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A Sharp Existence and Uniqueness Theorem for Linear Fuchsian Partial Differential Equations

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Abstract

This paper considers the equation $\mathcal{P}u = f$, where u and f are continuous with respect to t and holomorphic with respect to z, and $\mathcal{P}$ is the linear Fuchsian partial differential operator

$$\mathcal{P} = (tD_t)^m + \sum_{j=0}^{m-1} \sum_{|\alpha|\le m-j} a_{j,\alpha}(t,z)(\mu(t)D_z)^\alpha (tD_t)^j.$$

We will give a sharp form of unique solvability in the following sense: we can find a domain Ω such that if f is defined on Ω, then we can find a unique solution u also defined on Ω.

1 Introduction and Result

Denote by $\mathbb{N}$ the set of nonnegative integers, and let $(t,z) = (t, z_1, \ldots, z_n) \in \mathbb{R}\times\mathbb{C}^n$. Let $R > 0$ be sufficiently small, and for $\rho \in (0, R]$, let B_ρ be the polydisk $\{z \in \mathbb{C}^n;\ |z_i| < \rho \ \text{ for } i = 1, 2, \ldots, n\}$.

Given any bounded, open subset D of $\mathbb{C}^n$, we define by $\mathcal{A}(D)$ the Banach space of all functions $g(z)$ holomorphic in D and continuous up to $\overline{D}$; the norm in this space is given by $\|g\|_D = \max_{z\in\overline{D}} |g(z)|$. Let $T > 0$. Then we denote by $C^0([0,T], \mathcal{A}(D))$ the set of functions continuous on the interval $[0,T]$ and valued in the space $\mathcal{A}(D)$.

We say that a continuous, positive-valued function $\mu(t)$ on the interval $(0,T)$ is a *weight function* if $\mu(t)$ is increasing and the function

$$\varphi(t) = \int_0^t \frac{\mu(s)}{s}\, ds \tag{1.1}$$

is well-defined on $(0,T)$, i.e., the integral on the right is finite. (See Tahara [7].)

Consider now the linear partial differential operator

$$\mathcal{P} = (tD_t)^m + \sum_{j=0}^{m-1} \sum_{|\alpha|\le m-j} a_{j,\alpha}(t,z)(\mu(t)D_z)^\alpha (tD_t)^j. \tag{1.2}$$

Here, $D_t = \partial/\partial t$ and $D_z = (\partial/\partial z_1, \ldots, \partial/\partial z_n)$; $\mu(t)$ is a weight function; and the coefficients $a_{j,\alpha}(t,z)$ belong in the space $C^0([0,T], \mathcal{A}(B_R))$, i.e., for any

$s \in [0,T]$, each of the functions $a_{j,\alpha}(s,z)$, when viewed as a function of z, is holomorphic in B_R and continuous up to $\overline{B}_R$. We associate a polynomial with this operator, called the *characteristic polynomial* of $\mathcal{P}$, and we define it by

$$\mathcal{C}(\lambda, z) = \lambda^m + a_{m-1,0}(0,z)\lambda^{m-1} + \cdots + a_{0,0}(0,z). \tag{1.3}$$

Its roots $\lambda_1(z), \ldots, \lambda_m(z)$ will be referred to as *characteristic exponents*. In what follows, we will assume that there exists a positive number L such that

$$\Re\lambda_j(z) \le -L < 0 \qquad \text{for all } z \in B_R \text{ and } 1 \le j \le m. \tag{1.4}$$

Baouendi and Goulaouic [1] studied the above operator in the case when $\mu(t) = t^a$ $(a > 0)$. They called such operator a Fuchsian partial differential operator, which for them is the "natural" generalization of a Fuchsian ordinary differential operator. In their paper, they gave some generalizations of the classical Cauchy-Kowalewski and Holmgren theorems for this type of operators. Their method has been applied and extended to various cases as can be seen, for example, in Tahara [6], Mandai [5] and Yamane [8].

In a previous paper [4], the author proved existence and uniqueness theorems similar to those given in [1], but for general $\mu(t)$. Essentially, he proved the following unique solvability result.

