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A phase field system with memory: global existence *

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Abstract. In the present note we discuss a phase field system with memory:

$$(\mathbf{PFM})\begin{cases} u_t + \frac{l}{2}\phi_t = \int_{-\infty}^{t} a_1(t-s)\,\Delta u(s)\,ds \quad (x,t)\in\Omega\times(0,T), \\ \tau\phi_t = \int_{-\infty}^{t} a_2(t-s)\,[\xi^2\Delta\phi + \frac{\phi-\phi^3}{\eta} + u](s)\,ds \quad (x,t)\in\Omega\times(0,T), \\ \mathbf{n}\cdot\nabla u = \mathbf{n}\cdot\nabla\phi = 0 \quad (x,t)\in\partial\Omega\times(0,T), \\ u(x,0) = u_0(x), \quad \phi(x,0)=\phi_0(x) \quad x\in\Omega, \end{cases}$$

for $T > 0$, which has been proposed [18] as a phenomenological model to describe phase transitions in the presence of slowly relaxing internal variables. The system yields motion by mean curvature with memory under suitable assumptions in a sharp interface limit. We outline here a proof of global existence of a solution $(u, \phi) \in C([0, T]; L^2(\Omega) \times H^1(\Omega))$ for **(PFM)** assuming that Ω is a smooth bounded domain in R^n, $n = 1, 2,$ or 3, the kernels $a_1, a_2 \in L^1(R^+)$ are of positive type, the initial data is in $L^2(\Omega) \times H^1(\Omega)$, and the history is in $L^1(-\infty, 0; H^2(\Omega))$ and $L^1(-\infty, 0; H^3(\Omega)) \cap L^5(-\infty, 0; L^6(\Omega))$, respectively. Our methodology combines results from the theory of Volterra integral equations with Galerkin methods and energy estimates.

1 Introduction

A proof of global existence is outlined of a $C([0, T]; L^2(\Omega) \times H^1(\Omega))$, $T > 0$, solution for the system:

$$u_t + \tfrac{l}{2}\phi_t = a_1 * \Delta u + f_1 \qquad (x,t)\in\Omega\times(0,T), \tag{1.1}$$
$$\tau\phi_t = a_2 * [\xi^2\Delta\phi + \tfrac{\phi-\phi^3}{\eta} + u] + f_2 \qquad (x,t)\in\Omega\times(0,T), \tag{1.2}$$
$$\mathbf{n}\cdot\nabla u = \mathbf{n}\cdot\nabla\phi = 0 \qquad (x,t)\in\partial\Omega\times(0,T), \tag{1.3}$$
$$u(x,0) = u_0(x), \quad \phi(x,0) = \phi_0(x) \qquad x\in\Omega, \tag{1.4}$$

where $u = u(x, t)$ represents a dimensionless temperature and $\phi = \phi(x, t)$ is a nonconserved order parameter. For further details, see [14]. The constant l is a dimensionless latent

* **Key words:** phase field equations, integro-differential equations, Galerkin methods, phase transitions.

heat, τ is a dimensionless relaxation time, ξ is a dimensionless interaction length, and η is a dimensionless potential well depth. Equation (1.1) constitutes an energy balance equation, and equation (1.2) is a type of phase relaxation equation. The underlying constitutive assumption here is that the system responds in a delayed or time averaged fashion to thermal gradients and to deviations from equilibrium [18]. We shall assume Ω to be a bounded domain in R^n, $n = 1, 2$, or 3, with a sufficiently smooth boundary, and we shall take the initial data $\{u_0, \phi_0\}$ to be prescribed in $L^2(\Omega) \times H^1(\Omega)$.

In (1.1)–(1.2), the first terms on the right hand side are convolution terms, and

$$(a_i * \Psi)(t) := \int_0^t a_i(t-s)\,\Psi(s)\,ds \quad i = 1, \text{ or } 2,$$

for $0 \le t < T$, $\Psi \in L^p(0, T; L^p(\Omega))$, $1 \le p \le \infty$. Here a_i, $i = 1, 2$ act as "memory kernels" mediating the delayed or averaged response of the system. With regard to the memory kernels, we shall assume throughout that $a_i \in L^1(R^+)$, $i = 1, 2$, and that the kernels a_i are of *positive type.*

Definition 1 *A kernel a is said to be of positive type on the interval* $[0, T]$ *for* $T > 0$ *if* $a \in L^1(0, T)$, *and*

$$\int_0^T < \psi, a_i * \psi > dt \ge 0, \quad \forall\, \psi \in L^2(0, T;\, L^2(\Omega)). \tag{1.5}$$

