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Solvability of a class of differential equations in the sheaf of microfunctions with holomorphic parameters

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1 Introduction

We study solvability of some class of differential equations in the sheaf of 2-analytic functions, that is, microfunctions with holomorphic parameters. For that purpose, we introduce an integral formula of Mellin's type for holomorphic functions.

Let V and Σ be the following regular involutive and Lagrangian submanifolds of T_M^*X with $M=\mathbb{R}^n$, $X=\mathbb{C}^n$ respectively:

$$V=\left\{(x,\sqrt{-1}\xi\cdot dx)\in \dot{T}_M^*X;\xi_1=\cdots=\xi_{n-1}=0\right\},$$
$$\Sigma=\left\{(x,\sqrt{-1}\xi\cdot dx)\in \dot{T}_M^*X;\xi_1=\cdots=\xi_{n-1}=x_n=0\right\},$$

where $\dot{T}_M^*X=T_M^*X\setminus M$. One sets $x=(x',x_n)$ with $x'=(x_1,\ldots,x_{n-1})$ and $\xi=(\xi',\xi_n)$ with $\xi'=(\xi_1,\ldots,\xi_{n-1})$. Let P be a differential operator with analytic coefficients defined near a point $0\in M$. Assume P is transversally elliptic in a neighborhood of $p_\circ=(0,\sqrt{-1}\,dx_n)\in\Sigma$, that is, P satisfies the property:

$$|\sigma(P)(x,\sqrt{-1}\xi/|\xi|)|\sim(|x_n|+|\xi'|/|\xi|)^l$$

for some non-negative integer l in a neighborhood of $p_\circ$. Here $\sigma(P)$ denotes the principal symbol of P. Grigis-Schapira-Sjöstrand [3] has given a theorem on the propagation of analytic singularities for this operator P along the bicharacteristic leaf of V passing through $p_\circ$.

On the other hand, assume P satisfies the property:

$$|\sigma(P)(x,\sqrt{-1}\xi/|\xi|)| \sim (|x_n|^k + |\xi'|/|\xi|)^l$$

for some non-negative integers k and l in a neighborhood of $p_\circ \in \Sigma$. We have proved in [1] unique solvability in the sheaf $\tilde{\mathcal{C}}_V^2$ of small second microfunctions for this operator P. This result was obtained by using our elementary construction of $\tilde{\mathcal{C}}_V^2$ and the estimate of the support of solution complexes with coefficients in $\tilde{\mathcal{C}}_V^2$. In this case, the structure of solutions of $Pu = f$ in the sheaf $\mathcal{C}_M$ of Sato microfunctions is reduced to that in the sheaf $\mathcal{A}_V^2$ of 2-analytic functions. Therefore our result implies the above theorem due to Grigis-Schapira-Sjöstrand [3] because any section of $\mathcal{A}_V^2$ has the property of the uniqueness of analytic continuation along the bicharacteristic leaves of V.

In connection with these operators, we consider a new class of differential operators with analytic coefficients defined near $0 \in M$:

$$P(x, D_{x'}, x_n D_{x_n}) = \sum_{|\alpha|\le m} a_\alpha(x) D_{x'}^{\alpha'}(x_n D_{x_n})^{\alpha_n}, \tag{1.1}$$

where $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $D_x^\alpha = D_{x_1}^{\alpha_1} \ldots D_{x_n}^{\alpha_n}$, and $D_{x_j} = \partial/\partial x_j$ for $\alpha = (\alpha', \alpha_n) = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$. Recall that the sheaf $\mathcal{A}_V^2$ of second analytic functions on V is defined by:

$$\mathcal{A}_V^2 = H^1(\mu_N(\mathcal{O}_X))|_V,$$

where $N = \{z \in X; \operatorname{Im} z_n = 0\}$ and μ_N denotes the functor of Sato's microlocalization along N. Any germ $f(x) \in \mathcal{A}_V^2$ at $p_\circ = (0, \sqrt{-1}\,dx_n)$ is obtained as boundary value of a holomorphic function:

$$f(x) = b_{D_r^{n-1}\times U_r}(F(z)), \tag{1.2}$$

where $F(z) \in \mathcal{O}(D_r^{n-1} \times U_r)$ for some $r > 0$, open sets:

$$\begin{aligned} D_r^{n-1} &= \{z \in \mathbb{C}^{n-1}; |z_j| < r, j = 1, \ldots, n-1\}, \\ U_r &= \{z_n \in \mathbb{C}; |z_n| < r, \operatorname{Im} z_n > 0\}. \end{aligned}$$