Theorem 1. *Let $\mathcal{P}$ be as in (1.2). Then given any $\rho \in (0,R)$, there exists an $\varepsilon \in (0,T]$ such that for any $f(t,z) \in C^0([0,T], \mathcal{A}(B_R))$, the equation $\mathcal{P}u = f$ has a unique solution $u(t,z) \in C^0([0,\varepsilon], \mathcal{A}(B_\rho))$ satisfying for $1 \le p \le m$ the relation $(tD_t)^p u \in C^0([0,\varepsilon], \mathcal{A}(B_\rho))$.*

We remark that although $f(t,z)$, viewed as a function of z, is defined on B_R, the existence of the solution $u(t,z)$ is only guaranteed up to B_ρ, with $\rho < R$. Moreover, any two solutions of $\mathcal{P}u = f$ can only be shown to coincide in a neighborhood of the origin which is smaller than the neighborhood on which the two are defined.

In this paper, we shall present a formulation leading to an existence and uniqueness result sharper than the one given above. The result is sharper in the sense that the solution $u(t,z)$ of the equation $\mathcal{P}u = f$ will now have the same domain of definition as the inhomogeneous part $f(t,z)$.

To proceed, we will need the following definitions.

Definition 1. Let $\tau \in (0,T)$, $\gamma > 0$ and $\varphi(t)$ be the one in (1.1). We define

(*i*) $\omega_\tau[\gamma] = \{z \in \mathbb{C}^n;\ |z_i| < R - \gamma\varphi(\tau) \quad \text{for } i = 1,2,\ldots,n\}$, and

(*ii*) $\Omega_T[\gamma] = \{(\tau, z) \in \mathbb{R} \times \mathbb{C}^n;\ 0 \le \tau \le T \text{ and } z \in \omega_\tau[\gamma]\}$.

Definition 2. Let $p \in \mathbb{N}$ and $\gamma > 0$.

(*i*) We say that $f(t,z)$ belongs in $\mathcal{K}_0(\Omega_T[\gamma])$ if for each $\tau \in [0,T]$, we have $f(t) \in C^0([0,\tau], \mathcal{A}(\omega_\tau[\gamma]))$.

(*ii*) We say that $w(t,z)$ belongs in $C_p^0([0,\tau],\mathcal{A}(\omega_\tau[\gamma]))$ if for all $0 \le j \le p$, we have $(tD_t)^j w(t) \in C^0([0,\tau],\mathcal{A}(\omega_\tau[\gamma]))$.

(*iii*) We say that $u(t,z)$ belongs in $\mathcal{K}_p(\Omega_T[\gamma])$ if for each $\tau \in [0,T]$, we have $u(t) \in C_p^0([0,\tau],\mathcal{A}(\omega_\tau[\gamma]))$.

Under the above assumptions, we now state the following main result.

Theorem 2. *Let $\mathcal{P}$ be the operator given in* (1.2). *Then there exist constants $T_0 > 0$ and $\gamma_0 > 0$ depending on $\mathcal{P}$ such that for any $f(t,z) \in \mathcal{K}_0(\Omega_{T_0}[\gamma_0])$, the equation*

$$\mathcal{P}u = f \qquad in \quad \Omega_{T_0}[\gamma_0] \tag{1.5}$$

has a unique solution $u(t,z)$ in $\mathcal{K}_m(\Omega_{T_0}[\gamma_0])$.

Moreover, the solution satisfies the a priori estimate

$$\sum_{p=0}^{m} \max_{\Delta}\left|(tD_t)^p u\right| \le C \max_{\Delta}|f|, \tag{1.6}$$

where Δ is the closure of $\Omega_{T_0}[\gamma_0]$ and $C > 0$ is some constant dependent on the above equation and on the domain $\Omega_{T_0}[\gamma_0]$.