The terms f_1 and f_2 reflect the influence of the "history" of the system. We shall assume that $\{f_1, f_2\} \in L^1(R^+;\, L^1(\Omega) \times H^1(\Omega))$. It shall be assumed, moreover, that f_1 and f_2 are of the form:

$$f_1(x, t) = \int_{-\infty}^0 a_1(t-s)\,\Delta u(x, s)\,ds \qquad (x, t) \in \Omega \times [0, T] \tag{1.6}$$

and

$$f_2(x, t) = \int_{-\infty}^0 a_2(t-s)\,[\xi^2 \Delta\phi + \frac{\phi - \phi^3}{\eta} + u](x, s)\,ds \qquad (x, t) \in \Omega \times [0, T], \tag{1.7}$$

respectively, where

$$u(x, t) = u_h(x, t) \text{ and } \phi(x, t) = \phi_h(x, t) \quad (x, t) \in \Omega \times (-\infty, 0)$$

for prescribed functions u_h and ϕ_h, the history, where

$$u_h \in L^1(-\infty, 0; H^2(\Omega)) \text{ and } \phi_h(x, t) \in L^1(-\infty, 0; H^3(\Omega)) \cap L^5(-\infty, 0; L^6(\Omega)).$$

Thus (1.1)–(1.2) may be written equivalently as they appear in (**PFM**). We remark here that though the effects of possible body heating and boundary heating have been neglected for simplicity in **(PFM)**, they can be included in the system (1.1)–(1.4) by incorporating appropriate forcing terms into f_1 and f_2, and the analysis which we present here may be suitably modified accordingly. See for example the discussion in [13], where possible boundary heating was taken into account.

Note that if the kernels are chosen as $a_i(t) = \alpha_i\,\delta(t)$, where α_i is a constant, then the system **(PFM)** reduces to the classical phase field equations which were first treated by Caginalp [3] and which have their roots in Landau-Ginzburg theory [11, 10]. Classical phase field equations were designed to describe nonisothermal phase transitions, and the literature treating their analysis and predictions is vast. We only remark here that existence and uniqueness results for the classical phase field equations are given in Bates & Zheng [2] for initial data in $L^2(\Omega) \times H^1(\Omega)$.

There is also a large literature concerning phase field equations in which memory effects have been included in the energy balance equation, but not in the phase relaxation equation. The rationale behind the inclusion of the memory effects in the energy balance equation is to rid the system of the anomaly of infinite speed of heat propagation, essentially incorporating a Gurtin-Pipkin type formulation [9] for memory effects into an energy balance equation in a phase field setting. With regard to papers which have been published treating such systems, we note that existence was first proven by Aizicovici & Barbu [1] under somewhat restrictive assumptions and for Dirichlet boundary conditions. In [6] existence of weak solutions is proven assuming the initial data to be in $L^2(\Omega) \times H^1(\Omega)$, the thermal history in $L^1(0,\infty; L^2(\Omega))$ and the thermal memory kernel to belong to $L^1(0,\infty)$ and to be of positive type. In [7], uniqueness is established under similar assumptions.

The present formulation essentially constitutes a phenomenological extension of the classical phase field equations in which memory effects are taken into account both in the energy balance equation and in the phase relaxation equation. The rationale for including memory effects in the phase relaxation equation is to take into account in an averaged way the presence of slowly relaxing "internal variables" which are troublesome to represent explicitely. Such internal variables could represent for example configurational degrees of freedom which are important in polymer melts during phase separation in the proximity of the glass transition temperature. Equation (1.2) can be seen to have the structure

$$\tau\phi_t = -\int_{\infty}^{t} a_2(t-s)\frac{\delta\mathcal{F}(u,\phi)}{\delta\phi}(s)ds, \tag{1.8}$$

where $\mathcal{F}(u,\phi)$ is an appropriately defined free energy, as opposed to classical phase relaxation which has the form

$$\tau\phi_t = -\frac{\delta\mathcal{F}(u,\phi)}{\delta\phi}.$$

While the effects of delayed or averaged response in the energy balance equation are predicted to be noticeable under extreme thermal conditions – such as very very high temperatures or very very low temperatures, the effects of delayed or averaged response in the phase relaxation equation should not require such extreme conditions to be influential. For a lengthier discussion of the derivation and the implications of (1.8), see [18]. We believe that we present here the first proof of existence for such a system.

