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Witten Laplacian and Twisted de Rham Cohomology

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Abstract

This paper presents a survey of a recent work by the author on an intersection theory for twisted de Rham cohomology and its applications to special function theory. While Witten used a twisted Laplacian, twisted by a Morse function, to develop his Morse theory as a super-symmetric quantum mechanics, we use a twisted Laplacian, twisted by an isolated singularity, in a quite different situation.

Key words: Witten Laplacian, isolated singularity, twisted de Rham cohomology, duality, super-symmetry, generalized Airy function, Schur polynomial, skew-Schur polynomial.

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1 Introduction

In his famous paper [24], Witten used a twisted Laplacian, twisted by a Morse function, to develop his Morse theory as a super-symmetric quantum mechanics. Afterwards, in their studies on *puits multiples en mecanique semi-classique*, Helffer and Sjöstrand [8] made a detailed investigation of Witten's complexes in a mathematically rigorous formulation.

Recently, with a similar philosophy but in a quite different situation, Iwasaki [9] made use of a twisted Laplacian, twisted by a versal deformation of an isolated singularity (invariant under a finite unitary reflection group), to construct a duality between a pair of polynomial twisted de Rham cohomology groups associated with the isolated singularity. Then Iwasaki and Matsumoto [11] applied this construction to calculate the intersection matrix of a generalized Airy function in terms of skew-Schur polynomials. The aim of this expository article is to survey the results in the papers [9, 11].

A motivation for developing a twisted de Rham cohomology theory comes from its (possible) applications to the theory of hypergeometric functions in their broadest sense. This is because, from the standpoint of Euler integral representations, a hypergeometric function is defined to be an integral of a closed twisted differential form along a twisted cycle. See Aomoto and Kita [2] and Matsumoto and Yoshida [16] for more information.

Let $Z = \mathbb{C}^n$ be the complex n-space with coordinates $z = (z_1, \dots, z_n)$, and let W be a finite subgroup of the unitary group $U(n)$. The case where W is a finite unitary reflection group is of particular interest to us. Consider a W-invariant polynomial $f \in \mathbb{C}[z]^W$ of degree $N = \deg f$ satisfying the following assumption, which will be made throughout the paper.

Assumption 1.1 The top homogeneous component f_0 of f has the origin $0 \in Z$ as the only isolated singular point of it.

The first aim of this paper is to discuss a W-invariant duality between the pair of twisted polynomial de Rham cohomology groups $H^{\cdot}(\Omega^{\cdot}_Z, d_{\pm f})$, where $\Omega^{\cdot}_Z$ denotes the space of polynomial differential forms on Z and $d_{\pm f}$ are the twisted exterior differentials defined by

$$d_{\pm f} = e^{\mp f}\, d\, e^{\pm f} = d \pm df \wedge .$$

Before entering into the duality, we should describe some elementary properties of the cohomology group $H^{\cdot}(\Omega^{\cdot}_Z, d_f)$ and its W-invariant component $H^{\cdot}(\Omega^{\cdot}_Z, d_f)^W$. Let $\Omega^{\cdot}_{Z/W}$ be the space of polynomial differential forms on Z invariant under the action of W. Standard techniques in finite group actions, such as transfer and the vanishing of the first cohomology of W, imply that the inclusion of cochain complexes $(\Omega^{\cdot}_{Z/W}, d_f) \hookrightarrow (\Omega^{\cdot}_Z, d_f)$ induces an isomorphism of cohomology groups (the transfer isomorphism):

$$H^{\cdot}(\Omega^{\cdot}_{Z/W}, d_f) \xrightarrow{\sim} H^{\cdot}(\Omega^{\cdot}_Z, d_f)^W. \tag{1}$$

When W is a finite unitary reflection group, a detailed description of $H^{\cdot}(\Omega^{\cdot}_Z, d_f)^W$ is possible. A classical theorem of Chevalley [6] and Shephard and Todd [21] tells us that if W is a finite unitary reflection group, then there exists an n-tuple of algebraically independent, homogeneous, W-invariant polynomials, say, $t = (t_1, \dots, t_n)$, that generates the invariant algebra $\mathbb{C}[z]^W = \mathbb{C}[t]$. Thus f can be thought of as a polynomial of t. Let $T = \mathbb{C}^n$ be the complex n-space with coordinates t. Then the twisted de Rham complex $(\Omega^{\cdot}_T, d_f)$ makes sense, and there exists an inclusion of cochain complexes $(\Omega^{\cdot}_T, d_f) \hookrightarrow (\Omega^{\cdot}_{Z/W}, d_f)$. A theorem of Solomon [22] asserts that this inclusion is an isomorphism. This, together with (1), yields isomorphisms:

$$H^{\cdot}(\Omega^{\cdot}_T, d_f) \xrightarrow{\sim} H^{\cdot}(\Omega^{\cdot}_{Z/W}, d_f) \xrightarrow{\sim} H^{\cdot}(\Omega^{\cdot}_Z, d_f)^W. \tag{2}$$

