows that

$$K_{\max}\int_{B_{2R_0}} g'(w_{0,K_{\max}})w_{0,K_{\max}}\,dy = K_{\max}\int_{B_{2R_0}} g'(w_{0,K_{\max}})w_{0,K_{\max}}\phi_0 - g(w_{0,K_{\max}})\phi_0\,dy.$$

Consequently, by noting $K_{\max} - K(x_\mu) = o(1)$ and $W_{0,\mu} - w_{0,K_{\max}} = o(1)$ again, we have

$$\begin{aligned} \frac{E_\mu}{\mu^3} =& K(x_\mu)^{-\frac{1}{2}}E_{0,1} + \frac{bK_{\max}^{\frac{1}{2}}\mu t(x_\mu)}{2}\int_{B_{2R_0}} g(w_{0,K_{\max}})\,dy + o(\mu t(x_\mu)) \\ \ge& K_{\max}^{-\frac{1}{2}}E_{0,1} + \frac{K_{\max}k_0\mu t(x_\mu)}{2}\int_{B_{2R_0}} g(w_{0,K_{\max}})\,dy + o(\mu t(x_\mu)). \end{aligned}$$

To prove Lemma 4.5, we prepare the following Lemma. For the proof of it, see [5].

Lemma 4.7. *For $q > 3$, we define L by $Lv = \Delta v + g'(w_{0,c})v$ for $v \in W^{2,q}(\mathbf{R}^3)$. Then we have the following formula*

$$\ker L = \mathrm{Span}\left\{ \frac{\partial w_{0,c}}{\partial x_1}, \dots, \frac{\partial w_{0,c}}{\partial x_3} \right\}.$$

Proof of Lemma 4.5. Put $\tilde{\phi}_\mu := \phi_\mu / \left(bK(x_\mu)^{-\frac{1}{2}} t(x_\mu)\right)$. By Lemma 4.3, we have

$$L_\mu \tilde{\phi}_\mu + \tilde{g}_\mu(\tilde{\phi}_\mu) = 0 \quad \text{in } \Omega_\mu, \qquad\qquad \tilde{g}_\mu(\tilde{\phi}_\mu) = 0 \quad \text{in } B^c_{2R_0}.$$

Here $\tilde{g}_\mu(\tilde{\phi}_\mu) = g_\mu(\phi_\mu)/\left(bK(x_\mu)^{-\frac{1}{2}}t(x_\mu)\right)$. Dividing (4.12) by $bK(x_\mu)^{-\frac{1}{2}}t(x_\mu)$, it follows

$$\left|\tilde{g}_\mu(\tilde{\phi}_\mu)\right| \leq C \left| \tilde{\phi}_\mu - \frac{h_{x_\mu}}{t(x_\mu)} \right| |w_\mu - W_{0,\mu}|^\sigma + o(1) + K(x_\mu) g'(W_{0,\mu}) \frac{h_{x_\mu}}{t(x_\mu)}.$$

Since $h_{x_\mu}/t(x_\mu)$ is a bounded function by Lemma 4.6, we can find

$$\left|\tilde{g}_\mu(\tilde{\phi}_\mu)\right| \leq C(\left|\tilde{\phi}_\mu\right| + 1)\, |w_\mu - W_{0,\mu}|^\sigma + C$$

for some constant $C > 0$. Now, we show the following claim.

Claim. $\|\tilde{\phi}_\mu\|_{L^\infty(\Omega_\mu)}$ is uniformly bounded for sufficiently small μ.

Put $M_\mu := \|\tilde{\phi}_\mu\|_{L^\infty(\Omega_\mu)}$ and suppose that there exist a subsequence $\{\mu_j\}_{j=1}^\infty$ of $\mu \to 0$ such that $M_{\mu_j} \to \infty$ as $j \to \infty$. Put $\overline{\phi}_{\mu_j} := \tilde{\phi}_{\mu_j}/M_{\mu_j}$. Then $\overline{\phi}_{\mu_j}$ satisfies the following properties:

$$L_\mu \overline{\phi}_{\mu_j} + \frac{\tilde{g}_{\mu_j}(\tilde{\phi}_{\mu_j})}{M_{\mu_j}} = 0 \quad \text{in } B_{2R_0}, \qquad\qquad \Delta \overline{\phi}_{\mu_j} = 0 \quad \text{in } \Omega_{\mu_j} \setminus B_{2R_0},$$

$$\left|\overline{\phi}_{\mu_j}\right| \leq 1, \left|\overline{\phi}_{\mu_j}(y)\right| \leq \frac{c}{|y|} \quad \text{in } \Omega_{\mu_j}. \qquad\qquad \overline{\phi}_{\mu_j} = 0 \quad \text{on } \partial\Omega_{\mu_j},$$