To gain intuition into our expectations from the system (**PFM**), it is possible to consider the long time behavior. In terms of general analysis, a discussion of attractors and inertial sets for the classical phase field equations is given in [2]. Convergence of the classical phase field equations to sharp interface limiting motion is considered in [5]. See also references mentioned therein. It is shown in [5], that depending on which particular distinguished limit is considered, the predicted limiting motion may be the classical Stefan problem, a type of surface tension model- with or without attachment kinetics, a two phase Hele-Shaw model, or motion by mean curvature. In particular, we note that motion by mean curvature is predicted when $\tau = \mathcal{O}(\epsilon)$, $\xi = \mathcal{O}(\epsilon^{1/2})$, $\eta = \mathcal{O}(\epsilon)$, and $l = \mathcal{O}(\epsilon)$, for $0 < \epsilon \ll 1$. For the standard phase field model with memory, where memory effects are included in the energy balance equation, but not in the phase relaxation equation, analytical work has been undertaken regarding certain aspects of the long time behavior, see e.g. [1] and [8]. Recently Giorgi, Grasselli & Pata [8] have demonstrated the existence of absorbing sets for the standard phase field model with memory under somewhat restrictive assumptions, by considering it as a non-autonomous dynamical system. To best of our knowledge, however, a study of limiting motions for this model has yet to be undertaken.

With regard to the system (**PFM**), we remark that while the long time behavior has yet to be studied carefully, certain sharp interface limiting motions have been worked out formally. For example, if the distinguished limit $\tau = \mathcal{O}(\epsilon)$, $\xi = \mathcal{O}(\epsilon^{1/2})$, $\eta = \mathcal{O}(\epsilon)$, and $l = \mathcal{O}(\epsilon^2)$ is considered, and the kernels are taken to be exponential functions, then the limiting motion is given by [18, 16]

$$V_t + \gamma V(1 - V^2) = \kappa(1 - V^2), \tag{1.9}$$

where V denotes the normal velocity of the front, κ denotes the mean curvature, and γ is the rate of exponential decay of the phase memory kernel, or if the distinguished limit $\tau = \mathcal{O}(\epsilon)$, $\xi = \mathcal{O}(\epsilon^{1/2})$, $\eta = \mathcal{O}(\epsilon^2)$, and $l = \mathcal{O}(\epsilon)$ is considered, and the kernels are taken to be "weakly singular" exponentials; i.e., $a_i = b_i\gamma_i \exp(-\gamma_i t)$ where $b_i = \mathcal{O}(1)$ and $\gamma_i = \mathcal{O}(\epsilon^{3/2})$, then the (scaled) limiting motion:

$$\epsilon^{3/2} V_t + V = \kappa \tag{1.10}$$

[15] is predicted. A crystalline algorithm was contructed to study equations such as (1.9) and (1.10), [16]. Implementing the crystalline algorithm for initially convex polygonal phase boundaries with vanishing initial velocity, it could be seen that while monotone melting occurred for regular polygonal initial conditions, for certain sufficiently irregular polygonal initial conditions two-dimensional damped oscillations appeared [17, 18]. Further study of the equations (1.9),(1.10) is forthcoming.

We now focus on existence. For simplicity and without loss of generality, we set $\tau = \eta = 1$. In the next and main section of this note we prove:

Theorem 1 *Suppose that* $\{u_0, \phi_0\} \in L^2(\Omega) \times H^1(\Omega)$ *and* $\{f_1, f_2\} \in L^1(0, T; L^2(\Omega) \times H^1(\Omega))$, *then there exists a global solution to (1.1)–(1.4) in the sense of Definition 2.*

Definition 2 *We shall say that* $\{u, \phi\}$ *constitutes a solution to (1.1)–(1.4) on the interval* $[0, T]$, $0 < T < \infty$, *if*

$$\{u, \phi\} \in \mathcal{C}([0, T]; L^2(\Omega) \times H^1(\Omega))$$

$$\{u_t, \phi_t\} \in L^\infty(0, T; H^{-2}(\Omega) \times H^{-1}(\Omega)) + L^1(0, T; L^2(\Omega) \times H^1(\Omega)),$$

$\{u, \phi\}$ *satisfy the initial conditions (1.4), and*

$$\int_0^T \int_\Omega y(x, t)\, [u_t + \frac{l}{2}\phi_t - f_1](x, t)\, dxdt - \int_0^T \int_\Omega \Delta y(x, t)\, (a_1 * u)(x, t)\, dxdt = 0,$$

$$\int_0^T \int_\Omega z(x, t)\, [\phi_t - a_2 * (\phi - \phi^3 - u) - f_2](x, t)\, dxdt +$$

$$+\xi^2 \int_0^T \int_\Omega \nabla z(x, t) \cdot a_2 * \nabla \phi(x, t)\, dxdt = 0,$$

for any $y \in L^1(0, T; H^2(\Omega)) \cap L^\infty(0, T; L^2(\Omega))$ *and* $z \in L^1(0, T; H^1(\Omega)) \cap L^\infty(0, T; L^2(\Omega))$.

