Microlocalization
of Topological Boundary Value Morphism
and Regular-Specializable Systems

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Introduction

In microlocal analysis, it is one of the main subjects to give an appropriate formulation of the boundary value problems for hyperfunction or microfunction solutions to a system of linear partial differential equations with analytic coefficients (that is, a coherent (left) \( D \)-Module, here in this article, we shall write Module with a capital letter, instead of sheaf of modules). If the system is regular-specializable, the nearby-cycle of the system can be defined in the theory of \( D \)-Modules. After the results by Kashiwara and Oshima [K-O], Oshima [Os] and Schapira [Sc 2], [Sc 3], for any hyperfunction solutions to regular-specializable system Monteiro Fernandes [MF 1] defined a boundary value morphism which takes values in hyperfunction solutions to the nearby-cycle of the system instead of the induced system. This morphism is injective (cf. [MF 2]) and a generalization of the non-characteristic boundary value morphism (for the non-characteristic case, see Komatsu and Kawai [Ko-K], Schapira [Sc 1] and further Kataoka [Kat]). Moreover recently Laurent and Monteiro Fernandes [L-MF 2] reformulated this boundary value morphism and discussed the solvability under a kind of hyperbolicity condition (the near-hyperbolicity). However, since this morphism is defined only for hyperfunction solutions, a microlocal boundary value problem is not considered. Therefore in this article, we shall state a microlocalization of their result in the framework of Oaku [Oa 2] and Oaku-Yamazaki [O-Y].

The details of this article will be given in our forthcoming paper [Y].

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1 Notation

We denote the set of integers, of real numbers and of complex numbers by $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ respectively as usual. Moreover we set $\mathbb{N} := \{n \in \mathbb{Z}; n \geq 1\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

All the manifolds are assumed to be paracompact. Let $\tau: E \rightarrow Z$ a vector bundle over a manifold $Z$. Then, set $\dot{E} := E \setminus Z$ and $\dot{\tau}$ the restriction of $\tau$ to $\dot{E}$. Let $M$ be an $(n+1)$-dimensional real analytic manifold and $N$ a one-codimensional closed real analytic submanifold of $M$. Let $X$ and $Y$ be complexifications of $M$ and $N$ respectively such that $Y$ is a closed submanifold of $X$ and that $Y \cap M = N$. Moreover, we assume the existence of a partial complexification of $M$ in $X$; that is, there exists a $(2n+1)$-dimensional real analytic submanifold $L$ of $X$ containing both $M$ and $Y$ such that the triplet $(N, M, L)$ is locally isomorphic to $\left(\mathbb{R}^n \times \{0\}, \mathbb{R}^{n+1}, \mathbb{C}^n \times \mathbb{R}\right)$ by a local coordinate system $(z, \tau) = (x + \sqrt{-1} y, t + \sqrt{-1} s)$ of $X$ around each point of $N$. We say such a coordinate system admissible. We shall mainly follow the notation in Kashiwara-Schapira [K-S]; we denote the normal deformations of $N$ and $Y$ in $M$ and $L$ by $\overline{M}_N$ and $\tilde{L}_Y$ respectively and regard $\overline{M}_N$ as a closed submanifold of $\tilde{L}_Y$. We have the following commutative diagram:

![Commmutative Diagram]

and by admissible coordinates we have locally the following relation:

$$N = \mathbb{R}_x^n \times \{0\} \hookrightarrow M = \mathbb{R}_z^n \times \mathbb{R}_t$$

$$Y = \mathbb{C}_z^n \times \{0\} \hookrightarrow L = \mathbb{C}_t^n \times \mathbb{R}_t \quad X = \mathbb{C}_z^n \times \mathbb{C}_t.$$

With these coordinates, we often identify $T_YX$ and $T_YL$ with $X$ and $L$ respectively.