Now one makes the hypothesis:

$$a_{(m,0,\ldots,0)}(0) \neq 0, \quad a_{(0,\ldots,0,m)}(0) \neq 0. \tag{1.3}$$

By introducing an integral formula of Mellin's type for holomorphic functions, one has obtained the following theorem in [2] on the solvability for the operator $P\colon \mathcal{A}_V^2 \to \mathcal{A}_V^2$ at $p_\circ$.

Theorem 1.1. *Assume* (1.3) *for the differential operator* (1.1). *We assume, furthermore, a germ* $f \in \mathcal{A}^2_{V,p_\circ}$ *represented by* (1.2) *satisfies the following growth condition. There exist positive constants* $p < 1$, C *such that*

$$|F(z)| \leq C|\operatorname{Im} z_n|^{-p}, \quad z \in D_r^{n-1} \times U_r. \tag{1.4}$$

Then we can find a solution $u \in \mathcal{A}^2_{V,p_\circ}$ *of* $Pu = f$.

In Theorem 1.1, we need the growth condition (1.4) because of some condition in the integral formula. Here we will remove the growth condition (1.4) by improving an integral formula of Mellin's type.

Wakabayashi [5] also proved solvability of microhyperbolic operators and some second order operators in a different way.

2 Statements of the main theorems

Let D, $D' \subset \mathbb{C}^{n-1}$ be pseudoconvex domains with $D' \subset\subset D$ and let r, α, β be constants with $0 < r < 1$, $0 < \beta - \alpha < 2\pi$. We set $I_+ = (0, \pi/2)$, $I_- = (-\pi/2, 0)$.

Theorem 2.1. *Let* $f(z)$ *be a holomorphic function on* $D \times \{z_n \in \mathbb{C}; \alpha < \arg z_n < \beta, 0 < |z_n| < r\}$. *Then there exist* $\delta > 0$, $f_0(z) \in \mathcal{O}(D' \times \{z_n \in \mathbb{C}; |z_n| < \delta\})$, $g_\pm(z', \lambda) \in \mathcal{D}'(\{(z', \rho, \theta) \in D' \times \mathbb{R} \times I_\pm\})$ *with* $\lambda = \rho e^{i\theta}$ *such that for* $z' \in D'$, $|z_n| < \delta$, $\alpha < \arg z_n < \beta$, *we have:*

$$\begin{aligned} f(z) = f_0(z) &+ \int_{\Gamma_+} (z_n e^{-i\alpha})^{i\lambda} g_+(z', \lambda)\, d\lambda \\ &+ \int_{\Gamma_-} (z_n e^{-i\beta})^{-i\lambda} g_-(z', \lambda)\, d\lambda, \end{aligned}$$

and the following conditions are fulfilled.

(1) $\operatorname{supp} g_\pm \subset \{(z', \rho, \theta) \in D' \times \mathbb{R} \times I_\pm; \rho \geq 0\}$.

(2) $(\rho\partial/\partial\rho + i\partial/\partial\theta) g_\pm = 0$, $\partial g_\pm/\partial \bar{z}_j = 0$ *for* $j = 1, \ldots, n-1$; *in particular,* $g_\pm$ *are holomorphic functions of* (z', λ) *in* $\{\lambda \neq 0\}$.

(3) *For any* $\varepsilon > 0$ *there exists a positive constant* C_ε *such that one has* $|g_\pm(z', \rho e^{i\theta})| \leq C_\varepsilon$ *for* $z' \in D'$, $\rho \geq 1$, $(\pi/2) - \varepsilon \geq |\theta| \geq \varepsilon$.