We remark that if $u_h \in L^1(0, T; H^2(\Omega))$ and $\phi_h \in L^1(0, T; H^3(\Omega)) \cap L^5(0, T; L^6(\Omega))$, then $\{f_1, f_2\}$ satisfy the assumptions stated in Theorem 1.

In a short final section, we give a few closing remarks.

2 Existence

In this section we outline the proof of Theorem 1 stated in the Introduction. Our method of proof relies on a Galerkin approximation based on the eigenfunctions of the linear operator $\mathcal{A} : L^2(\Omega) \to H^{-2}(\Omega)$,

$$\mathcal{A}\Psi = -\Delta\Psi, \quad x \in \Omega \qquad \mathbf{n} \cdot \nabla\Psi = 0 \quad x \in \partial\Omega. \tag{2.1}$$

Let $\{\Psi_i\}$ denote an $L^2(\Omega)$–orthonormal sequence of eigenfunctions of the linear operator $\mathcal{A}$ which are ordered sequentially so that the associated eigenvalues λ_i satisfy

$$0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \ldots,$$

and note that $\Psi_0 = |\Omega|^{-\frac{1}{2}}$. We shall seek approximations for u and ϕ of the form:

$$u_N(x, t) := \sum_{i=0}^{N} c_{Ni}(t)\Psi_i(x) \tag{2.2}$$

$$\phi_N(x, t) := \sum_{i=0}^{N} d_{Ni}(t)\Psi_i(x). \tag{2.3}$$

Let $Sp(N) := span\{\Psi_0, \ldots, \Psi_N\}$. We denote by $P^i : L^2(\Omega) \to Sp(N)$ the projection of $L^2(\Omega)$ onto Ψ_i, and by $P_N : L^2(\Omega) \to Sp(N)$ the projection of $L^2(\Omega)$ onto the span of the first $N+1$ modes; i.e., $P_N = \sum_{i=0}^{N} P^i$.

The functions $\{u_N, \phi_N\}$ shall constitute an approximation to a solution $\{u, \phi\}$ of (**PFM**) in that they shall be required to satisfy

$$\int_0^T b_{1i} < \Psi_i, u_{Nt} + \frac{l}{2}\phi_{Nt} - a_1 * \Delta u_N - P_N f_1 >_0 dt, \tag{2.4}$$

$$\int_0^T b_{2i} < \Psi_i, \phi_{Nt} - a_2 * [\xi^2 \Delta\phi_N + \phi_N - P_N(\phi_N)^3 + u_N] + P_N f_2 >_0 dt, \tag{2.5}$$

$$< \Psi_i, u_N >_0 = P^i u_0 \quad < \Psi_i, \phi_N >_0 = P^i \phi_0, \tag{2.6}$$

for $i = 0, 1, \ldots, N$ and for all $b_{1i}, b_{2i} \in L^\infty(0, T)$, where $< \cdot, \cdot >_0$ denotes the $L^2(\Omega)$ inner product. Equations (2.4)–(2.6) imply that $\{c_{Ni}, d_{Ni}\}$ satisfy

$$c_{Nit} + \tfrac{l}{2} d_{Nit} = -\lambda_i \xi^2 a_1 * c_{Ni} + P^i f_1, \tag{2.7}$$

$$d_{Nit} = -\lambda_i \xi^2 a_2 * d_{Ni} + a_2 * (c_{Ni} + d_{Ni}) + g_{Ni}(d_{N0}, \ldots, d_{NN}) + P^i f_2, \tag{2.8}$$

$$c_{Ni}(0) = P^i u_0 \quad d_{Ni}(0) = P^i \phi_0, \tag{2.9}$$

for $i = 0, 1, \ldots, N$, in the $L^1(0, T)$ sense, where g_{Ni} denotes a nonlinear term which can be expressed explicitely as:

$$g_{Ni} := g_{Ni}(d_{N0}, d_{N1}, \ldots, d_{NN}) = - < \Psi_i, a_2 * P_N(\phi_N)^3 > .$$

With regard to existence, uniqueness, and regularity of solutions to (2.7)–(2.9), we can state the following:

Lemma 2.1 *For $(u_0, \phi_0) \in L^2(\Omega) \times L^2(\Omega))$ and $f_1, f_2 \in L^1(0, \infty; L^2(\Omega))$, there exists a unique solution $\{c_{Ni}, d_{Ni}\} \in [C(0, T_N)]^{2(N+1)}$ to the system (2.7)–(2.9) for $N = 0, 1, \ldots$, where the interval $(0, T_N)$ is maximal; i.e., either $T_N = \infty$ or else the solution becomes unbounded as $t \uparrow T_N$. Moreover, the solution is differentiable with values in $L^1(0, T_N)$.*

Proof: Substituting (2.8) into (2.7), integrating over the interval $(0, t)$, for $t > 0$, and formally exchanging the order of integration, yields the system:

$$x(t) = \int_0^t \tilde{b}(t,\, s,\, x(s))\, ds + \tilde{f}(t) + x(0), \tag{2.10}$$

where $\tilde{b}, \tilde{f} \in R^{2(N+1)}$, $x = (c_{N0}, \ldots, c_{NN}, d_{N0}, \ldots, d_{NN})$, and $\tilde{b} = \tilde{b}(t, s, x)$ and $\tilde{f} = \tilde{f}(t)$ depend continuously on their arguments. Thus written, standard theorems on Volterra integral equations of the second kind can be invoked. In particular, by [12, Theorems 1.1 & 2.2], there exists a continuous solution $x(t)$ to (2.10) on a maximal interval. The continuity of the solution to (2.10) allows us to re-exchange the order of integration, yielding a continuous solution $\{c_{Ni},\, d_{Ni}\}_{i=0,\ldots,N}$ to the integrated form of (2.7)–(2.8).

It can be readily checked that, in fact, (2.10) may be written more specifically as

$$x(t) = \alpha_1 * \tilde{h}_1(x(t)) + \alpha_2 * \tilde{h}_2(x(t)) + \tilde{f}(t) + x(0), \tag{2.11}$$

where $\alpha_j(t) = 1 * a_j(t)$, and $\tilde{h}_j(x)$, and $\tilde{f}(t)$ are continuous functions of their arguments. The assumptions that $a_j \in L^1(R^+)$ and $f_1,\, f_2 \in L^1(0, \infty;\, L^2(\Omega))$ in conjunction with (2.11) imply the L^1 differentiability of $x(t)$, and hence of $\{c_{Ni},\, d_{Ni}\}_{i=0,\ldots,N}$. Moreover, it follows now from [12, Theorem 2.3] that the solution $x(t)$ is unique. This readily implies in turn that $\{c_{Ni},\, d_{Ni}\}_{i=0,\ldots,N}$ constitute a unique continuous solution to (2.7)–(2.8).□

It is now not difficult to prove that in fact for $\{u_0,\, \phi_0\} \in L^2(\Omega) \times H^1(\Omega)$ and $\{f_1,\, f_2\} \in L^1(0, \infty;\, L^2(\Omega) \times H^1(\Omega))$, $T_N = \infty$ for any $N = 0, 1, 2, \ldots$, where T_N denotes the maximal interval of existence of the solution attained in Lemma 2.1. This is accomplished by establishing an a priori estimate which is uniform in N and T which is stated below as lemma 2.2.

Lemma 2.2 *If $\{c_{Ni},\, d_{Ni}\}_{i=0,\ldots,N}$ denotes the solution of (2.7)–(2.9), $(u_0,\, \phi_0) \in L^2(\Omega) \times H^1(\Omega)$ and $\{f_1,\, f_2\} \in L^1(0, \infty;\, L^2(\Omega) \times H^1(\Omega))$, then*

$$\sum_{i=0}^{N} |c_{Ni}(T)|^2 \leq C \quad \sum_{i=0}^{N} |d_{Ni}(T)|^2 \leq C, \tag{2.12}$$

for any $0 < T < T_N$, where C depends on initial conditions and history, but is independent of N and T.

Proof: Let us now multiply (2.7) by $\frac{2}{l} c_{Ni}$ and (2.8) by $\lambda_i \xi^2 d_{Ni} - d_{Ni} + < \Psi_i,\, P_N(\phi_N^3) > -c_{Ni}$, sum over i, add together the two resultant expressions, and integrate from 0 to T, for $0 < T < T_N$. Subsequently integrating over Ω yields

$$\int_\Omega \Big[\frac{\xi^2}{2}|\nabla\phi_N|^2 - \frac{1}{2}|\phi_N|^2 + \frac{1}{4}\phi_N^4 + \frac{1}{l}u_N^2\Big](T) =$$