for some constant $C > 0$. Here we used the maximal principle to obtain the last inequality. Since $w_{\mu_j} - W_{0,\mu_j} \to 0$ on B_{2R_0} as $j \to \infty$ and

$$\frac{\tilde{g}_{\mu_j}(\tilde{\phi}_{\mu_j})}{M_{\mu_j}} \leq C \left(\left|\overline{\phi}_{\mu_j}\right| + \frac{1}{M_{\mu_j}} \right) \left|w_{\mu_j} - W_{0,\mu}\right|^\sigma + \frac{C}{M_{\mu_j}},$$

we obtain $\lim_{j\to\infty} \|\tilde{g}_{\mu_j}(\tilde{\phi}_{\mu_j})/M_{\mu_j}\|_{L^\infty(\Omega_{\mu_j})} = 0$. It follows from standard elliptic estimates that $\left|\overline{\phi}_{\mu_j}\right|_{1,\alpha;K} \leq C$ for each $\alpha \in (0,1)$ and $K \subset\subset \mathbf{R}^3$. By using Ascoli-Arzela's Theorem and the diagonal argument assert that, by passing to a subsequence if necessary, $\overline{\phi}_{\mu_j} \to \overline{\phi}_0$ in $C^1_{\mathrm{loc}}(\mathbf{R}^3)$. for some $\overline{\phi}_0 \in C^1(\mathbf{R}^3)$. By using the standard interior Schauder estimate, we obtain $\overline{\phi}_{\mu_j} \to \overline{\phi}_0$in $C^{2,\alpha}_{\mathrm{loc}}(\mathbf{R}^3)$. So we have

$$L_0 \overline{\phi}_0 = 0, \quad \left|\overline{\phi}_0\right| \leq 1, \quad \left|\overline{\phi}_0(y)\right| \leq c\,|y|^{-1} \quad \text{in } \mathbf{R}^3, \qquad \Delta\overline{\phi}_0 = 0 \quad \text{in } B^c_{2R_0}.$$

It follows that $\phi_0 \in W^{2,q}(\mathbf{R}^3)$ for some $q > 3$ and $\phi_0 \in \ker L_0$. So we can apply Lemma 4.7 to obtain that there exist constants a_1, a_2, a_3 such that $\overline{\phi}_0 = \sum_{i=1}^3 a_i \frac{\partial}{\partial y_i} W_{0,\mu}$. Consequently, we have $\nabla\overline{\phi}_0 = \sum_{i=1}^3 a_i \nabla \frac{\partial}{\partial y_i} W_{0,\mu}$.

Now, we show $\nabla\phi_0(O) = 0$. Recall that

$$\frac{W_{0,\mu_j} - v_{\mu_j}}{\mu_j bK(x_{\mu_j})^{-\frac{1}{2}} t(x_{\mu_j})} = \frac{h_{x_{\mu_j}}}{t(x_{\mu_j})}.$$

Here, the right hand side is uniformly bounded for μ in B_{2R_0} by Lemma 4.6. It follows from $\Delta(W_{0,\mu_j} - v_{\mu_j}) = 0$ in Ω_{μ_j} and the interior Schauder estimate that $\left|(W_{0,\mu_j} - v_{\mu_j})/(\mu_j bK(x_{\mu_j})^{-\frac{1}{2}} t(x_{\mu_j}))\right|_{2,\alpha;B_{2R_0}}$ is uniformly bounded for μ_j. Hence, there exists a constant $C > 0$ independent on μ_j such that

$$\left|\frac{\nabla W_{0,\mu_j}(O) - \nabla v_{\mu_j}(O)}{\mu_j bK(x_{\mu_j})^{-\frac{1}{2}} t(x_{\mu_j})}\right| < C.$$

Since $\nabla w_{\mu_j}(O) = \nabla W_{0,\mu_j}(O) = 0$, we have

$$\left|\frac{\nabla W_{0,\mu_j}(O) - \nabla v_{\mu_j}(O)}{\mu_j bK(x_{\mu_j})^{-\frac{1}{2}} t(x_{\mu_j})}\right| = \left|\frac{\nabla w_{\mu_j}(O) - \nabla v_{\mu_j}(O)}{\mu_j bK(x_{\mu_j})^{-\frac{1}{2}} t(x_{\mu_j})}\right| = \left|\nabla\tilde{\phi}_{\mu_j}(O)\right|.$$