Note that $f(t,z)$ and $u(t,z)$ both have $\Omega_{T_0}[\gamma_0]$ as their domain of definition. This fact allows us to restate our theorem in the following manner: for any T, $\gamma > 0$, let $X_{T,\gamma}$ and $Y_{T,\gamma}$ be the spaces $\mathcal{K}_m(\Omega_T[\gamma])$ and $\mathcal{K}_0(\Omega_T[\gamma])$, respectively. Let $W_{T,\gamma}$ be the subspace of $X_{T,\gamma}$ consisting of functions $u \in X_{T,\gamma}$ such that $\mathcal{P}u$ belongs in $Y_{T,\gamma}$. Define a linear operator Ψ from $X_{T,\gamma}$ to $Y_{T,\gamma}$ with domain $W_{T,\gamma}$ by $\Psi u = \mathcal{P}u$. Let $|||\cdot|||_{T,\gamma}$ denote the maximum norm in the closure of $\Omega_T[\gamma]$. Then $X_{T,\gamma}$ and $Y_{T,\gamma}$ are Banach spaces; given $u \in X_{T,\gamma}$ and $f \in Y_{T,\gamma}$, we define their norms by $\sum_{p=0}^{m} |||(tD_t)^p u|||_{T,\gamma}$ and $|||f|||_{T,\gamma}$, respectively. Note further that the operator Ψ is a closed linear operator from $X_{T,\gamma}$ to $Y_{T,\gamma}$. The above theorem can now be stated as

Theorem 2′. *There exist T_0, $\gamma_0 > 0$ depending on $\mathcal{P}$ such that the operator Ψ is a one-one, closed linear operator from X_{T_0,γ_0} onto Y_{T_0,γ_0}.*

Since Ψ is an injection, Ψ^{-1} exists and is also closed. The Closed Graph Theorem further implies that Ψ^{-1} is continuous. The estimate given in (1.6) is just a consequence of the continuity of Ψ^{-1}.

2 Preliminary Discussion

We can rewrite the operator $\mathcal{P}$ as

$$\mathcal{P} = \mathcal{Q} + \sum_{j=0}^{m-1} \sum_{|\alpha| \le m-j} c_{j,\alpha}(t,z)(\mu(t)D_z)^\alpha (tD_t)^j,$$

where the operator $\mathcal{Q}$ is defined by

$$\mathcal{Q} = (tD_t)^m + a_{m-1,0}(0,z)(tD_t)^{m-1} + \cdots + a_{0,0}(0,z) \tag{2.1}$$

and

$$c_{j,\alpha}(t,z) = \begin{cases} a_{j,\alpha}(t,z) & \text{if } |\alpha| \neq 0, \\ a_{j,\alpha}(t,z) - a_{j,\alpha}(0,z) & \text{if } |\alpha| = 0. \end{cases}$$

Note that the coefficients of $\mathcal{Q}$ are holomorphic functions of z in B_R. Note further that the characteristic exponents of $\mathcal{Q}$ are the same as that of $\mathcal{P}$, and hence satisfy (1.4).

Lemma 1. *Fix $\tau > 0$ and let $g(t) \in C^0([0,\tau], \mathcal{A}(\omega_\tau[\gamma]))$. Then the equation $\mathcal{Q}u = g$ has a unique solution $u(t) \in C^0_m([0,\tau], \mathcal{A}(\omega_\tau[\gamma]))$ given by*

$$\begin{aligned} u(t) &= \frac{1}{m!}\sum_{\sigma \in S_m} \int_0^t \int_0^{s_m} \cdots \int_0^{s_2} \left(\frac{s_m}{t}\right)^{-\lambda_{\sigma(m)}} \left(\frac{s_{m-1}}{s_m}\right)^{-\lambda_{\sigma(m-1)}} \cdots \\ &\qquad \times \left(\frac{s_1}{s_2}\right)^{-\lambda_{\sigma(1)}} g(s_1)\, \frac{ds_1}{s_1}\frac{ds_2}{s_2}\cdots\frac{ds_m}{s_m}. \end{aligned} \tag{2.2}$$

Here, S_m is the group of permutations of $\{1, 2, \ldots, m\}$.