Here, we choose the infinite paths $\Gamma_\pm$ *as follows:*

$$\Gamma_\pm \colon \lambda = \lambda_\pm(\rho) = \rho e^{i\theta_\pm(\rho)}, \quad \rho \in \mathbb{R}, \tag{2.1}$$

where each $\theta_\pm(\rho) \in C^\infty(\mathbb{R})$ *satisfies the following conditions respectively:*

$$\begin{cases} 0 < \pm\theta_\pm(\rho) < \pi/2 \\ \pm\theta_\pm(\rho) \downarrow 0,\ \mp\theta'_\pm(\rho) \downarrow 0 & \textit{as} \quad \rho \to +\infty \\ \rho^{-1} \log C_{|\theta_\pm(\rho)|} \to 0 & \textit{as} \quad \rho \to +\infty. \end{cases} \tag{2.2}$$

We apply Theorem 2.1 to the explicit construction of microlocal solutions for some differential operators treated in [2]. Let $p_\circ = (0, \sqrt{-1}\, dx_n) \in \Sigma$. We consider the following differential operator with analytic coefficients defined near $0 \in M$:

$$P(x, D_{x'}, x_n D_{x_n}) = \sum_{|\alpha| \le m} a_\alpha(x) D_{x'}^{\alpha'} (x_n D_{x_n})^{\alpha_n}, \tag{2.3}$$

where $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $D_x^\alpha = D_{x_1}^{\alpha_1} \ldots D_{x_n}^{\alpha_n}$, and $D_{x_j} = \partial/\partial x_j$ for $\alpha = (\alpha', \alpha_n) = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$. This type of operators covers the transversally elliptic operators treated by Grigis-Schapira-Sjöstrand [3] as for the symbols under the following condition:

$$a_{(m,0,\ldots,0)}(0) \neq 0, \quad a_{(0,\ldots,0,m)}(0) \neq 0. \tag{2.4}$$

Before giving the statements of theorems, we recall the sheaf $\mathcal{C}^{\mathbb{R}}_{Y|X}$ of holomorphic microfunctions on $T^*_Y X$ defined by:

$$\mathcal{C}^{\mathbb{R}}_{Y|X} = H^1(\mu_Y(\mathcal{O}_X)),$$

where $Y = \{z \in X; z_n = 0\}$. Any germ $f(x) \in \mathcal{C}^{\mathbb{R}}_{Y|X, p_\circ}$ is written:

$$f(x) = b_{D_r^{n-1} \times V_r}(F(z)),$$

where $F(z) \in \mathcal{O}(D_r^{n-1} \times V_r)$ for some $r > 0$, open sets:

$$\begin{aligned} D_r^{n-1} &= \{z \in \mathbb{C}^{n-1}; |z_j| < r, j = 1, \ldots, n-1\}, \\ V_r &= \{z_n \in \mathbb{C}; |z_n| < r, \operatorname{Im} z_n > -r|\operatorname{Re} z_n|\}. \end{aligned}$$

Then we have natural inclusion morphisms:

$$\mathcal{C}^{\mathbb{R}}_{Y|X}|_\Sigma \hookrightarrow \mathcal{A}^2_V|_\Sigma \hookrightarrow \mathcal{C}_M|_\Sigma,$$

where $\mathcal{C}_M (= \mu_M(\mathcal{O}_X)[n])$ is the sheaf of Sato microfunctions on M.

Let us consider the following Cauchy problem:

$$\begin{cases} P(z, D_{z'}, z_n D_{z_n}) u(z) = f(z) \\ \partial_{z_1}^j u(0, z_2, \ldots, z_n) = h_j(z_2, \ldots, z_n), \quad j = 0, \ldots, m-1, \end{cases} \tag{2.5}$$

where $P(z, D_{z'}, z_n D_{z_n})$ is the complexification of P at (2.3) satisfying the condition (2.4). We set complex submanifolds X', Y' of X as follows:

$$X \supset X' = \{z \in X; z_1 = 0\} \supset Y' = Y \cap X' = \{z \in X'; z_n = 0\}.$$

Further we set

$$\begin{aligned}\Sigma' = \big\{(z_2, \dots, z_n; \zeta_2, \dots, \zeta_n) \in T^*X'; \operatorname{Im} z_2 = \cdots = \operatorname{Im} z_{n-1} = z_n = 0,\\ \zeta_2 = \cdots = \zeta_{n-1} = \operatorname{Re}\zeta_n = 0\big\} \simeq \Sigma \cap \pi^{-1}(X')\end{aligned}$$

with a natural projection $\pi : T^*X \to X$.