Therefore, we obtain $\left|\nabla\overline{\phi}_{\mu_j}(O)\right| \le C/M_{\mu_j}$ which apples $\nabla\overline{\phi}_0(O) = 0$. It asserts that

$$0 = \nabla\overline{\phi}_0(O) = \sum_{i=1}^{3} a_i \nabla \frac{\partial}{\partial y_i} W_0(O).$$

Since W_0 is radially symmetric about the origin, $\frac{\partial^2}{\partial y_i^2} W_0(O) \neq 0$ and $\frac{\partial^2}{\partial y_i \partial y_j} W_0(O) \neq 0$ if $i \neq j$, It yields $0 = \sum_{i=1}^3 a_i e_i$. Here, e_i is some basis of $\mathbf{R}^3$. So we have $a_1 = a_2 = a_3 = 0$ hence $\overline{\phi}_0 \equiv 0$. On the other hand, $\overline{\phi}_{\mu_j}$ satisfies $\left|\overline{\phi}_{\mu_j}\right| \le 1$ on Ω_{μ_j} and $\left|\overline{\phi}_{\mu_j}\right| \le c\,|y|^{-1} < 1$ on $\Omega_{\mu_j} \setminus B_{2R_0}$. So we have $1 = \|\overline{\phi}_{\mu_j}\|_{L^\infty(\Omega_{\mu_j})} = \|\overline{\phi}_{\mu_j}\|_{L^\infty(B_{2R_0})}$. As $j \to \infty$, we obtain $\|\phi_0\|_{L^\infty(B_{2R_0})} = 1$ and it is a contradiction. Consequently, we establish the uniform boundedness of $\|\tilde{\phi}_\mu\|_{L^\infty(\Omega_\mu)}$.

Now, we continue the proof of Lemma 4.5. We write

$$\begin{aligned} \tilde{g}_\mu(\tilde{\phi}_\mu) + g'(W_{0,\mu})K(x_\mu) =& \frac{1}{bK(x_\mu)^{-\frac{1}{2}} t(x_\mu)} \left(g_\mu(\phi_\mu) + g'(W_{0,\mu})bK(x_\mu)^{-\frac{1}{2}} h_{x_\mu} K(x_\mu)\right) \\ &+ K(x_\mu) g'(W_{0,\mu}) \left(1 - \frac{h_{x_\mu}}{t(x_\mu)}\right) =: (\mathrm{I}) + (\mathrm{II}). \end{aligned}$$

It follows from Lemma 4.6 that $\lim_{\mu\to 0} (\mathrm{II}) = 0$. By using Lemma 4.4, we have

$$|(\mathrm{I})| \le C\left|\left|\tilde{\phi}_\mu\right| + C\right| |w_\mu - W_{0,\mu}|^\sigma + o(1) \qquad \text{in } B_{2R_0}.$$

Since $|\tilde{\phi}_\mu|$ is bounded and $w_\mu - W_{0,\mu} = o(1)$, we obtain (I) $\to 0$ as $\mu \to 0$. It asserts that

$$\lim_{\mu\to 0} -\tilde{g}_\mu(\tilde{\phi}_\mu) = g'(w_{0,K_{\max}})K_{\max} \qquad\qquad \text{in } C^0(\overline{B_{2R_0}}).$$

Note $\tilde{g}_\mu(\tilde{\phi}_\mu) = 0$ in $B^c_{2R_0}$ and by using the Schauder estimate, there exists ϕ_0 such that $\lim_{\mu\to 0}\tilde{\phi}_\mu = \phi_0$ in $C^2_{\text{loc}}(\mathbf{R}^3)$ and ϕ_0 satisfies

$$L_0\phi_0 = g'(W_{0,\mu})K_{\max}, \quad |\phi_0| \le c\,|y|^{-1} \quad \text{in } \mathbf{R}^3, \qquad \phi_0 \in L^\infty(\mathbf{R}^3).$$

Especially, we have $\lim_{\mu\to 0}\tilde{\phi}_\mu = \phi_0$ in $C^0\left(\overline{B_{2R_0}}\right)$. By using $\Delta(\tilde{\phi}_\mu - \phi_0) = 0$ in $\Omega_{\mu_j}\setminus B_{2R_0}$ and the maximal principle, we obtain

$$\sup_{\Omega_\mu\setminus B_{2R_0}} |\tilde{\phi}_\mu - \phi_0| \le \sup_{\partial\Omega_\mu\cup\partial B_{2R_0}} |\tilde{\phi}_\mu - \phi_0|.$$

It follows from $|\tilde{\phi}_\mu - \phi_0| \le 2c\,|y|^{-1}$ that $\lim_{\mu\to 0}\tilde{\phi}_\mu = \phi_0$ in $C^0(\mathbf{R}^3)$. It completes the proof of Lemma 4.5. □

The following Theorem 4.8 and Proposition 4.1, 4.2 completes the proof of Theorem B.