The projection $\tau_Y: T_YL \rightarrow Y$ and $s_L: T_YL \rightarrow \tilde{L}_Y$ induce natural mappings:

$$T^*_N Y \hookrightarrow T_N^* M \times T^*_Y \xrightarrow{r_Y} T^*_M T_Y L \xleftarrow{i_Y} T^*_N M \times T^*_M \tilde{L}_Y \xrightarrow{s_L^*} T^*_M \tilde{L}_Y,$$

and by these mappings, we identify $T^*_N T_Y L$ with $T_N^* Y$ and $T_N^* \tilde{L}_Y$.
$T_Y L \setminus T_Y Y$ has two components with respect to its fiber. We denote one of them by $T_Y L^+$ and represent (at least locally) by fixing an admissible coordinate system

$$T_Y L^+ = \{(z, t) \in T_Y L; t > 0\}.$$  
Moreover set $T_N M^+ := T_Y L^+ \cap T_N M$. Set an open embedding $f: T_Y L^+ \hookrightarrow T_Y L$ and $f_N := f|_{T_N M^+}: T_N M^+ \hookrightarrow T_N M$. We regard $T_N M^+ \times T^*_N Y$ as an open set of $T_T M Y$. Moreover $f$ induces mappings:

$$T_{T_N M^+}^* T_Y L^+ \to T_N M^+ \times T_T M^* T_Y L,$$

$$f^* \otimes \omega_L/X.$$  
Hence we identify $T_{T_N M^+}^* T_Y L^+$ with $T_N M^+ \times T^*_N Y$, and $f^*$ with $f \times \text{id}$. Let $\pi_{N,M}: T_T M \to \overline{M}_{N}$ and $\pi_{N|M}: T_{T_N M}^* T_Y L \to T_N M$, be the natural projections. We denote as usual by $\nu$ and $\mu$ the Sato specialization and microlocalization functors respectively.

## 2 General Boundary Values

By using an admissible coordinate system we define a continuous section $\sigma: Y \to \tilde{T}_Y X$ by $z \mapsto (z, 1)$. Similarly we define $\theta: Y \to \tilde{T}_Y X$ by $z \mapsto (z, 1)$. In general, let $Z$ be a complex manifold, $\tau: E \to Z$ a complex vector bundle. Then, denote by $\mathbf{D}^b_c(X)$ the subcategory of $\mathbf{D}^b(X)$ consisting of $\mathbb{C}^x$-conic objects.

### 2.1 Theorem

For any object $\mathcal{F}$ of $\mathbf{D}^b(X)$ such that $\nu_Y(\mathcal{F}) \in \text{Ob}(\mathbf{D}^b_c(T_Y X))$, there exists the following natural isomorphism:

$$f^*_\pi \mu_{T_N M} (\nu_Y(i_* \mathcal{F})) \to f^*_\pi \mu_{T_N M} \mu_N (\sigma^{-1} \nu_Y(\mathcal{F})) \otimes \omega_L/X.$$

### 2.2 Definition

For any object $\mathcal{F}$ of $\mathbf{D}^b(X)$ such that $\nu_Y(\mathcal{F}) \in \text{Ob}(\mathbf{D}^b_c(T_Y X))$, we define by virtue of Kashiwara-Schapira [K-S] and Theorem 2.1:

$$\beta: f^*_\pi s_* \mu_M (R j L \cdot \pi L_1 i_* \mathcal{F}) \to f^*_\pi \mu_{T_N M} (\nu_Y(i_* \mathcal{F}))$$

$$\approx f^*_\pi \pi^{-1} \nu_Y(\mathcal{F}) \otimes \omega_L/X.$$

### 2.3 Definition (Laurent-Monteiro Fernandes [L-MF 2])

We say an object $\mathcal{F}$ of $\mathbf{D}^b(X)$ is near-hyperbolic at $x_0 \in N$ (in $dt$-codirection) if there exist positive constants $C$ and $\epsilon_1$ such that

$$\text{SS}(\mathcal{F}) \cap \{(z, \tau; z^*, \tau^*) \in T^* X; |z - x_0|, |\tau| < \epsilon_1, 0 < \text{Re} \tau\}$$

$$\subset \{(z, \tau; z^*, \tau^*) \in T^* X; |\text{Re} \tau^*| < C(|\text{Im} z^*|(|\text{Im} z| + |\text{Im} \tau|) + |\text{Re} z^*|)\}$$

holds by an admissible coordinate system. Here $\text{SS}(\mathcal{F})$ denotes the microsupport of $\mathcal{F}$. 