Theorem 2.2. *Let $P(x, D_{x'}, x_n D_{x_n})$, $p_\circ$, X', Y', Σ' be as above, and*

$$f(z) \in \mathcal{C}^{\mathbb{R}}_{Y|X, p_\circ}, \quad h_j(z_2, \dots, z_n) \in \mathcal{C}^{\mathbb{R}}_{Y'|X', p'_\circ} \quad (j = 0, \dots, m-1)$$

with $p'_\circ = (0, \sqrt{-1}\, dx_n) \in T^*_{Y'}X'$ be any holomorphic microfunctions. We suppose the condition (2.4) for P. Then Cauchy problem (2.5) has a unique solution $u(z) \in \mathcal{C}^{\mathbb{R}}_{Y|X, p_\circ}$. In other words, we have the following exact sequence and isomorphism in a neighborhood of $p_\circ$:

$$\begin{gathered}0 \longrightarrow \mathcal{C}^{\mathbb{R}\ \ P}_{Y|X}|_\Sigma \longrightarrow \mathcal{C}^{\mathbb{R}}_{Y|X}|_\Sigma \xrightarrow{P} \mathcal{C}^{\mathbb{R}}_{Y|X}|_\Sigma \longrightarrow 0,\\ \mathcal{C}^{\mathbb{R}\ \ P}_{Y|X}|_{\Sigma\cap\pi^{-1}(X')} \xrightarrow{\sim} \left(\mathcal{C}^{\mathbb{R}}_{Y'|X'}\right)^m|_{\Sigma'},\end{gathered}$$

where $\mathcal{C}^{\mathbb{R}\ \ P}_{Y|X} := \operatorname{Ker}(\mathcal{C}^{\mathbb{R}}_{Y|X} \xrightarrow{P} \mathcal{C}^{\mathbb{R}}_{Y|X})$ and a natural trace morphism:

$$\mathcal{C}^{\mathbb{R}}_{Y|X}|_{\Sigma\cap\pi^{-1}(X')} \ni u(z) \longmapsto \left(\partial^j_{z_1} u(0, z_2, \dots, z_n)\right)_{j=0}^{m-1} \in \left(\mathcal{C}^{\mathbb{R}}_{Y'|X'}\right)^m|_{\Sigma'}.$$

Remark 2.3. According to Professor M. Uchida, this result is obtained also by the usual Cauchy-Kovalevski theorem and the method of the micro-support theory. However, our method is much more useful to get explicit forms of solutions; indeed, we use only once the Cauchy-Kovalevski theorem with a large parameter in solving the problems and never use arguments of analytic continuation.

Theorem 2.4. *Let $P(x, D_{x'}, x_n D_{x_n})$, $p_\circ$ be as above. We suppose the condition (2.4) for P. Then we have the following exact sequence and isomorphism in a neighborhood of $p_\circ$:*

$$\begin{gathered}0 \longrightarrow \mathcal{A}^{2\ P}_V|_\Sigma \longrightarrow \mathcal{A}^2_V|_\Sigma \xrightarrow{P} \mathcal{A}^2_V|_\Sigma \longrightarrow 0,\\ \mathcal{C}^{\mathbb{R}\ \ P}_{Y|X}|_\Sigma \xrightarrow{\sim} \mathcal{A}^{2\ P}_V|_\Sigma,\end{gathered}$$

where $\mathcal{A}^{2\ P}_V := \operatorname{Ker}(\mathcal{A}^2_V \xrightarrow{P} \mathcal{A}^2_V)$.