Assumption 1.1 tells us that f_0, as a polynomial of t, has the origin $0 \in T$ as the only isolated singular point of it. It is natural to compare $H^{\cdot}(\Omega_T^{\cdot}, d_f)$ with the cohomology group $H^{\cdot}(\Omega_T^{\cdot}, df_0\wedge)$ of the Koszul complex $(\Omega_T^{\cdot}, df_0\wedge)$. A classical result on isolated surface singularities (see e.g., Arnold, Gusein-Zade and Varchenko [4, Chap. 12]) implies that

$$H^p(\Omega_T^{\cdot}, df_0\wedge) \cong \begin{cases} J & (p = n), \\ 0 & (p \neq 0), \end{cases}$$

where $J = \mathbb{C}[t]/(\partial f_0/\partial t_1, \ldots, \partial f_0/\partial t_n)$ is the Jacobian ring of the isolated singularity $f_0 = 0$. Let $d = (d_1, \ldots, d_n)$ denote the degrees of W, namely, $d_j = \deg t_j$. By a formula of Milnor and Orlik [17] and Arnold [3], the complex dimension $\mu = \dim_{\mathbb{C}} J$ of J, or the Milnor number of f_0, is given by

$$\mu = \prod_{i=1}^{n} \left(\frac{N}{d_i} - 1 \right) < \infty.$$

If we provide the complex $(\Omega_T^{\cdot}, d_f)$ with the degree filtration by assigning degree d_j to t_j and dt_j, then the principal term of the exterior differential $d_f = d + df\wedge = d + df_0 \wedge + \cdots$ with respect to the filtration is $df_0\wedge$, and the graduation of the complex $(\Omega_T^{\cdot}, d_f)$ becomes isomorphic to the Koszul complex $(\Omega_T^{\cdot}, df_0\wedge)$. This observation readily leads to the formula:

$$\dim H^p(\Omega_T^{\cdot}, d_f) = \begin{cases} \mu & (p = n), \\ 0 & (p \neq n). \end{cases} \tag{3}$$

In particular, if W is the trivial group then $d = (1, \ldots, 1)$ and the Milnor number is given by $\mu = (N-1)^n$. Since $T = Z$ in this case, (3) implies that $H^p(\Omega_Z^{\cdot}, d_f) = 0$ unless $p = n$ and $\dim H^n(\Omega_Z^{\cdot}, d_f) = (N-1)^n$. So only the n-th cohomology group is nontrivial. Note that for any finite subgroup W of $U(n)$, which may or may not be a unitary reflection group, $H^n(\Omega_Z^{\cdot}, d_f)^W$ is finite-dimensional, as a subspace of the finite-dimensional space $H^n(\Omega_Z^{\cdot}, d_f)$.

Now our duality theorems are stated in the following manner.

Theorem 1.2 *Let W be a finite subgroup of $U(n)$. Under Assumption 1.1, there exists a natural W-invariant duality:*

$$H^n(\Omega_Z^{\cdot}, d_f) \times H^n(\Omega_Z^{\cdot}, d_{-f}) \to \mathbb{C}. \tag{4}$$

In view of (2), Theorem 1.2 leads to the following corollary.

Corollary 1.3 *Let W be a finite unitary reflection group and set $T = Z/W$. Under Assumption 1.1, there exists a natural duality:*

$$H^n(\Omega_T^{\cdot}, d_f) \times H^n(\Omega_T^{\cdot}, d_{-f}) \to \mathbb{C}. \tag{5}$$

Although the statement of the duality does not involve analysis, our construction requires analysis and PDE theory in an essential way. We remark that another approach based on algebraic D-module theory, which uses a result of Sabbah [19], is also possible. In §2 we explain how to construct the duality (4). The construction is based on two kinds of comparison theorems for several types of twisted de Rham cohomology groups. The first comparison theorem (Theorem 2.1) is proved by using Liouville's theorem in several complex variables and twisted versions of the Poincaré lemma (see §3). The second comparison theorem (Theorem 2.2) is based on a twisted version of Hodge-Kodaira decomposition (see §4). It involves a pseudo-differential calculus of Witten's twisted Laplacian $\Delta_f = d_f d_f^* + d_f^* d_f$. There are natural real structures on the cohomology groups compatible with the duality (Theorem 5.1). We introduce the notion of super-symmetry in §6. What we call a super-symmetry is such a transformation that permutes the $\pm$ cohomology groups $H^n(\Omega_Z^{\cdot}, d_{\pm f})$ and induces symmetric or skew-symmetric self-duality on $H^n(\Omega_Z^{\cdot}, d_f)^W$ (Theorem 6.2).