2.4 Theorem. Let \( \mathcal{F} \) be a object of \( \mathcal{D}^b(X) \). Assume that \( \nu_Y(\mathcal{F}) \in \text{Ob}(\mathcal{D}^b_{\mathbb{C}^\times}(T_YX)) \) and \( \mathcal{F} \) is near-hyperbolic at \( x_0 \in N \). Then, for any \( p^* \in T^*_N M^+T_YL^+ \)

\[
\beta: s_{L^+}^{-1}\mu_{M_N}(Rj_L\cdot \tilde{p}_L^{-1}i_L^1 F)_{p^*} \rightarrow \mu_N(\sigma^{-1}\nu_Y(\mathcal{F}))_{\tau_{Y*}(p^*)} \otimes \omega_{L/X}
\]

is an isomorphism.

3 Regular-Specializable Systems

In this section, we shall recall the basic results concerning the regular-specializable \( \mathcal{D} \)-Module and its nearby-cycle.

As usual, we denote by \( \mathcal{D}_X \) the sheaf on \( X \) of holomorphic differential operators, and by \( \{D^{(m)}_X\}_{m \in \mathbb{N}_0} \) the usual order filtration on \( \mathcal{D}_X \).

3.1 Definition. Denote by \( \mathcal{J}_Y \) the defining Ideal of \( Y \) in \( \mathcal{O}_X \) with a convention that \( \mathcal{J}_Y^j = \mathcal{O}_X \) for \( j \leq 0 \). The \( V \)-filtration \( \{V^k_Y(D_X)\}_{k \in \mathbb{Z}} \) (along \( Y \)) is a filtration on \( \mathcal{D}_X|_Y \) defined by

\[
V^k_Y(D_X) := \bigcap_{j \in \mathbb{Z}} \{P \in \mathcal{D}_X|_Y; P \mathcal{J}_Y^j \subset \mathcal{J}_Y^{j-k}\}.
\]

Let us denote by \( \vartheta \) the Euler operator. Note that \( \vartheta \in V^0_Y(D_X) \setminus V^{-1}_Y(D_X) \) and that \( \vartheta \) can be represented by \( \tau \partial_x \) or admissible coordinates.

3.2 Definition. A coherent \( \mathcal{D}_X|_Y \)-Module \( \mathcal{M} \) is said to be regular-specializable (along \( Y \)) if there exist locally a coherent \( \mathcal{O}_X \)-sub-Module \( \mathcal{M}_0 \) of \( \mathcal{M} \) and a non-zero polynomial \( b(\alpha) \in \mathbb{C}[\alpha] \) such that the following conditions are satisfied:

1. \( \mathcal{M}_0 \) generates \( \mathcal{M} \) over \( \mathcal{D}_X \); that is, \( \mathcal{M} = \mathcal{D}_X \mathcal{M}_0 \);
2. \( b(\vartheta) \mathcal{M}_0 \subset (\mathcal{D}^{(m)}_X \cap V^{-1}_Y(D_X)) \mathcal{M}_0 \), where \( m \) is the degree of \( b(\alpha) \).

In what follows, we shall omit the phrase "along \( Y \)" since \( Y \) is fixed.

3.3 Remark. (1) Let \( \mathcal{M} \) be a coherent \( \mathcal{D}_X|_Y \)-Module for which \( Y \) is non-characteristic. Then, it is easy to see that \( \mathcal{M} \) is regular-specializable.

(2) Kashiwara-Kawai [K-K] proved that every regular-holonomic \( \mathcal{D}_X|_Y \)-Module is regular-specializable.

3.4 Proposition. If \( \mathcal{M} \) is a regular-specializable \( \mathcal{D}_X|_Y \)-Module, \( R\text{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_Y) \) and \( R\text{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_Y) \) are objects of \( \mathcal{D}^b_{\mathbb{C}^\times}(T_YX) \) and \( \mathcal{D}^b_{\mathbb{C}^\times}(T_YX) \) respectively.

Let \( \iota: Y \rightarrow X \) be the natural inclusion. Then the induced system, or the inverse image in the sense of \( \mathcal{D} \)-Modules is defined by \( \mathcal{D} \iota^* \mathcal{M} := \mathcal{O}_Y \otimes_{\mathcal{O}_X} \iota^{-1} \mathcal{M} \).