Remark 2.5. i) The last isomorphism is already obtained in Theorem 3.1 of [2]. We quoted it here for the reader's convenience. ii) In Theorem 3.2 of the former paper [2], we needed essentially an additional hypothesis concerning the growth order of the defining function $F(z)$ of $f(x)$:

$$|F(z)| \leq C|\operatorname{Im} z_n|^{-p}$$

for some $p \in (0,1)$ as $\operatorname{Im} z_n \to +0$. We can remove this condition by the new idea in the decomposition of holomorphic functions, though the main arguments about the explicit construction of solutions are the same as in the former paper [2].

Together with our former results in [1], we obtain the following theorem as a direct corollary of Theorems 2.2 and 2.4.

Theorem 2.6. *Let $P(x, D_{x'}, x_n D_{x_n})$, $p_\circ$, X', Y', Σ' be as above. We suppose the transversal ellipticity for the principal symbol $\sigma(P)$:*

$$|\sigma(P)(x, \sqrt{-1}\xi/|\xi|)| \sim (|x_n| + |\xi'|/|\xi|)^m \tag{2.6}$$

in a neighborhood of $p_\circ$ in $T^*_M X$. Then we have the following exact sequence and isomorphisms in a neighborhood of $p_\circ$:

$$0 \longrightarrow {\mathcal{C}_M}^P|_\Sigma \longrightarrow \mathcal{C}_M|_\Sigma \xrightarrow{P} \mathcal{C}_M|_\Sigma \longrightarrow 0, \tag{2.7}$$

$${\mathcal{C}_M}^P|_{\Sigma\cap\pi^{-1}(X')} \xleftarrow{\sim} {\mathcal{C}^{\mathbb{R}}_{Y|X}}^P|_{\Sigma\cap\pi^{-1}(X')} \xrightarrow{\sim} \left(\mathcal{C}^{\mathbb{R}}_{Y'|X'}\right)^m|_{\Sigma'}, \tag{2.8}$$

where ${\mathcal{C}_M}^P := \operatorname{Ker}(\mathcal{C}_M \xrightarrow{P} \mathcal{C}_M)$.

Proof. By the solvability result of [1] in small second microfunctions for a transversally elliptic equation $Pu = f$, we have the isomorphisms

$${\mathcal{A}^2_V}^P|_\Sigma \xrightarrow{\sim} {\mathcal{C}_M}^P|_\Sigma, \quad (\mathcal{A}^2_V/P\mathcal{A}^2_V)|_\Sigma \xrightarrow{\sim} (\mathcal{C}_M/P\mathcal{C}_M)|_\Sigma$$

in a neighborhood of $p_\circ$. We remark here that condition (2.6) implies our main condition (2.4) for P. Therefore the exactness of (2.7) follows from Theorem 2.4. Further the isomorphisms (2.8) follow from Theorems 2.4 and 2.2. □

3 A sketch of proof of Theorem 2.1

We can suppose from the beginning that $0 \leq \alpha < \beta < 2\pi$. Further we choose a pseudoconvex open set D'' as $D' \subset\subset D'' \subset\subset D$. We set:

$$U_0 = \{z_n \in \mathbb{C}; |z_n| < r\},$$
$$U_1 = \mathbb{P}^1 \setminus \{z_n \in \mathbb{C}; |z_n| \leq r, \beta \leq \arg z_n \leq \alpha + 2\pi\}.$$

Proposition 3.1. *One can find functions $f_j(z) \in \mathcal{O}(D \times U_j)$ for $j = 1,2$ such that $f = f_0 + f_1$ in $D \times \{z_n \in \mathbb{C}; \alpha < \arg z_n < \beta,\ 0 < |z_n| < r\}$ and $f_1(z', \infty) \equiv 0$.*

Next, choose the system of local coordinates $(z', w) = (z_1, \ldots, z_{n-1}, w)$ with

$$w = \log z_n, \quad \alpha < \arg z_n < \beta,$$

and set $w = u + iv$. Then we will decompose the second function $f_1(z', e^w)$ into a sum $f_+(z', w) + f_-(z', w)$ of holomorphic functions $f_\pm \in \mathcal{O}(D'' \times \Omega_\pm)$ satisfying some growth order conditions. Here we set:

$$\begin{aligned}
\Omega &= \{w \in \mathbb{C}; \operatorname{Re} w > \log r \ \text{ or } \ \alpha < \operatorname{Im} w < \beta\}, \\
\Omega^+ &= \{w \in \mathbb{C}; \operatorname{Re} w > \log r \ \text{ or } \ \operatorname{Im} w > \alpha\}, \\
\Omega^- &= \{w \in \mathbb{C}; \operatorname{Re} w > \log r \ \text{ or } \ \operatorname{Im} w < \beta\}.
\end{aligned}$$