$$= \int_\Omega \Big[\frac{\xi^2}{2}|\nabla\phi_N|^2 - \frac{1}{2}\phi_N^2 + \frac{1}{4}\phi_N^4 + \frac{1}{l}u_N^2\Big](0) - \frac{2}{l}\int_0^T < \psi_1(s), \int_0^t a_1(t-s)\psi_1(s)\,ds > dt$$
$$- \int_0^T < \psi_2(s), \int_0^t a_2(t-s)\psi_2(s)\,ds > dt$$
$$+ \frac{2}{l}\int_0^T < u_N, f_1(x,t) > dt - \int_0^T < \psi_2(s), f_2(x,t) > dt \tag{2.13}$$

where

$$\psi_1 = \nabla u_N \text{ and } \psi_2 = \xi^2\Delta\phi_N + \phi_N - P_N(\phi_N)^3 + u_N. \tag{2.14}$$

Equation (2.13) can now be shown to imply the estimate:

$$\int_\Omega \Big[\frac{\xi^2}{2}|\nabla\phi_N|^2 - \frac{1}{2}\phi_N^2 + \frac{1}{4}\phi_N^4 + \frac{1}{l}u_N^2\Big](T) \le \int_\Omega \Big[\frac{\xi^2}{2}|\nabla\phi_0|^2 - \frac{1}{2}\phi_0^2 + \frac{1}{4}\phi_0^4 + \frac{1}{l}u_0^2\Big]$$
$$+C_1\int_0^T \int_\Omega \Big[\frac{\xi^2}{2}|\nabla\phi_N|^2 + \frac{1}{4}\phi_N^4 + \frac{1}{l}u_N^2\Big]\Big\{||f_1||_{L^2(\Omega)} + ||f_2||_{H^1(\Omega)}\Big\}\,dt$$
$$+C_2\Big\{||f_1||_{L^1(0,T;L^2(\Omega))} + ||f_2||_{L^1(0,T;H^1(\Omega))}\Big\}, \tag{2.15}$$

where C_1 and C_2 depend on Ω, l, and ξ only. Noting that

$$-\int_\Omega \frac{1}{8}\phi_N^4 - \frac{1}{2}|\Omega| \le \int_\Omega -\frac{1}{2}\phi_N^2,$$

and using Gronwall's inequality, we obtain

$$\int_\Omega \Big[\frac{1}{2}|\nabla\phi_N|^2 + \frac{1}{8}\phi_N^4 + \frac{1}{l}u_N^2\Big](T) \le C_3, \tag{2.16}$$

where C_3 is independent of N and T. From (2.16) it follows that

$$||u_N||_{L^\infty(0,T;L^2(\Omega))} < C_4 \text{ and } ||\phi_N||_{L^\infty(0,T;L^2(\Omega))} < C_5, \tag{2.17}$$

where C_4 and C_5 are independent of N and T, which implies in turn (2.12). □

To guarantee the existence of a solution to **(PFM)** in the sense indicated in **Definition 2**, we must ascertain convergence of a subsequence of the approximants $\{u_N, \phi_N\}$ in an appropriate sense. The uniform estimates implied by (2.12) and the compactness results of Simon [19, Corollary 4] enable us to conclude

Lemma 2.3 *For any $T > 0$, there exist functions u, ϕ, χ_0, χ_1, and χ_2, and a subsequence $\{u_{N'}, \phi_{N'}\}$, denoted for simplicity again as $\{u_N, \phi_N\}$, such that the following convergences hold:*

$$u_N \stackrel{*}{\rightharpoonup} u \text{ in } L^\infty(0,T;L^2(\Omega)), \tag{2.18}$$

$$\phi_N \overset{*}{\rightharpoonup} \phi \ \ in \ L^\infty(0, T; H^1(\Omega)), \tag{2.19}$$

$$u_{Nt} \rightharpoonup u_t \ \ in \ L^1(0, T; H^{-2}(\Omega)), \tag{2.20}$$

$$\phi_{Nt} \rightharpoonup \phi_t \ \ in \ L^1(0, T; H^{-1}(\Omega)), \tag{2.21}$$

$$u_{Nt} - P_N f_1 + P_N f_2 \overset{*}{\rightharpoonup} u_t - f_1 + f_2 \ \ in \ L^\infty(0, T; H^{-2}(\Omega)), \tag{2.22}$$

$$\phi_{Nt} - P_N f_2 \overset{*}{\rightharpoonup} \phi_t - f_2 \ \ in \ L^\infty(0, T; H^{-1}(\Omega)), \tag{2.23}$$

$$a_1 * u_N \overset{*}{\rightharpoonup} \chi_0 \ \ in \ \ L^\infty(0, T; L^2(\Omega)), \tag{2.24}$$

$$a_2 * \phi_N \overset{*}{\rightharpoonup} \chi_1 \ \ in \ \ L^\infty(0, T; H^1(\Omega)), \tag{2.25}$$

and

$$a_2 * {\phi_N}^3 \overset{*}{\rightharpoonup} \chi_2 \ \ in \ L^\infty(0, T; L^2(\Omega)). \tag{2.26}$$