For any regular-specializable \( \mathcal{D}_X \)-Module \( \mathcal{M} \), the nearby-cycle \( \Psi_Y(\mathcal{M}) \) of \( \mathcal{M} \) and the vanishing-cycle \( \Phi_Y(\mathcal{M}) \) of \( \mathcal{M} \) in the theory of \( \mathcal{D} \)-Modules can be defined. For the definitions of \( \Psi_Y(\mathcal{M}) \) and \( \Phi_Y(\mathcal{M}) \), we refer to Laurent [L], Mebkhout [Me]. We shall recall the following two results:
3.5 Proposition (Laurent [L], Mebkhout [Me]). Let $\mathcal{M}$ be a regular-specializable $\mathcal{D}_X|_Y$-Module. Then, $\Psi_\mathcal{Y}(\mathcal{M})$, $\Phi_\mathcal{Y}(\mathcal{M})$ and each cohomology of $\mathcal{D}_t^* \mathcal{M}$ are coherent $\mathcal{D}_Y$-Modules. Moreover, there exists the following distinguished triangle:

$$\Phi_\mathcal{Y}(\mathcal{M}) \xrightarrow{\text{Var}} \Psi_\mathcal{Y}(\mathcal{M}) \rightarrow \mathcal{D}_t^* \mathcal{M} \rightarrow +1.$$

Here, $\text{Var} := \varphi(\partial) \tau$ with $\varphi(\zeta) := (e^{2\pi \sqrt{-1} \zeta} - 1)/\zeta$.

3.6 Theorem (Laurent [L]). Let $\mathcal{E}_{\mathcal{Y}|X}$ be the sheaf of real holomorphic microfunctions on $T^*_X X$ as usual. Let $\mathcal{M}$ be a regular-specializable $\mathcal{D}_X|_Y$-Module. Then, there exists the following isomorphism of distinguished triangles:

$$\mathcal{R}\text{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)|_Y \rightarrow \mathcal{R}\text{Hom}_{\mathcal{D}_X}(\mathcal{M}, \sigma^{-1} \mathcal{E}_{\mathcal{Y}|X}) \rightarrow \mathcal{R}\text{Hom}_{\mathcal{D}_X}(\mathcal{M}, \psi^{-1} \mathcal{E}_{\mathcal{Y}|X}) \rightarrow +1.$$

3.7 Remark. (1) The isomorphism (the Cauchy-Kovalevskaja type theorem)

$$\mathcal{R}\text{Hom}_{\mathcal{D}_Y}(\mathcal{D}_t^* \mathcal{M}, \mathcal{O}_Y) \simeq \mathcal{R}\text{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)|_Y$$

holds for Fuchsian systems in the sense of Laurent-Monteiro Fernandes [L-MF 1].

(2) Recently Mandai [Man] extended the definition of boundary values to a general Fuchsian differential equation in the complex domain.

4 Boundary Values for Regular-Specializable System

We denote by $\mathcal{O}_X$, $\mathcal{B}_M$ and $\mathcal{C}_M$ the sheaf of holomorphic functions on $X$, of hyperfunctions on $M$ and of microfunctions on $T^*_M X$ respectively.

4.1 Definition (Oaku [Oa2], Oaku-Yamazaki [O-Y]). We set:

$$\mathcal{C}_{N|M} := s_{L*}^{-1} \mu_{\overline{M}_{N}}(R\pi_{L*} \tilde{p}_{L}^{-1} \mathcal{O}_X) \otimes \mathcal{O}_{M/X}[n + 1].$$

We can regard $\mathcal{C}_{N|M}$ as a microlocalization of $\nu_N(\mathcal{B}_M)$:

4.2 Proposition. (1) $\mathcal{C}_{N|M}$ is concentrated in degree zero; that is, $\mathcal{C}_{N|M}$ is regarded as a sheaf on $T^*_X M \times Y L$. Further $\mathcal{C}_{N|M}|_{T^*_X M} = \nu_N(\mathcal{B}_M)$ holds.

