数理解析研究所講究録 1210 巻 2001 年 24-28

A free boundary problem of wave equation

S.OMATA & S.KINAMI
小俣正朗 木南伸一

Department of Computational Science, Faculty of Science
Kanazawa University, Kanazawa, 920-1192 Japan
金沢大学

1 Introduction to the Mathematical Problem

In this paper we treat the following free boundary problem for hyperbolic equation numerically. Let $\Omega \subset \mathbf{R}^n$, $T > 0$ and put $\Omega_T = \Omega \times (0, T)$, find a non-negative solution to the following equalities:

$$(\mathrm{P}) \quad \begin{cases} u_{tt} - \Delta u = 0 & \text{in} \quad \Omega_T \cap \{u > 0\} \\ |\nabla u|^2 - u_t^2 = Q^2 & \text{on} \quad \Omega_T \cap \partial\{u > 0\}, \end{cases}$$

under the suitable initial and boundary condition. Here Q is a given positive constant. This problem was firstly introduced by K.Kikuchi and S.Omata (see [2]). In the case of $\Omega \subset \mathbf{R}^2$, physical image of this problem is to investigate the movement of soapy film which goes into soap water or of the membrane whose part adhered to the plane. It could be described by a stationary point of the action functional below:

$$J(u) = \int_{\Omega_T} \left(|\nabla u|^2 - (u_t)^2 \chi_{\{u>0\}} + Q^2 \chi_{\{u>0\}} \right) dz, \tag{1.1}$$

where $\chi_{\{u>0\}}$ is the characteristic function of the set $\{(x,t) \in \Omega_T; u(x,t) > 0\}$ and $z = (x,t)$. Equations are derived as Euler-Lagrange equations of J. However the functional J is not Gâteaux differentiable in general. We derive the equations in (P) just as a necessary condition for a smooth function u to be a stationary point of J. The first equation of (P) is derived from $\frac{d}{d\varepsilon} J(u + \varepsilon\zeta)|_{\varepsilon=0} = 0$ $(\zeta \in C_0^\infty(\Omega_T \cap \{u > 0\}))$. On the other hand, the second one is from $\frac{d}{d\varepsilon} J(u(\tau_\varepsilon^{-1}(z)))|_{\varepsilon=0} = 0$ (inner variation) where $\tau_\varepsilon(z)$ is diffeomorphism and in $C_0^\infty(\Omega_T; \Omega_T)$. We got the unique local existence of the solution (P) on some one dimensional cases. (See [2]).

2 Smoothing of Equations

In [1], we adopted the fixed domain method for numerical analysis to the one dimensional problem. It seems to keep good accuracy, but unfortunately it could not treat the case when the free boundary changes its topology. For this, we introduce a smoothing method for (P). Unfortunately we did not get any proof which guarantees the convergence to the original problem (P) from "smoothing" solution.

We consider the following equation:

$$\Delta u - u_{tt} = \beta_\varepsilon(u) \quad \text{in } \Omega_T \tag{2.1}$$

with some initial and boundary conditions. Here u^ε is a classical solution of (2.1) and $\beta_\varepsilon(f)$ defined in the following way:

$$B_\varepsilon(f) := \int_0^f \beta_\varepsilon(f)df,$$

where

$$B_\varepsilon(f) \longrightarrow \begin{cases} Q^2 \text{ (given constant) in } \{f>0\} \\ 0 \text{ in } \Omega \times (0,T) \setminus \{f>0\}. \end{cases}$$

This means that $B_\varepsilon(f)$ is a smoothing of the characteristic function $Q^2\chi_{\{f>0\}}(x)$.

If we assume $u^\varepsilon \longrightarrow \exists v$ in some suitable sense and that such v satisfies $\Delta v - v_{tt} = 0$ in $\Omega \times (0,T) \cap \{v > 0\}$, then we can say that v must satisfy the free boundary condition $|\nabla v|^2 - (v_t)^2 = Q^2$ on $\partial\{v>0\}$ automatically.