To this end, we will solve a $\bar{\partial}$-equation under some growth order condition as follows: We choose a C^∞-function $\psi\colon \mathbb{R} \to \mathbb{R}$ such that $0 \le \psi(v) \le 1$ for $v \in \mathbb{R}$, $\psi(v) = 0$ for $v \le \alpha + \delta_1$ and $\psi(v) = 1$ for $v \ge \beta - \delta_1$, where $\delta_1 > 0$ is a small constant. Using this function, we define:

$$g(z', w) = \frac{\partial}{\partial \bar{w}}(f_1(z', e^w)\psi(v)) = \frac{i}{2} f_1(z', e^w)\psi'(v)$$

for $\alpha < v < \beta$. We can consider $g(z', w)$ as a C^∞-function on $D \times \mathbb{C}$ by setting $g(z', w) \equiv 0$ for $\operatorname{Im} w \in \mathbb{R} \setminus (\alpha, \beta)$.

Lemma 3.2. *There exists a C^∞-function $\chi\colon \mathbb{R} \to \mathbb{R}$ such that $g(z', w) \in L^2(D'' \times \mathbb{C}, \chi)$, $\chi'(u) < 0$, $\chi''(u) \ge 0$ for any $u \in \mathbb{R}$ and that $\chi(u) = 1/2 - u$ for $u > 0$.*

Lemma 3.3. *There exists a subharmonic function $\varphi(w) \in C^2(\mathbb{C})$ such that $\varphi(w) \ge \chi(u)$ for $\alpha + \delta_1 \le v \le \beta - \delta_1$ and $\varphi(w) = 0$ for $w \notin \{w \in \mathbb{C}; u < 1,\ \alpha < v < \beta\}$.*

From Lemmas 3.2 and 3.3, it follows that $g(z', w) \in L^2(D'' \times \mathbb{C}, \varphi)$. Then we can apply Theorem 4.4.2 in Hörmander [4] to $g(z', w)\, d\bar{w} \in L^2_{(0,1)}(D'' \times \mathbb{C}, \varphi)$, that is to say, there is a solution $h(z', w) \in L^2(D'' \times \mathbb{C}, \mathrm{loc})$ of the equation $\bar{\partial} h = g\, d\bar{w}$ such that

$$\int_{D'' \times \mathbb{C}} |h|^2 e^{-\varphi} \left(1 + |(z', w)|^2\right)^{-2} dV \le \int_{D'' \times \mathbb{C}} |g|^2 e^{-\varphi}\, dV.$$

In fact, $h \in L^2(D'' \times \mathbb{C}, \phi)$, where $\phi(z', w) := \varphi(w) + 2\log(1 + |(z', w)|^2)$.

$$f_+(z',w) = f_1(z',e^w)(1-\psi(v)) + h(z',w),$$
$$f_-(z',w) = f_1(z',e^w)\psi(v) - h(z',w).$$

We find immediately that $f_\pm \in \mathcal{O}(D'' \times \Omega^\pm)$ and that

$$f_1(z',e^w) = f_+(z',w) + f_-(z',w) \quad \text{for} \quad (z',w) \in D'' \times \Omega.$$

Proposition 3.4. *There exist positive-valued locally bounded functions $C_\theta^\pm$ on $I_\pm$ such that one has*

$$|f_\pm(z',w)| \le C_\theta^\pm \left(1+|w|^2\right) \quad \textit{for}\, \forall z' \in D',\ w = i\gamma_\pm \pm (\mu + i\nu)e^{-i\theta}$$

with $\mu \in \mathbb{R}$, $\nu \ge 0$, $\gamma_+ = \alpha$, $\gamma_- = \beta$.