A result in symmetric entire functions asserts that $u(t,z)$ is holomorphic with respect to z. The fact that it belongs in $C^0_m([0,\gamma], \mathcal{A}(\omega_\tau[\gamma]))$ is seen in the integral expression, but may actually be obtained *a priori*. (See [1].)

To facilitate computation, we define for $\lambda = (\lambda_1, \ldots, \lambda_m)$ the function

$$G^t_\theta(\lambda) \overset{\text{def}}{=} \frac{1}{m!}\sum_{\sigma \in S_m} \left(\frac{s_m}{t}\right)^{-\lambda_{\sigma(m)}} \left(\frac{s_{m-1}}{s_m}\right)^{-\lambda_{\sigma(m-1)}} \cdots \left(\frac{\theta}{s_2}\right)^{-\lambda_{\sigma(1)}}, \tag{2.3}$$

for some dummy variables $s_2, \ldots, s_m$. Define, too, the integral operator

$$\int_{[t;\theta]}^{(m)} g \overset{\text{def}}{=} \int_0^t \int_0^{s_m} \cdots \int_0^{s_2} g(\theta)\, \frac{d\theta}{\theta}\frac{ds_2}{s_2}\cdots\frac{ds_m}{s_m} \tag{2.4}$$

Using the above, we can now write the solution $u(t)$ of the equation $\mathcal{Q}u = g$ as

$$u(t) = \int_{[t;s]}^{(m)} G^t_s(\lambda)\, g\,.$$

In our proof of the main theorem, it will be necessary to consider the action of the differential operator $(tD_t)^p$ on integral expressions similar to the one in (2.2). One can easily verify the following

Lemma 2. *Let $u(t)$ be the solution of $\mathcal{Q}u = g$. Then for a natural number p less than m, we have*

$$\begin{aligned} (tD_t)^p u &= \sum_{i=m-p}^{m} \int_{[t;s_1]}^{(i)} g \times \left\{ \frac{1}{m!}\sum_{\sigma \in S_m} h_i(\sigma,\lambda) \left(\frac{s_i}{t}\right)^{-\lambda_{\sigma(i)}} \right. \\ &\qquad \left. \times \left(\frac{s_{i-1}}{s_i}\right)^{-\lambda_{\sigma(i-1)}} \cdots \left(\frac{s_1}{s_2}\right)^{-\lambda_{\sigma(1)}} \right\}, \end{aligned} \tag{2.5}$$

where the functions $h_i(\sigma,\lambda)$ are suitable polynomial functions of the characteristic exponents $\lambda_1(z),\dots,\lambda_m(z)$.

For brevity, let us set, for a natural number k,

$$H_\theta^t(k,\lambda) = \frac{1}{m!}\sum_{\sigma\in S_m} h_k(\sigma,\lambda)\left(\frac{s_k}{t}\right)^{-\lambda_{\sigma(k)}}\left(\frac{s_{k-1}}{s_k}\right)^{-\lambda_{\sigma(k-1)}}\cdots\left(\frac{\theta}{s_2}\right)^{-\lambda_{\sigma(1)}}. \tag{2.6}$$

By symmetry, the functions $H_s^t(k,\lambda)$ are holomorphic with respect to z and thus belong in $\mathcal{A}(B_R)$.

The next lemma is useful in evaluating some integral expressions in the proof.

Lemma 3. *Let k be natural number. Then the following equalities hold:*

(a) $$\int_0^{s_k}\int_0^{s_{k-1}}\cdots\int_0^{s_1}\left(\frac{s_0}{s_k}\right)^L \frac{ds_0}{s_0}\cdots\frac{ds_{k-1}}{s_{k-1}} = \frac{1}{L^k}$$

(b) $$\int_0^t\int_0^{s_k}\cdots\int_0^{s_1}\frac{\mu(s_k)}{s_k}\frac{\mu(s_{k-1})}{s_{k-1}}\cdots\frac{\mu(s_1)}{s_1} \times\left(\frac{s_0}{t}\right)^L\frac{s_0^{-1}}{[\varphi(t)-\varphi(s_0)]^k}\,ds_0\dots ds_k = \frac{1}{L\,k!}$$

The first equality is obvious. The second can be proved by reversing the order of integration and recalling that $t\varphi'(t)=\mu(t)$.