Theorem 4.8. *Let $\{\mu_j\}_{j=1}^\infty$ be subsequence of $\mu \to 0$ which satisfies $x_{\mu_j} \to x_0$ as $j \to \infty$. Then $t(x_0) = \min_{x\in\Omega_K} t(x)$ holds.*

Proof. By using proposition 4.1 and proposition 4.2, we obtain

$$\frac{E_\mu}{c_1\mu^3} - \frac{K_{\max}^{-\frac{1}{2}}E_{0,1}}{c_1} \ge \mu t(x_\mu) + o\left(\mu t(x_\mu)\right), \quad \frac{E_\mu}{c_1\mu^3} - \frac{K_{\max}^{-\frac{1}{2}}E_{0,1}}{c_1} \le \mu \min_{x\in\Omega_K} t(x) + o(\mu).$$

Put $\mu = \mu_j$ and taking the limit $j \to \infty$, we have $t(x_0) \le \min_{x\in\Omega_K} t(x)$. Since $x_0 \in \Omega_K$ by Lemma 3.3, $t(x_0) = \min_{x\in\Omega_K} t(x)$ holds. □

5 Proof of Theorem C,D

In this section, we give the proof of Theorem C and D.

Proof of Theorem C. Let $\tilde{K} \in C^2(\Omega)$ be satisfying $K \equiv 1$ on some neighborhood U of M and $K \equiv 1/2$ on all other local minimal points. From Theorem B, for each μ, there exist a solution u_μ of $\mu^2\Delta u + \tilde{K}(x)(u-1)_+^p = 0$, $u > 0$ in Ω, $u = 0$ on $\partial\Omega$, which satisfying (i), (ii), (iii) and (iv) of this Lemma. It follows from (ii) and (iii) that the core of u_μ is contained in U for sufficiently small μ. So we obtain $\mu^2\Delta u + (u-1)_+^p = 0$ in Ω and it completes the proof of this Lemma. □

Proof of Theorem D. By using similar way to the proof of Theorem C and by using Theorem A, we can prove Theorem D. □

References

[1] C. Bandle and M. Flucher. Harmonic radius and concentrarion of energy; hyperbolic radius and Liouville's equations $\Delta U = e^U$ and $\Delta U = U^{\frac{n+2}{n-2}}$. *SIAM Review*, 38(2):191–238, 1996.

[2] Luis A. Caffarelli and Avner Friedman. Asymptotic estimates for the plasma problem. *Duke Math. J.*, 47(3):705–742, 1980.

[3] Ya-Zhe Chen and Lan-Cheng Wu. *Second Order Elliptic Equations and Elliptic Systems.* Number 174 in Translations of Mathematical Monographs. American Mathematical Society, 1998.

[4] M.J. Esteban and P. L. Lions. Existence and non-existence results for semilinear elliptic problems in unbounded domains. *Proceedings of the Royal Society of Edinburgh*, 93A:1–14, 1982.

[5] M. Flucher and J. Wei. Asymptotic shape and location of small cores in elliptic free-boundary problems. *Math. Zeitschr.*, 228:683–703, 1998.

[6] B. Gidas, Wei-Ming Ni, and L. Nirenberg. Symmetry and related properties via the maximam principle. *Comm. Math Phys.*, 68:209–243, 1979.

[7] D. Gilberg and N. S. Trudinger. *Elliptic Partial Differential Equations of Second Order.* Springer-Verlag, second edition, 1983.

[8] Yi Li and Wei-Ming Ni. Radial symmetry of positive solutions of nonlinear elliptic equations in $\mathbf{R}^n$. *Comm. in Partial Differential Equations*, 18:1043–1054, 1993.

[9] C.-S. Lin, W.-M. Ni, and I. Takagi. Large amplitude stationary solutions to a chemotaxis system. *J. Diff. Equations*, 72:1–27, 1988.

[10] P. H. Rabinowitz. *Minimax Methods in Critical Point Theory with Applications to Differential Equations.* Number 65 in CBMS Regional Conference Series in Mathematics. American Mathematical Society, 1986.

[11] Xiaofeng Ren. Least-energy solutions to a non-autonomous semilinear problem with small diffusion. *Electronic J. of Diff. Eq.*, pages 1–21, 1993.

[12] R. Temam. A nonlinear eigenvalue problem: The shape at equilibrium of a confined plasma. *Arch. Rat. Mech. and Analysis*, 60:51–73, 1975.

[13] R. Temam. Remarks on a free boundary value problem arising in plasma physics. *Comm. in Partial Differential Equations*, 2:563–585, 1977.