The construction of Iwasaki [9] was applied to the theory of (generalized) Airy functions by Iwasaki and Matsumoto [11]. The second aim of this article is to introduce their result to the reader (see §7). Further applications to singularity theory and special function theory should be made in the future.

2 Comparison Theorems and Duality

The construction of the duality (4) is based on two kinds of comparison theorems. The first comparison theorem is stated as follows.

Theorem 2.1 *Let $\mathcal{T}_Z^{\cdot}$ be the space of tempered currents on Z. Then the inclusion of complexes $(\Omega_Z^{\cdot}, d_f) \hookrightarrow (\mathcal{T}_Z^{\cdot}, d_f)$ induces a W-equivariant isomorphism of cohomology groups:*

$$H^{\cdot}(\Omega_Z^{\cdot}, d_f) \xrightarrow{\sim} H^{\cdot}(\mathcal{T}_Z^{\cdot}, d_f). \tag{6}$$

We remark that this theorem holds *without* Assumption 1.1; the only essential assumption needed is the condition that f is a *complex* polynomial. The second comparison theorem is stated as follows.

Theorem 2.2 *Let $\mathcal{S}_Z^{\cdot}$ be the space of smooth differential forms of Schwartz class on Z. Then the inclusion of complexes $(\mathcal{S}_Z^{\cdot}, d_f) \hookrightarrow (\mathcal{T}_Z^{\cdot}, d_f)$ induces a W-equivariant isomorphism of cohomology groups:*

$$H^{\cdot}(\mathcal{S}_Z^{\cdot}, d_f) \xrightarrow{\sim} H^{\cdot}(\mathcal{T}_Z^{\cdot}, d_f). \tag{7}$$

We remark that Assumption 1.1 is essential in this theorem.

Once Theorems 2.1 and 2.2 are established, the construction of the duality (4) proceeds along the same line as in the classical counterparts: the Poincaré duality for de Rham cohomology, and the Serre duality for Dolbeault cohomology on a compact (complex in the latter case) manifold. To explain this, consider the diagram:

$$\begin{array}{ccccc} H^n(\Omega^{\cdot}_Z, d_f) & \xrightarrow{\text{(a)}} & H^n(\mathcal{T}^{\cdot}_Z, d_f) & & H^n(\Omega^{\cdot}_Z, d_{-f}) \\ & & \Big\downarrow \text{(d)} & & \Big\downarrow \text{(b)} \\ & & H^n(\mathcal{S}^{\cdot}_Z, d_{-f}) & \xrightarrow{\text{(c)}} & H^n(\mathcal{T}^{\cdot}_Z, d_{-f}), \end{array}$$

where the arrows (a), (b), (c) are W-equivariant isomorphisms; (a) and (b) come from Theorem 2.1, and (c) comes from Theorem 2.2, respectively. The arrow (d) indicates a natural W-invariant duality between $H^n(\mathcal{T}^{\cdot}_Z, d_f)$ and $H^n(\mathcal{S}^{\cdot}_Z, d_{-f})$ induced from the topological duality between the space of tempered distributions and that of smooth functions of Schwartz class. The reduction to the former situation from the latter is made by a standard functional-analytical method as in Serre [20], based on the finite dimensionality of $H^n(\mathcal{T}^{\cdot}_Z, d_f)$ and $H^n(\mathcal{S}^{\cdot}_Z, d_{-f})$, which in turn follows from the finite dimensionality of $H^n(\Omega^{\cdot}_Z, d_{\pm f})$ through the isomorphisms (6) and (7). In this manner, establishing the duality (4) has been reduced to proving the comparison theorems, Theorems 2.1 and 2.2.