(2) There exists the following exact sequence on $T^*_X M$:

$$0 \rightarrow \nu_Y(\mathcal{B}_L)|_{T^*_X M} \rightarrow \nu_N(\mathcal{B}_M) \rightarrow \pi_{N|M*} \mathcal{C}_{N|M} \rightarrow 0.$$
4.3 Definition. Let \( M \) be a regular-specializable \( \mathcal{D}_X \mid_Y \)-Module. By Proposition 3.4, \( R\mathcal{H}om_{\mathcal{D}_X}(M, \mathcal{O}_X) \) satisfies the assumption of Theorem 2.1. Thus, by Definition 2.2 and Theorem 3.6, we define:

\[
\beta: f_{\pi}^{-1}R\mathcal{H}om_{\mathcal{D}_X}(M, \mathcal{C}_{N|M}) \rightarrow f_{\pi}^{-1}\tau_{Y\pi}^{-1}R\mathcal{H}om_{\mathcal{D}_Y}(\Psi_Y(M), \mathcal{C}_N).
\]

4.4 Theorem. (1) The morphism \( \beta \) gives a monomorphism

\[
\beta^0: f_{\pi}^{-1}\mathcal{H}om_{\mathcal{D}_X}(M, \mathcal{C}_{N|M}) \rightarrow f_{\pi}^{-1}\tau_{Y\pi}^{-1}\mathcal{H}om_{\mathcal{D}_Y}(\Psi_Y(M), \mathcal{C}_N).
\]

(2) The restriction of \( \beta^0 \) to the zero-section \( T_NM^+ \) coincides with the boundary value morphism in the sense of Monteiro Fernandes [MF1].

4.5 Definition. Let \( M \) be a coherent \( \mathcal{D}_X \mid_Y \)-Module. Then we say \( M \) is near-hyperbolic at \( x_0 \in N \) (in \( dt \)-codirection) if \( R\mathcal{H}om_{\mathcal{D}_X}(M, 0_X) \) is near-hyperbolic in the sense of Definition 2.3. Here, we remark that \( \mathcal{S}\mathcal{S}(R\mathcal{H}om_{\mathcal{D}_X}(M, 0_X)) = \text{char}(M) \).

The following theorem is a direct consequence of Theorem 2.4:

4.6 Theorem. Let \( M \) be a regular-specializable \( \mathcal{D}_X \mid_Y \)-Module. Assume that \( M \) is near-hyperbolic at \( x_0 \in N \). Then, for any \( p^* \in T^*_T M^+ \)

\[
\beta: R\mathcal{H}om_{\mathcal{D}_X}(M, \mathcal{C}_{N|M})_{p^*} \rightarrow R\mathcal{H}om_{\mathcal{D}_Y}(\Psi_Y(M), \mathcal{C}_N)_{\tau_{Y\pi}(p^*)}
\]

is an isomorphism.

4.7 Remark. Let \( \mathcal{C}_{N|M}^F \) be the sheaf of \( F \)-mild microfunctions on \( T^*_T M^+ T_Y L^+ \), and set \( \mathcal{C}_{N|M}^A := \mathcal{H}^n(\mu_N(\mathcal{O}_X \mid_Y)) \otimes \mathcal{O}_{N/Y} \) (see Oaku [Oa1], [Oa2], and Oaku-Yamazaki [O-Y]). Let \( M \) be a regular-specializable \( \mathcal{D}_X \mid_Y \)-Module. Set \( M_Y := \mathcal{H}^0(D\iota^* M) = \mathcal{O}_Y \otimes \iota^{-1} M \).

By the argument in Oaku-Yamazaki [O-Y] we have the following commutative diagram:

\[
\begin{array}{ccc}
\mathcal{H}om_{\mathcal{D}_X}(M, \mathcal{C}_{N|M}^F) & \rightarrow & \mathcal{H}om_{\mathcal{D}_X}(M, \mathcal{C}_{N|M}^A) \\
\downarrow & & \downarrow \\
\tilde{\mathcal{H}}om_{\mathcal{D}_X}(M, \mathcal{C}_{N|M}) & \rightarrow & \mathcal{H}om_{\mathcal{D}_Y}(\Psi_Y(M), \mathcal{C}_N)
\end{array}
\]