Moreover,

$$u_N \to u \ \ in \ L^p(0, T; H^s(\Omega)), \quad 1 \le p < \infty, \quad -1 \le s < 0, \tag{2.27}$$

$$\phi_N \to \phi \ \ in \ L^p(0, T; H^s(\Omega)), \quad 1 \le p < \infty, \quad 0 \le s < 1. \tag{2.28}$$

From Lemma 2.3 and equations (2.4)–(2.6), it is not difficult to conclude that $u \in C([0, T]; H^{-2})$ and $\phi \in C([0, T]; H^{-1})$. These results fall short of guaranteeing continuity from initial data in $L^2(\Omega) \times H^1(\Omega)$ as claimed in Theorem 1. Also, to guarantee the existence of a solution in the sense of **Definition 2**, it is necessary to be able to identify the limiting functions χ_0, χ_1, and χ_2.

We address the latter issue first. Young's inequality for convolutions and Lemma 2.3 imply that χ_0 and $a_1 * u$ belong to $L^\infty(0, T; H^{-2}(\Omega))$, and χ_1 and $a_2 * \phi$ belong to $L^\infty(0, T; H^{-1}(\Omega))$. Using (2.27)–(2.28) and weak lower semicontinuity, then allows us to make the appropriate identifications. The proof that $\chi_2 = a_2 * \phi^3$ is a little more involved, but basically relies on noting that χ_2 and $a_2 * \phi^3$ both belong to $L^\infty(0, T; L^2(\Omega))$, and hence to identify the limit it suffices to prove that the two functions coincide in the weaker space $L^{4/3}(0, T; L^{4/3}(\Omega))$. Using weak lower semicontinuity, the embedding estimate

$$||\phi_N - \phi||_{L^{4/3}(0,T;L^4(\Omega))} \le C_1 ||\phi_N - \phi||_{L^{4/3}(0,T;H^s(\Omega))} + C_2 ||\phi_N - \phi||_{L^{4/3}(0,T;L^2(\Omega))}$$

for any $s \in [3/4, 1)$, which follows from Gagliardo-Nirenberg, and (2.28), allows us to complete the proof of the third identification.

It remains now only to prove the desired continuity. With this end in mind, we first note that it can be proved, in a manner which is roughly analogous to the proof of Lemma 1.2 in Temam [20, Chapter III], that

Lemma 2.4 $\phi \in C([0, T]; L^2(\Omega))$.

We now prove:

Lemma 2.5 $u \in \mathcal{C}([0,T]; L^2(\Omega))$ *and* $\phi \in \mathcal{C}([0,T]; H^1(\Omega))$.

Proof: The methodology employed here is a little reminscent of the proof of continuity given in [4]. By Lemma 2.4, $\phi \in \mathcal{C}([0,T]; L^2(\Omega))$, and hence in particular ϕ is weakly continuous in $L^2(\Omega)$. From the weak continuity which has been demonstrated for ϕ and since by Lemma 2.3, $\phi \in L^\infty(0,T; H^1(\Omega))$, it follows from Lemma 1.4 in [20, Chapter III] that ϕ is weakly continuous in $H^1(\Omega)$. Since by assumption $\phi_0 \in H^1(\Omega)$, this implies in turn that

$$0 \le \liminf_{T\to 0} \int_\Omega |\nabla\phi(T) - \nabla\phi_0|^2 =$$

$$= \liminf_{T\to 0}\Big\{||\nabla\phi(T)||^2_{L^2(\Omega)} - 2 < \nabla\phi(T),\, \nabla\phi_0 >_0 + ||\nabla\phi_0||^2_{L^2(\Omega)}\Big\}$$

$$= \liminf_{T\to 0}||\nabla\phi(T)||^2_{L^2(\Omega)} - ||\nabla\phi_0||^2_{L^2(\Omega)}.$$

Therefore,

$$\int_\Omega |\nabla\phi_0|^2 \le \liminf_{T\to 0}\int_\Omega |\nabla\phi(T)|^2. \tag{2.29}$$