To estimate the derivatives with respect to z, we have the following lemma. (For a proof, see Hörmander [3], Lemma 5.1.3.)

Lemma 4. *Let the function $v(z)$ be holomorphic in B_R, and suppose there are positive constants K and c such that*

$$\|v\|_\rho \le \frac{K}{(R-\rho)^c} \qquad \text{for every } \rho\in(0,R). \tag{2.7}$$

Then we have

$$\|D_z^\alpha v\|_\rho \le \frac{Ke^{|\alpha|}(c+1)_{|\alpha|}}{(R-\rho)^{c+|\alpha|}} \qquad \text{for every } \rho\in(0,R). \tag{2.8}$$

In the above, we define $(c)_p=(c)(c+1)\cdots(c+p-1)$.

3 Proof of Main Theorem

Let f be any element of $\mathcal{K}_0(\Omega_{T_0}[\gamma_0])$. Here, the constants $T_0>0$ and $\gamma_0>0$ satisfy some conditions which will later be specified. For convenience, we will drop the subscript in both and instead use T and γ; we will again use the subscript upon stating the conditions that these constants need to satisfy.

We will use the method of successive approximations to solve the equation $\mathcal{P}u = f$. Define the approximate solutions as follows:

$$u_0(t) = \int_{[t;s]}^{(m)} G_s^t(\lambda)\, f \tag{3.1}$$

and for $k \geq 1$,

$$u_k(t) = \int_{[t;s]}^{(m)} G_s^t(\lambda)\big[f - \mathcal{S}(s)u_{k-1}\big]. \tag{3.2}$$

Here, $t \in [0,T]$, and for brevity, we have set $\mathcal{S}(t) = \sum_{j=0}^{m-1} \sum_{|\alpha| \leq m-j} c_{j,\alpha}(t,z) \cdot (\mu(t)D_z)^\alpha (tD_t)^j$. Note that for all k, the approximate solutions $u_k(t,z)$ are defined on $\Omega_{T_0}[\gamma_0]$. Furthermore, they are continuous with respect to t and holomorphic with respect to z on this region.

For each k, we also define the sequence of functions $v_k(t) = u_k(t) - u_{k-1}(t)$, with $u_{-1} \equiv 0$. Then the $v_k(t,z)$'s are also defined on the same region as the $u_k(t,z)$'s, and are also continuous with respect to t and holomorphic with respect to z. Using the expression for $u_k(t)$, we have $v_0(t) = \int_{[t;s]}^{(m)} G_s^t(\lambda)\, f$ and for $k \geq 1$,

$$v_k(t) = -\int_{[t;s]}^{(m)} G_s^t(\lambda)\mathcal{S}(s)v_{k-1}. \tag{3.3}$$

To prove that the approximate solutions converge to the real solution, we will henceforth fix one $t \in [0,T]$, and estimate the functions $v_k(t)$. Let C be the bound on $[0,T] \times \overline{B}_R$ of all $c_{j,\alpha}(t,z)$, and K be the bound in $\Omega_T[\gamma]$ of $f(t,z)$. As $G_s^t(\lambda)$ and $H_s^t(k,\lambda)$, we have for $1 \leq k \leq m$ and for some $D > 0$:

$$\sup_{z \in \overline{B}_R} \left|G_s^t(\lambda)\right| \leq \left(\frac{s}{t}\right)^L \quad \text{and} \quad \sup_{z \in \overline{B}_R} \left|H_s^t(k,\lambda)\right| \leq D\left(\frac{s}{t}\right)^L. \tag{3.4}$$