Now, we define the following holomorphic functions:

$$F_\pm(z',w) = \frac{f_\pm(z',w)}{(w - i\gamma_\pm \pm i)^4}. \tag{3.1}$$

By Proposition 3.4, we can get the following estimates.

Corollary 3.5. *There exist positive-valued locally bounded functions $C_\theta^{\pm\prime}$ on $I_\pm$ such that one has*

$$|F_\pm(z',w)| \le \frac{C_\theta^{\pm\prime}}{1+\mu^2+\nu^2} \quad \textit{for} \quad z' \in D',\ w = i\gamma_\pm \pm (\mu + i\nu)e^{-i\theta}$$

with $\mu \in \mathbb{R}$, $\nu \ge 0$.

Definition 3.6. *One defines*

$$G_\pm(z',\lambda) = e^{-i\theta} \int_{-\infty}^{\infty} F_\pm(z', i\gamma_\pm \pm \mu e^{-i\theta}) e^{-i\mu\rho}\, d\mu, \tag{3.2}$$

for $z' \in D'$, $\lambda = \rho e^{i\theta}$ with $\rho \in \mathbb{R}$, $\theta \in I_\pm$.

Note that the integrals in (3.2) absolutely converge by Corollary 3.5 and that these functions are continuous in (z',ρ,θ). Note, moreover, that $G_\pm$ is written as:

$$G_\pm(z',\lambda) = \pm \int_{C_\pm(\theta)} F_\pm(z',w) e^{\mp i(w - i\gamma_\pm)\lambda}\, dw,$$

where $C_\pm(\theta)$ is the path $C_\pm(\theta)\colon w = i\gamma_\pm \pm \mu e^{-i\theta}$, $\mu \in \mathbb{R}$.

Lemma 3.7. (1) $\operatorname{supp} G_\pm \subset \{(z',\rho,\theta) \in D' \times \mathbb{R} \times I_\pm; \rho \geq 0\}$.

(2) $(\rho\partial/\partial\rho + i\partial/\partial\theta)G_\pm = 0$, $\partial G_\pm/\partial\bar{z}_j = 0$ *for* $j = 1, \ldots, n-1$; *in particular,* $G_\pm$ *are holomorphic functions of* (z',λ) *in* $\{\lambda \neq 0\}$.

(3) *There exist positive-valued locally bounded functions* $C_\theta^{\pm''}$ *on* $I_\pm$ *such that one has* $|G_\pm(z',\rho e^{i\theta})| \leq C_\theta^{\pm''}$ *for* $\forall z', \forall \rho$.

Definition 3.8. *We set the distributions* $g_\pm(z',\lambda) \in \mathcal{D}'(\{(z',\rho,\theta) \in D' \times \mathbb{R} \times I_\pm\})$ *with* $\lambda = \rho e^{i\theta}$ *in the statement of Theorem* 2.1 *by*

$$g_\pm(z',\lambda) = \frac{1}{2\pi}\Big(e^{-i\theta}\frac{\partial}{\partial\rho} + 1\Big)^4 G_\pm(z',\lambda).$$

Further we give the constant C_ε *by*

$$C_\varepsilon = C_{\pi/2-\varepsilon} := \frac{4!}{2\pi}\Big(\frac{1}{\sin(\varepsilon/2)} + 1\Big)^4 \cdot \sup\{C_\theta^{\pm''}; \frac{\varepsilon}{2} \leq |\theta| \leq \frac{\pi}{2} - \frac{\varepsilon}{2}\}$$

for $0 < \varepsilon \leq \pi/4$.

Then, since $\rho\partial_\rho + i\partial_\theta$ commutes with $e^{-i\theta}\partial_\rho$, we obtain the conditions (1) $\sim$ (3) of $g_\pm$ in Theorem 2.1 directly from Lemma 3.7 and the Cauchy estimates. Hereafter, let $\Gamma_\pm$ be any paths satisfying conditions (2.1), (2.2).