that is, the boundary value morphism

\[
\gamma^F: f_{\pi}^{-1}\mathcal{H}om_{\mathcal{D}_X}(M, \mathcal{C}_{N|M}^F) \rightarrow f_{\pi}^{-1}\tau_{Y\pi}^{-1}\mathcal{H}om_{\mathcal{D}_Y}(\Psi_Y(M), \mathcal{C}_N)
\]

and \( \beta^0 \) are compatible. In particular, if \( Y \) is non-characteristic for \( M \), then it is known that \( \Psi_Y(M) \simeq D\iota^* M \simeq M_Y \) and by Oaku [Oa2] (cf. Oaku-Yamazaki [O-Y]) we have:

\[
\tilde{\gamma}_{N|M}: R\mathcal{H}om_{\mathcal{D}_X}(M, \mathcal{C}_{N|M}) \rightarrow \tau_{Y\pi}^{-1}R\mathcal{H}om_{\mathcal{D}_Y}(M_Y, \mathcal{C}_N).
\]
In this case we see that $\beta^0$ is equivalent to the non-characteristic boundary value morphism (see Kataoka [Kat] and Oaku [Oa 1]). In particular, the restriction of $\beta^0$ to the zero-section $T^*_NM^+$ is equivalent to Komatsu-Kawai [Ko-K] and Schapira [Sc 1]. Further, if $Y$ is non-characteristic for $M$ and $\pm dt \in T^*_NM$ is hyperbolic for $M$, then the nearly-hyperbolic condition is satisfied and $\beta$ is an isomorphism.

4.8 Example. Assume that $X = \mathbb{C}^{n+1}$ and so on by an admissible coordinate system.

1) Let $b(\alpha)$ be a non-zero polynomial with degree $m$, and $Q \in \mathcal{D}_X^{(m)} \cap V^{-1}_{Y}(\mathcal{D}_X)$. Set $M := \mathcal{D}_X/\mathcal{D}_X(b(\vartheta) + Q)$. Then $M$ is regular-specializable. Assume that $b(\alpha) = \prod_{j=1}^{\mu}(\alpha - \alpha_j)^{\nu_j}$, $\alpha_i - \alpha_j \notin \mathbb{Z}$ for $1 \leq i \neq j \leq \mu$, note that $\sum_{j=1}^{\mu} \nu_j = m$. Then a direct calculation shows that $\mathcal{F}_Y(M) \simeq \mathcal{D}_Y^{\oplus m}$, and $\beta^0$ is equivalent to $\gamma$ in Oaku [Oa 2]: Let $p^* = (x_0, t_0; \sqrt{-1}\langle \xi_0, dx \rangle)$ be a point of $T^*_{N,M} T^*_Y L^+$, and $f(x,t)$ a germ of $\mathcal{H}om_{\mathcal{D}_X} (M, C_N|_M)$ at $p^*$. Then, we can see that $f(x,t)$ has a defining function

$$F(z, \tau) = \sum_{j=1}^{\mu} \sum_{k=1}^{\nu_j} F_{jk}(z, \tau) \tau^{\alpha_j} (\log \tau)^{k-1}.$$  

Here each $F_{jk}(z, \tau)$ is holomorphic on a neighborhood of $\{(0,0) \in X; |x_0 - z| < \varepsilon, \operatorname{Im} z \in \Gamma \}$ with a positive constant $\varepsilon$ and an open convex cone $\Gamma$ such that $\xi_0 \in \operatorname{Int}(\Gamma^o)$ (the interior of the dual cone $\Gamma^o$ of $\Gamma$). Then, $\beta^0(f)$ is equivalent to $\{\sigma \mathcal{N}(F_{jk}(x + \sqrt{-1}\Gamma 0,0)); 1 \leq j \leq \nu_j, 1 \leq j \leq \mu \}$. Moreover, if the principal symbol of $b(\vartheta) + Q$ is written as $\tau^m P(z, \tau; z^*, \tau^*)$ for a hyperbolic polynomial $P$ at $dt$-codirection, the nearly-hyperbolic condition is satisfied. Note that this operator is a special case of Fuchsian hyperbolic operators due to Tahara [T].