We can easily see that $\|v_0(t)\|_{\omega_t}$ is bounded by KL^{-m} for any $0 \leq t \leq T$. Here, we have written for convenience $\|\cdot\|_{\omega_t}$ in place of $\|\cdot\|_{\omega_t[\gamma]}$. For general k, we note that $v_k(t)$ is given by the iterated integral

$$\begin{aligned} v_k(t) &= (-1)^k \int_{[t;s_k]}^{(m)} G_{s_k}^t(\lambda)\mathcal{S}(s_k) \int_{[s_k;s_{k-1}]}^{(m)} G_{s_{k-1}}^{s_k}(\lambda)\mathcal{S}(s_{k-1}) \cdots \\ &\quad \cdots \int_{[s_2;s_1]}^{(m)} G_{s_1}^{s_2}(\lambda)\mathcal{S}(s_1) \int_{[s_1;s_0]}^{(m)} G_{s_0}^{s_1}(\lambda) f(s_0). \end{aligned} \tag{3.5}$$

The expression above can be expanded using Lemma 2, and thus obtain a finite sum whose number of terms is less than $(mJ)^k$, where J is the cardinality of the set $\{(j,\alpha);\ 0 \leq j \leq m-1 \text{ and } |\alpha| \leq m-j\}$. Each term of the finite sum has the form

$$
\begin{aligned}
I \;=\; & (-1)^k \int_{[t;s_k]}^{(m)} G^{t}_{s_k}(\lambda)\, c_{j_k,\alpha_k}(\mu D_z)^{\alpha_k} \int_{[s_k;s_{k-1}]}^{(i_k)} H^{s_k}_{s_{k-1}}(i_k,\lambda)\, c_{j_{k-1},\alpha_{k-1}}(\mu D_z)^{\alpha_{k-1}} \\
& \cdots \int_{[s_2;s_1]}^{(i_2)} H^{s_2}_{s_1}(i_2,\lambda)\, c_{j_1,\alpha_1}(\mu D_z)^{\alpha_1} \int_{[s_1;s_0]}^{(i_1)} H^{s_1}_{s_0}(i_1,\lambda)\, f(s_0), \qquad (3.6)
\end{aligned}
$$

where for each p, the relations $m - j_p \le i_p \le m$ and $|\alpha_p| \le m - j_p$ hold. (Here, α_p is a multi-index and should not be confused with the pth component of α.) The above is further equal to

$$
\begin{aligned}
I \;=\; & (-1)^k \int_{[t;s_k]}^{(m)} \int_{[s_k;s_{k-1}]}^{(i_k)} \cdots \int_{[s_1;s_0]}^{(i_1)} G^{t}_{s_k}\, c_{j_k,\alpha_k}(s_k)(\mu(s_k) D_z)^{\alpha_k} \\
& \times H^{s_k}_{s_{k-1}} c_{j_{k-1},\alpha_{k-1}}(s_{k-1})(\mu(s_{k-1}) D_z)^{\alpha_{k-1}} \cdots \\
& \times H^{s_2}_{s_1} c_{j_1,\alpha_1}(s_1)(\mu(s_1) D_z)^{\alpha_1} H^{s_1}_{s_0}\, f(s_0). \qquad (3.7)
\end{aligned}
$$

Let $F_k(s)$ denote the integrand of the above integral. Let $R_{s_0} = R - \gamma\varphi(s_0)$. Then all the functions above, when viewed as a function of z, belong in $\mathcal{A}(\omega_{s_0}[\gamma])$. (This explains the necessity of the assumption that the coefficients be defined up to B_R, for all t in the interval $[0,T]$.)

We can therefore apply Lemma 4 repeatedly, starting from the rightmost expression, to obtain the following estimate: for any $\rho \in (0, R_{s_0})$, we have

$$
\begin{aligned}
\|F_k(s)\|_{B_\rho} \;\le\; & K(CD)^k \mu(s_1)^{|\alpha_1|} \cdots \mu(s_k)^{|\alpha_k|} \left(\frac{s_0}{t}\right)^L \times \\
& \left(\frac{e}{R_{s_0} - \rho}\right)^{|\alpha_1 + \cdots + \alpha_k|} |\alpha_1 + \cdots + \alpha_k|!. \qquad (3.8)
\end{aligned}
$$