Lemma 3.9. *For any* $z' \in D'$, $w = i\gamma_\pm \pm (\mu + i\nu)e^{-i\theta}$ *with* $\mu \in \mathbb{R}$, $\nu > 0$ *and with* $\theta \in I_\pm$, *we have in a classical sense*

$$F_\pm(z',w) = \frac{e^{i\theta}}{2\pi}\int_{-\infty}^{\infty} G_\pm(z',\rho e^{i\theta})e^{i(\mu+i\nu)\rho}\,d\rho.$$

Further by the change of the path of the integration, we finally obtain that

$$F_\pm(z',w) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G_\pm(z',\rho e^{i\theta_\pm(\rho)})e^{\pm i(w-i\gamma_\pm)\rho e^{i\theta_\pm(\rho)}}(1 + i\rho\theta'_\pm(\rho))e^{i\theta_\pm(\rho)}d\rho$$

for any $z' \in D', w \in \{\pm \operatorname{Im} w > \pm\gamma_\pm\}$.

Here, recalling the relationship (3.1) between $f_\pm$ and $F_\pm$, we have

$$\begin{aligned}
f_\pm(z',w) &= (w - i\gamma_\pm \pm i)^4 F_\pm(z',w) \\
&= \int_{-\infty}^{\infty} \frac{\lambda'_\pm(\rho)}{2\pi} G_\pm(z',\lambda_\pm(\rho)) \cdot \Big(1 - \frac{1}{\lambda'_\pm(\rho)}\frac{\partial}{\partial\rho}\Big)^4 e^{\pm i(w-i\gamma_\pm)\lambda_\pm(\rho)}d\rho \\
&= \int_{-\infty}^{\infty} e^{\pm i(w-i\gamma_\pm)\lambda_\pm(\rho)} \cdot \Big(1 + \frac{\partial}{\partial\rho}\frac{1}{\lambda'_\pm(\rho)}\Big)^4 \Big(\frac{\lambda'_\pm(\rho)}{2\pi}G_\pm(z',\lambda_\pm(\rho))\Big)d\rho \\
&= \int_{-\infty}^{\infty} \frac{\lambda'_\pm(\rho)}{2\pi}e^{\pm i(w-i\gamma_\pm)\lambda_\pm(\rho)} \cdot \Big(1 + \frac{1}{\lambda'_\pm(\rho)}\frac{\partial}{\partial\rho}\Big)^4 G_\pm(z',\lambda_\pm(\rho))d\rho.
\end{aligned}$$

$$
\begin{aligned}
\partial_\rho G_\pm(z',\lambda_\pm(\rho)) &= [\partial_\rho G_\pm(z',\rho e^{i\theta}) + \theta'_\pm(\rho)\partial_\theta G_\pm(z',\rho e^{i\theta})]|_{\theta=\theta_\pm(\rho)} \\
&= [\partial_\rho G_\pm(z',\rho e^{i\theta}) + i\rho\theta'_\pm(\rho)\partial_\rho G_\pm(z',\rho e^{i\theta})]|_{\theta=\theta_\pm(\rho)} \\
&= e^{-i\theta_\pm(\rho)}\lambda'_\pm(\rho)[\partial_\rho G_\pm(z',\rho e^{i\theta})]|_{\theta=\theta_\pm(\rho)},
\end{aligned}
$$

we get

$$
\begin{aligned}
(1+\lambda'_\pm(\rho)^{-1}\partial_\rho)^4 G_\pm(z',\lambda_\pm(\rho)) &= [(1+e^{-i\theta}\partial_\rho)^4 G_\pm(z',\rho e^{i\theta})]|_{\theta=\theta_\pm(\rho)} \\
&= 2\pi g_\pm(z',\lambda_\pm(\rho)).
\end{aligned}
$$

Therefore we obtain

$$
\begin{aligned}
f_\pm(z',w) &= \int_{-\infty}^{\infty} e^{\pm i(w-i\gamma_\pm)\lambda_\pm(\rho)} \cdot g_\pm(z',\lambda_\pm(\rho))\lambda'_\pm(\rho)d\rho \\
&= \int_{\Gamma_\pm} e^{\pm i(w-i\gamma_\pm)\lambda} \cdot g_\pm(z',\lambda)d\lambda.
\end{aligned}
$$

This completes the proof of Theorem 2.1.

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