We will show this. Multiply $\zeta u_k (\equiv \zeta \frac{\partial u}{\partial x_k})$ to both side of (2.1) and integrate on Ω_T, $(\zeta \in C_0^\infty(\Omega_T))$, we got the following equality:

$$\int_{\Omega_T} \zeta u_k(\Delta u - u_{tt})dz = \int_{\Omega_T} \zeta u_k \beta_\varepsilon(u)dz. \tag{2.3}$$

Noting that $\left[B_\varepsilon(u)\right]_{x_k} = \beta_\varepsilon(u)u_k$ and the integration by parts, the right hand side of (2.3) can be calculated the following

$$\begin{aligned} &= -\int_{\Omega_T} \zeta_k B_\varepsilon(u)dz \\ &\longrightarrow -\int_{\Omega_T\cap\{v>0\}} \zeta_k Q^2 dz \quad (\varepsilon \to 0) \\ &= -\int_{\Omega_T\cap\partial\{v>0\}} \zeta Q^2 \nu_k dS \end{aligned}$$

where ν_k is a k-th element of outer normal $\nu = (\nu_1 \cdots \nu_{n+1})$ to the set $\{v>0\}$.

On the other hand, left hand side of (2.3),

$$\begin{aligned} &= -\int_{\Omega_T} (\nabla(\zeta u_k)\nabla u - (\zeta u_k)_t u_t)dz \longrightarrow -\int_{\Omega_T} (\nabla(\zeta v_k)\nabla v - (\zeta v_k)_t v_t)dz \quad (\varepsilon \to 0) \\ &= -\int_{\Omega_T\cap\partial\{v>0\}} \zeta v_k(\nabla v, -v_t)\cdot \nu dS + \int_{\Omega_T\cap\{v>0\}} \zeta v_k(\Delta v - v_{tt})dz \\ &= -\int_{\Omega_T\cap\partial\{v>0\}} \zeta v_k(\nabla v, -v_t)\cdot \nu dS + 0. \end{aligned}$$

Note that outer normal to $\{v>0\}$ become

$$\nu = \frac{-Dv}{|Dv|} = \frac{-(v_{x_1}, \cdots, v_{x_n}, v_t)}{\sqrt{\sum (v_k)^2 + (u_t)^2}}$$

then $v_k = -\nu_k|Dv|$. Then the left hand side of (2.3) become

$$\begin{aligned} &= -\int_{\Omega_T\cap\partial\{v>0\}} \zeta v_k(\nabla v, -v_t)\cdot \nu dS \\ &= -\int_{\Omega_T\cap\partial\{v>0\}} \zeta |Du| \cdot \nu_k \left[|\nabla v|^2 - (v_t)^2\right] \cdot \frac{1}{|Du|} dS \\ &= -\int_{\Omega_T\cap\partial\{v>0\}} \zeta \left[|\nabla v|^2 - (v_t)^2\right] \nu_k dS. \end{aligned}$$

Thus from (2.3), we got the equation

$$\int_{\Omega_T \cap \partial\{v>0\}} \zeta Q^2 \nu_k dS = \int_{\Omega_T \cap \partial\{v>0\}} \zeta \left[|\nabla v|^2 - (v_t)^2 \right] \nu_k dS$$

then

$$|\nabla v|^2 - (v_t)^2 = Q^2 \quad \text{on} \quad \partial\{v > 0\}.$$

Thus we got a free boundary condition in (P).

3 Numerical Examples

Here we consider an one dimensional problem. Let l be a positive constant and let us set $\Omega = (0,1)$ and $0 < l_1 < l_2 < 1$. Here for comparison, we mention a trivial linear solution.

Linear Solution *Let a be a given positive constant and put $l = \dfrac{a}{\sqrt{Q^2 + a^2}}$. Consider Problem (P) for initial and boundary conditions*

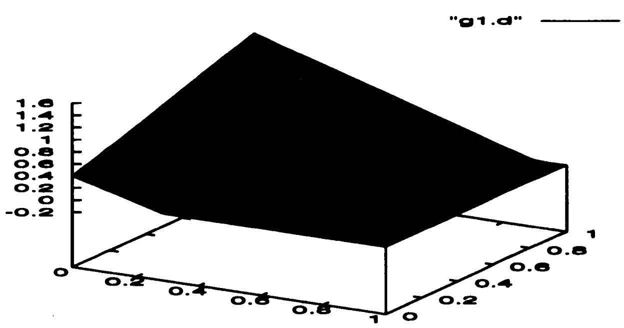
$$\begin{array}{llllll} u(x,0) & = -\sqrt{Q^2+a^2}\,(x - l_1) & x \in [0, l_1], & = 0 & x \in (l_1, 1], \\ u_t(x,0) & = a & x \in [0, l_1), & = 0 & x \in [l_1, 1]. \\ u(0,t) & = a(t+1) & & & \\ u(1,t) & \equiv 0 & & & \end{array}$$

The function $u(x,t) = \max(-\sqrt{Q^2+a^2}\,(x - l_1) + at, 0)$ satisfies (P) and then it is the unique solution.