2) Take an operator $A(z; \partial_z) \in \mathcal{D}_Y^{(1)}$ at the origin and set $A^0 := \{0,1\} A \circ A^{(j-1)} \in \mathcal{D}_Y^{(j)}$ for $j \geq 1$. Let $p^* = (0,1; \sqrt{-1}\langle \xi, dx \rangle)$ be a point of $T^*_{N,M} T^*_Y L^+$ and set $p_0 := (0, \sqrt{-1}\langle \xi_0, dx \rangle) \in T^*_{N,Y}$. Set $P := (\vartheta - \alpha_1)(\vartheta - \alpha_2) - \tau A(z; \partial_z) \vartheta \in \mathcal{D}_X|_Y$, where $(\alpha_1, \alpha_2) \in \mathbb{C}^{\oplus 2}$. Consider $M := \mathcal{D}_X/\mathcal{D}_X P = \mathcal{D}_X u$, where $u := 1 \bmod P$. Let $f(x,t)$ be a germ of $\mathcal{K}om_{\mathcal{D}_X} (M, C_N|_M)$ at $p^*$. Then:

(i) If $(\alpha_1, \alpha_2) = (-1,0)$, then

$$\phi_Y(M) = \frac{V^0_Y(D_X)u + V^1_Y(D_X)(\vartheta + 1)u}{V^{-1}_Y(D_X)u + V^0_Y(D_X)(\vartheta + 1)u} = D_Y[u] + D_Y[\vartheta + 1]u \simeq D_Y^{\oplus 2},$$

$$\psi_Y(M) = \frac{V^{-1}_Y(D_X)u + V^0_Y(D_X)(\vartheta + 1)u}{V^{-2}_Y(D_X)u + V^{-1}_Y(D_X)(\vartheta + 1)u} = D_Y[\tau u] + D_Y[(\vartheta + 1)u] \simeq D_Y^{\oplus 2},$$

and $\operatorname{Var}: ([u], [\partial_z(\vartheta - 1)u]) \mapsto ([\tau u], 0)$. Hence $M_Y \simeq D_Y[(\vartheta + 1)u] \simeq D_Y$. In this case $f(x,t)$ has the following defining function:

$$F(z, \tau) = U_0(z) + \frac{U_{-1}(z)}{\tau} - \sum_{j=1}^{\infty} \frac{A^{(j)} U_{-1}(z)}{j-1} \tau^{j-1} - A U_{-1}(z) \log \tau,$$
and $\beta^0(f(x,t))$ is given by $\{\text{sp}_N(U_i(x))\}_{i=-1,0}$ at $p_0$. If $f(x,t)$ is $F$-mild at $p_0$, then $U_{-1}(z) = 0$ and $\gamma^F(f(x,t)) = \{f(x,+0)\} = \{\text{sp}_N(U_0(x))\}$.

(ii) If $(\alpha_1, \alpha_2) = (0,1)$, then:

$$\Phi_Y(M) = \frac{V_Y^1(D_X)u + V_Y^2(D_X)\partial u}{V_Y^1(D_X)u + V_Y^2(D_X)\partial u} = D_Y[\partial \tau u] + D_Y[\partial^2 \tau u] \simeq D_Y^{\oplus 2},$$

$$\Psi_Y(M) = \frac{V_Y^0(D_X)u + V_Y^1(D_X)\partial u}{V_Y^0(D_X)u + V_Y^1(D_X)\partial u} = D_Y[u] + D_Y[\partial \tau u] \simeq D_Y^{\oplus 2},$$

and $\text{Var}[\partial \tau u] = \text{Var}[\partial^2 \tau u] = 0$. Hence $M_Y \simeq D_Y[u] + D_Y[\partial \tau u] \simeq D_Y^{\oplus 2}$. In this case $f(x,t)$ has the following defining function:

$$F(z,\tau) = U_0(z) + \sum_{j=0}^{\infty} A(j)U_1(z) \tau^{j+1},$$

and $f(x,t)$ is always $F$-mild. Hence $\beta^0(f(x,t))$ at $p_0$ coincides with $\gamma^F(f(x,t)) = \{\partial t f(x,+0)\}_{i=0,1} = \{\text{sp}_N(U_i(x))\}_{i=0,1}$ (if $\tau \neq 0$, $M$ is isomorphic to $D_X/\mathcal{D}_X(\partial^2 - A(z;\partial_z)\partial \tau)$ for which $Y$ is non-characteristic).