If $|\alpha_1 + \cdots + \alpha_k| = 0$, then for sufficiently small $T = T_0$, the bound for any $c_{j,0}(t,z) = a_{j,0}(t,z) - a_{j,0}(0,z)$ is actually small, since $a_{j,0}(t,z)$ is continuous with respect to t. In other words, by choosing a small $T = T_0$, we could find a small constant δ such that for any $t \in [0, T_0]$ and $0 \le s \le t$, the following holds:

$$
\|F_k(s)\|_{\omega_t} \le K\delta^k \left(\frac{s_0}{t}\right)^L. \qquad (3.9)
$$

Going back to the integral, we have

$$
\begin{aligned}
\|I\|_{\omega_t} \;\le\; & \int_{[t;s_k]}^{(m)} \int_{[s_k;s_{k-1}]}^{(i_k)} \cdots \int_{[s_1;s_0]}^{(i_1)} K\delta^k \left(\frac{s_0}{t}\right)^L \\
=\; & K \frac{\delta^k}{L^{m+i_1+\cdots+i_k}} \;\le\; K\left(\frac{\delta}{L_0}\right)^k, \qquad (3.10)
\end{aligned}
$$

for some constant L_0 dependent on L. This is possible since $i_p \le m$ for all p.

If $|\alpha_1 + \cdots + \alpha_k| \neq 0$, set the ρ in (3.8) to be equal to $R - \gamma\varphi(t)$. This gives

$$\begin{aligned} \|F_k(s)\|_{\omega_t} &\leq K(CD)^k \mu(s_1)^{|\alpha_1|} \cdots \mu(s_k)^{|\alpha_k|} \left(\frac{s_0}{t}\right)^L \\ &\times |\alpha_1 + \cdots + \alpha_k|! \left(\frac{e}{\gamma[\varphi(t) - \varphi(s_0)]}\right)^{|\alpha_1 + \cdots + \alpha_k|}. \end{aligned} \tag{3.11}$$

By renaming if necessary, assume that for $p = 1, \ldots, q$, we have $|\alpha_p| \neq 0$. Note that $q \geq 1$. We will again use the continuity of $a_{j,0}(t, z)$ to estimate those expressions which are not acted upon by D_z, i.e., the $k - q$ cases when $|\alpha_p| = 0$. Just like before, we can show that for small δ,

$$\begin{aligned} \|F_k(s)\|_{\omega_t} &\leq K(CD)^q \delta^{k-q} \mu(s_1)^{|\alpha_1|} \cdots \mu(s_q)^{|\alpha_q|} \left(\frac{s_0}{t}\right)^L \\ &\times |\alpha_1 + \cdots + \alpha_q|! \left(\frac{e}{\gamma[\varphi(t) - \varphi(s_0)]}\right)^{|\alpha_1 + \cdots + \alpha_q|}. \end{aligned} \tag{3.12}$$

Thus, the integral I can now be estimated as follows:

$$\begin{aligned} \|I\|_{\omega_t} &\leq K(CD)^q \delta^{k-q} \left(\frac{e}{\gamma}\right)^{|\alpha_1 + \cdots + \alpha_q|} |\alpha_1 + \cdots + \alpha_q|! \\ &\times \int_{[t;s_k]}^{(m)} \int_{[s_k;s_{k-1}]}^{(i_k)} \cdots \int_{[s_1;s_0]}^{(i_1)} \left(\frac{s_0}{t}\right)^L \frac{\mu(s_1)^{|\alpha_1|} \cdots \mu(s_q)^{|\alpha_q|}}{[\varphi(t) - \varphi(s_0)]^{|\alpha_1 + \cdots + \alpha_q|}}. \end{aligned} \tag{3.13}$$