Using similar arguments for u and ϕ^2 and the results of Lemma 2.4, we obtain

$$\int_\Omega\Big[\frac{\xi^2}{2}|\nabla\phi_0|^2 - \frac{1}{2}|\phi_0|^2 + \frac{1}{4}\phi_0^4 + u_0^2\Big] \le \liminf_{T\to 0}\int_\Omega\Big[\frac{\xi^2}{2}|\nabla\phi|^2 - \frac{1}{2}|\phi|^2 + \frac{1}{4}\phi^4 + u^2\Big](T). \tag{2.30}$$

To obtain an estimate in the opposite direction, we return to the estimate (2.15) obtained earlier which we write now as

$$\int_\Omega\Big[\frac{\xi^2}{2}|\nabla\phi_N|^2 + \frac{1}{4}\phi_N^4 + \frac{1}{l}u_N^2\Big](T) \le \int_\Omega\Big[\frac{\xi^2}{2}|\nabla\phi_N|^2 + \frac{1}{4}\phi_N^4 + \frac{1}{l}u_N^2\Big](0)$$

$$+\frac{1}{2}\int_\Omega[\phi_N^2(T) - \phi_N^2(0)] + C_1\int_0^T\Big\{||f_1||_{L^2(\Omega)} + ||f_2||_{H^1(\Omega)}\Big\}\,dt$$

$$+C_2\int_0^T\Big\{\frac{\xi^2}{2}|\nabla\phi_N|^2 + \frac{1}{4}\phi_N^4 + \frac{1}{l}u_N^2\Big\}\Big\{||f_1||_{L^2(\Omega)} + ||f_2||_{H^2(\Omega)}\Big\}\,dt, \tag{2.31}$$

where the coefficients C_1 and C_2 depend on Ω, l, and ξ only and are independent of N and T. By establishing the estimate

$$\int_\Omega[\phi_N^2(T) - \phi_N^2(0)] \le \bar{C}\int_0^T\Big\{1 + ||f_2||_{H^1(\Omega)}\Big\}\,dt$$

where $\bar{C}$ is independent of N and T, we can conclude from (2.31) that

$$\int_\Omega \Big[\frac{\xi^2}{2}|\nabla\phi_N|^2 + \frac{1}{4}\phi_N^4 + \frac{1}{l}u_N^2\Big](T) \le \int_\Omega \Big[\frac{\xi^2}{2}|\nabla\phi_N|^2 + \frac{1}{4}\phi_N^4 + \frac{1}{l}u_N^2\Big](0)$$

$$+C_6\int_0^T \Big\{1 + ||f_1||_{L^2(\Omega)} + ||f_2||_{H^1(\Omega)}\Big\}\,dt$$

$$+C_2\int_0^T \Big\{\frac{\xi^2}{2}|\nabla\phi_N|^2 + \frac{1}{4}\phi_N^4 + \frac{1}{l}u_N^2\Big\}\Big\{||f_1||_{L^2(\Omega)} + ||f_2||_{H^2(\Omega)}\Big\}\,dt.$$

Adding C_6/C_2 to both sides of the above equation and applying Gromwall's Lemma:

$$\int_\Omega \Big[\frac{\xi^2}{2}|\nabla\phi_N|^2 + \frac{1}{4}\phi_N^4 + \frac{1}{l}u_N^2 + \frac{C_6}{C_2}\Big](T) \le$$

$$\le \int_\Omega \Big[\frac{\xi^2}{2}|\nabla\phi_N|^2 + \frac{1}{4}\phi_N^4 + \frac{1}{l}u_N^2 + \frac{C_6}{C_2}\Big](0)e^{C_2\int_0^T\{1+||f_1||_{L^2(\Omega)}+||f_2||_{H^1(\Omega)}\}\,dt}, \qquad (2.32)$$

which implies

$$\limsup_{T\to 0}\int_\Omega \Big[\frac{\xi^2}{2}|\nabla\phi|^2 + \frac{1}{4}\phi^4 + \frac{1}{l}u^2\Big](T) \le \int_\Omega \Big[\frac{\xi^2}{2}|\nabla\phi|^2 + \frac{1}{4}\phi^4 + \frac{1}{l}u^2\Big](0). \qquad (2.33)$$

Combining (2.33) with (2.30) completes the proof of the lemmà. □

3 Some concluding remarks

Our proof of existence should be viewed as a first step in putting the phase field equation with memory (**PFM**) on sound analytical grounds. Questions of uniqueness and long time behavior should be approachable under appropriate assumptions, and further numerical methods are under development. Long term goals include rigorous justification of the predicted limiting motions.

Remark: It has just come to our attention that weaker existence results have been obtained independently by Rotstein & Grasselli (JMAA, to appear) and Grasselli (Proceeding of FBP99).

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