(iii) If $(\alpha_1, \alpha_2) = (1,1)$, then:

$$\Phi_Y(M) = \frac{V_Y^2(D_X)u}{V_Y^1(D_X)u} = D_Y[\partial^2 \tau u] + D_Y[\partial^2(\partial - 1)u] \simeq D_Y^{\oplus 2},$$

$$\Psi_Y(M) = \frac{V_Y^1(D_X)u}{V_Y^0(D_X)u} = D_Y[\partial \tau u] + D_Y[\partial(\partial - 1)u] \simeq D_Y^{\oplus 2},$$

and $\text{Var}([\partial^2 \tau u], [\partial^2(\partial - 1)u]) \mapsto (2\pi^{-1}[\partial \tau(\partial - 1)u], 0)$. Hence $M_Y \simeq D_Y[\partial \tau u] \simeq D_Y$. In this case $f(x,t)$ has the following defining function:

$$F(z,\tau) = \sum_{j=0}^{\infty} A(j)U_0(z) \tau^{j+1} - \sum_{j=1}^{\infty} \sum_{k=1}^{j} \frac{A(j)U_1(z)}{k} \tau^{j+1} + \sum_{j=0}^{\infty} A(j)U_1(z) \tau^{j+1} \log \tau,$$

and $\beta^0(f(x,t))$ is given by $\{\text{sp}_N(U_i(x))\}_{i=0,1}$ at $p_0$. If $f(x,t)$ is $F$-mild at $p_0$, then $U_0(z) = 0$ and $\gamma^F(f(x,t)) = \{\partial_t f(x,+0)\} = \{\text{sp}_N(U_1(x))\}$.

(iv) If $(\alpha_1, \alpha_2) = (1,2)$, then:

$$\Phi_Y(M) = \frac{V_Y^2(D_X)u + V_Y^3(D_X)(\partial - 1)u}{V_Y^1(D_X)u + V_Y^2(D_X)(\partial - 1)u} = D_Y[\partial^2 \tau u] + D_Y[\partial^3(\partial - 1)u] \simeq D_Y^{\oplus 2},$$

$$\Psi_Y(M) = \frac{V_Y^1(D_X)u + V_Y^2(D_X)(\partial - 1)u}{V_Y^0(D_X)u + V_Y^1(D_X)(\partial - 1)u} = D_Y[\partial \tau u] + D_Y[\partial^2(\partial - 1)u] \simeq D_Y^{\oplus 2},$$

and $\text{Var}([\partial^2 \tau u], [\partial^3(\partial - 1)u]) \mapsto (0, 2A[\partial \tau u])$. Hence

$$\mathcal{M}_Y \simeq \frac{D_Y[\partial \tau u] + D_Y[\partial^2(\partial - 1)u]}{D_Y A[\partial \tau u]}.$$
In this case $f(x,t)$ has the following defining function:

$$F(z, \tau) = \sum_{j=0}^{\infty} A^{(j)} U_2(z) \tau^{j+2} + U_1(z) \tau - \sum_{j=2}^{\infty} \sum_{k=1}^{j-1} \frac{j A^{(j)} U_1(z)}{k} \tau^{j+1} + \left( \sum_{j=0}^{\infty} (j+1) A^{(j+1)} U_1(z) \tau^{j} \right) \tau^2 \log \tau,$$

and $\beta^0(f(x,t))$ is given by $\{\text{sp}_N(U_1)(x)\}_{i=1,2}$ at $p_0$. $f(x,t)$ is $F$-mild under the condition that $AU_1(z) = 0$, and in this case $\gamma^F(f(x,t))$ at $p_0$ is given by $\gamma^F(f_3(x,t)) = \{\partial_t f(x, +0)\}_{i=1,2} = \{\text{sp}_N(U_1)(x), 2\text{sp}_N(U_2)(x)\}$ with $A\partial_t f(x, +0) = A\text{sp}_N(U_1)(x) = 0$.

References