We investigate the following numerical calculations

Case 1 (ε = 0.02)

$$\begin{array}{llllll} u(0,t) & = t + 0.4 & & & \\ u(x,0) & = -\sqrt{2}x + 0.4 & x \in [0, \frac{0.4}{\sqrt{2}}], & = 0 & x \in (\frac{0.4}{\sqrt{2}}, 1], \\ u_t(x,0) & = 1 & x \in [0, \frac{0.4}{\sqrt{2}}], & = 0 & x \in (\frac{0.4}{\sqrt{2}}, 1]. \end{array}$$



Unfortunately, the accuracy is not so good.

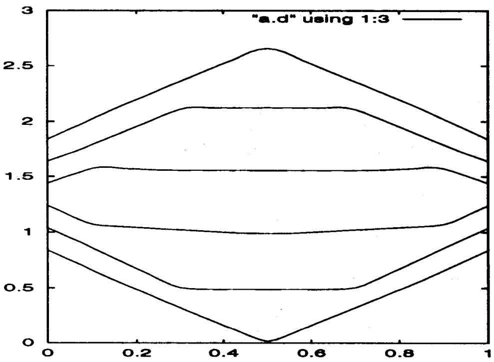
Case 2 ($\varepsilon = 0.02$)

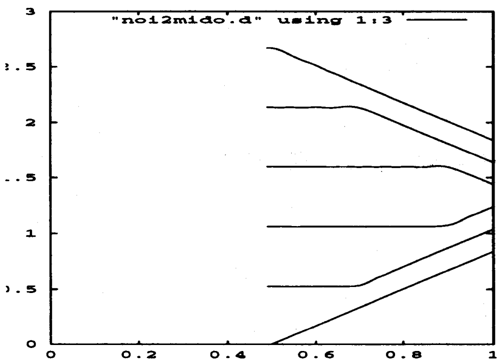
$$\begin{aligned} u(0,t) &= u(1,t) = t + 0.4 \\ u(x,0) &= \max(-\sqrt{2}x + 0.4, 0, \sqrt{2}x + 0.4 - \sqrt{2}) \\ u_t(x,0) &= 1 \quad \text{if} \quad u(x,0) > 0, \qquad = 0 \quad \text{otherwise} \end{aligned}$$

For comparison, we consider the following initial and boundary problem;

$$\begin{cases} u_{tt} = \Delta u \ \text{ in } \ (\frac{1}{2}, 1) \times (0, t_0) \\ u(x,0) = \frac{0.839}{0.5}(x - 0.5) \ \text{ at } \ (\frac{1}{2}, 1) \times \{0\} \\ u_t(x,0) = 1 \ \text{ at } \ (\frac{1}{2}, 1) \times \{0\} \\ u(1,t) = t + 0.839 \\ u_x(\frac{1}{2}, t) = 0 \end{cases} \tag{4.1}$$

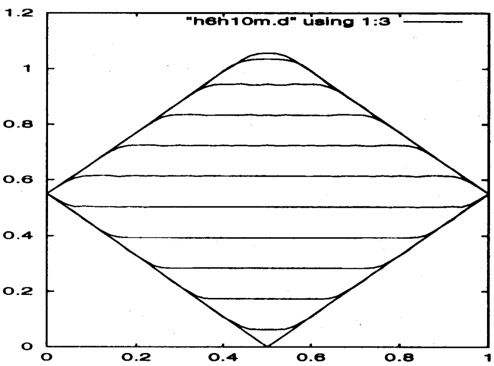
After peeling off, we can compare the case 2 and solution of (4.1). The results are the following:

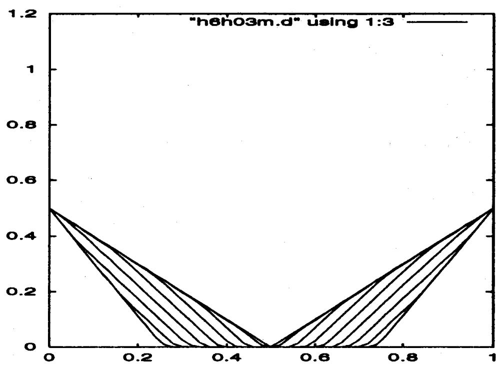




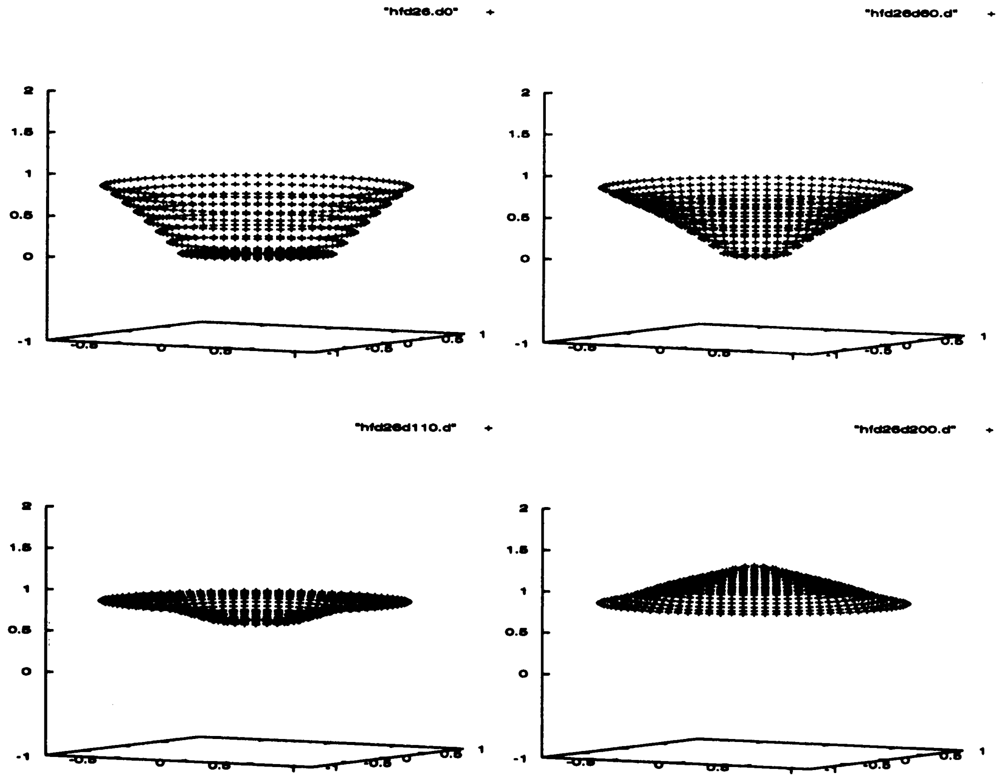
Case 3 ($\varepsilon = 0.02$)

There is a "threshold" for the boundary condition. In this case, initial data are "V"-shaped function without initial velocity.





Case 4 ($\Omega = B_1(0) \subset \mathbf{R}^2$, $\varepsilon = 0.05$)
In a 2-dimensional case, we can see peeling off phenomena and vibration.



References

[1] H. Imai - K. Kikuchi - K. Nakane - S. Omata - T. Tachikawa, "*A numerical approach to the asymptotic behavior of solutions of a one-dimensional hyperbolic free boundary problems*", to appear in JJIAM.

[2] K. Kikuchi - S. Omata, "*A free boundary problem for a one dimensional hyperbolic equation*", Adv. Math. Sci. Appl. **9** No.2 (1999) 775-786.

[3] S. Omata, "*A free boundary problem for a quasilinear elliptic equation Part I: Rectifiability of free boundary*", Differential and Integral Equations. **6** No.6 (1993) 1299-1312.

[4] S. Omata - Y. Yamaura, "*A free boundary problem for quasilinear elliptic equations*", Proc. Japan Acad. Ser. A Math. 21(1990), 281-286.

[5] S. Omata - Y. Yamaura, "*A free boundary problem for quasilinear elliptic equations Part II: $C^{1,\alpha}$- regularity of free boundary*", Funkcialaj Ekvacioj **42** No.1 (1999) 9-70.