Let $d = m + i_1 + \ldots + i_k$ and $b = |\alpha_1 + \ldots + \alpha_q|$. Note that $b \geq q$. Since for each p, we have $|\alpha_p| \leq m - j_p \leq i_p$, and using the fact that both $\varphi(t)$ and $\mu(t)$ are increasing on $(0, T_0)$, we have

$$\begin{aligned} \|I\|_{\omega_t} &\leq K\,(CD)^q\, \delta^{k-q} \left(\frac{e}{\gamma}\right)^b b! \\ &\times \int_0^t \int_0^{\xi_b} \cdots \int_0^{\xi_1} \frac{\mu(\xi_b)}{\xi_b} \cdots \frac{\mu(\xi_1)}{\xi_1} \left(\frac{\xi_0}{t}\right)^L \frac{1}{[\varphi(t) - \varphi(\xi_0)]^b} \frac{d\xi_0}{\xi_0}\, d\xi_1 \cdots d\xi_b \\ &\times \int_0^{\xi_0} \int_0^{\eta_1} \cdots \int_0^{\eta_{d-b-2}} \left(\frac{s_0}{\xi_0}\right)^L \frac{ds_0}{s_0} \cdots \frac{d\eta_1}{\eta_1} \end{aligned} \tag{3.14}$$

By (a) of Lemma 3, the second integral is equal to L^{-d+b+1}. Thus, the above simplifies into

$$\begin{aligned} \|I\|_{\omega_t} &\leq K(CD)^q \delta^{k-q} \left(\frac{e}{\gamma}\right)^b L^{-d+b+1}\, b! \\ &\times \int_0^t \int_0^{\xi_b} \cdots \int_0^{\xi_1} \frac{\mu(\xi_b)}{\xi_b} \cdots \frac{\mu(\xi_1)}{\xi_1} \left(\frac{\xi_0}{t}\right)^L \frac{\xi_0^{-1}}{[\varphi(t) - \varphi(\xi_0)]^b}\, d\xi_0 \cdots d\xi_b. \end{aligned} \tag{3.15}$$

The last integral is equal to $(Lb!)^{-1}$, by (b) of Lemma 3. Meanwhile, since $d \leq m(k+1)$, we can find a constant L_1, depending on L, such that $L^{-d} \leq L_1^k$.

Substituting these results into the above equation, we get

$$\|I\|_{\omega_t} \;\leq\; K(CD)^q\delta^{k-q}\Big(\frac{eL}{\gamma}\Big)^b L_1^k \;=\; K\Big(\frac{CD}{\delta}\Big)^q(\delta L_1)^k\Big(\frac{eL}{\gamma}\Big)^b. \qquad (3.16)$$

By taking a sufficiently small T_0, we can find a δ small enough such that δL_1 above and δL_0^{-1} in (3.10) are both less than $(mJ)^{-1}$. Now, since $q \leq b$, we can make the remaining expression less than one by choosing a large $\gamma = \gamma_0$.

To summarize, we have shown that if T_0 is sufficiently small and γ_0 is sufficiently large, some constants $K > 0$ and $\delta_0 < 1$ exist such that for all k, we have

$$\|v_k(t)\|_{\omega_t[\gamma_0]} \;\leq\; K\delta_0^k \qquad \text{for any } t \in [0, T_0]. \qquad (3.17)$$

It follows that the series $\sum_{k=0}^{\infty} v_k(t,z)$ is majorized by a convergent geometric series, and hence is itself convergent in $C^0([0,\tau], \mathcal{A}(\omega_\tau[\gamma_0]))$ for all $\tau \in [0, T_0]$. This means that $u_k(t)$ converges uniformly to $u(t)$ on $\Omega_{T_0}[\gamma_0]$.

By following the steps above, we can also show that for $1 \leq p \leq m-1$, the sequence $(tD_t)^p u_k(t)$ converges uniformly to $(tD_t)^p u(t)$ on $\Omega_{T_0}[\gamma_0]$. Thus, it follows that on a compact subset of $\Omega_{T_0}[\gamma_0]$, the sequence $D_z^\alpha(tD_t)^p u_k(t)$ converges to $D_z^\alpha(tD_t)^p u(t)$. This implies the convergence of the approximate solutions to the true solution $u(t)$.

Uniqueness may be proved in a similar manner.

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