Putting Theorems 2.1 and 2.2 together yields natural W-equivariant isomorphisms of cohomology groups:

$$H^n(\Omega^{\cdot}_Z, d_{\pm f}) \xrightarrow{\sim} H^n(\mathcal{S}^{\cdot}_Z, d_{\pm f}). \tag{8}$$

Therefore every cohomology classes $\phi_{\pm} \in H^n(\Omega^{\cdot}_Z, d_{\pm f})$ can be represented by some $d_{\pm f}$-closed smooth n-forms of Schwartz class on Z, say, $\psi_{\pm}$. Then the duality (4) is expressed by the integrals:

$$\langle \phi_+, \phi_- \rangle = \int_Z \psi_+ \wedge \phi_- = \int_Z \phi_+ \wedge \psi_- = \int_Z \psi_+ \wedge \psi_-.$$

Note that these integrals are convergent, since $\phi_{\pm}$ are of polynomial growth and $\psi_{\pm}$ are rapidly decreasing as $|z| \to \infty$.

3 Twisted Poincaré Lemmas

We present an outline of the proof of Theorem 2.1. A main idea is to represent the polynomial de Rham complex $(\Omega^{\cdot}_Z, d_f)$ in terms of the double complex

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & & 0 \arrow[d] & \\
0 \arrow[r] & \Omega_Z^0 \arrow[r, "d_f"] \arrow[d] & \Omega_Z^1 \arrow[r, "d_f"] \arrow[d] & \Omega_Z^2 \arrow[r, "d_f"] \arrow[d] & \cdots \arrow[r, "d_f"] & \Omega_Z^n \arrow[r] \arrow[d] & 0 \\
 & \mathcal{T}_Z^{0,0} \arrow[r, "\partial_f"] \arrow[d, "\bar\partial"] & \mathcal{T}_Z^{1,0} \arrow[r, "\partial_f"] \arrow[d, "-\bar\partial"] & \mathcal{T}_Z^{2,0} \arrow[r, "\partial_f"] \arrow[d, "\bar\partial"] & \cdots \arrow[r, "\partial_f"] & \mathcal{T}_Z^{n,0} \arrow[r] \arrow[d, "\pm\bar\partial"] & 0 \\
 & \mathcal{T}_Z^{0,1} \arrow[r, "\partial_f"] \arrow[d, "-\bar\partial"] & \mathcal{T}_Z^{1,1} \arrow[r, "\partial_f"] \arrow[d, "\bar\partial"] & \mathcal{T}_Z^{2,1} \arrow[r, "\partial_f"] \arrow[d, "-\bar\partial"] & \cdots \arrow[r, "\partial_f"] & \mathcal{T}_Z^{n,1} \arrow[r] \arrow[d, "\mp\bar\partial"] & 0 \\
 & \vdots \arrow[d, "\pm\bar\partial"] & \vdots \arrow[d, "\mp\bar\partial"] & \vdots \arrow[d, "\pm\bar\partial"] & & \vdots \arrow[d, "\bar\partial"] & \\
 & \mathcal{T}_Z^{0,n} \arrow[r, "\partial_f"] \arrow[d] & \mathcal{T}_Z^{1,n} \arrow[r, "\partial_f"] \arrow[d] & \mathcal{T}_Z^{2,n} \arrow[r, "\partial_f"] \arrow[d] & \cdots \arrow[r, "\partial_f"] & \mathcal{T}_Z^{n,n} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 & & 0 &
\end{tikzcd}$$

Figure 1: Double Complex

$(\mathcal{T}_Z^{\cdot,\cdot}, \partial_f, \bar\partial)$ in a manner indicated in Figure 1, where $\mathcal{T}_Z^{p,q}$ is the space of tempered (p,q)-currents on Z, and ∂_f is the holomorphic twisted exterior differential defined by

$$\partial_f = e^{-f}\,\partial\, e^f = \partial + df \wedge . \tag{9}$$

Since f is a complex polynomial, the exterior differential d_f is decomposed into the $(1,0)$- and $(0,1)$-components as $d_f = \partial_f + \bar\partial$. The key ingredients of the proof consist of the Liouville theorem in several complex variables and two kinds of Poincaré lemmas.

Lemma 3.1 (Liouville Theorem) *There following sequence is exact:*

$$0 \longrightarrow \Omega_Z^{p,0} \longrightarrow \mathcal{T}_Z^{p,0} \xrightarrow{\ \bar\partial\ } \mathcal{T}_Z^{p,1}.$$

Lemma 3.2 ($\bar{\partial}$-Poincaré Lemma) *The following sequence is exact:*

$$\mathcal{T}_Z^{p,0} \xrightarrow{\bar{\partial}} \mathcal{T}_Z^{p,1} \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \mathcal{T}_Z^{p,n} \longrightarrow 0. \tag{10}$$

Lemma 3.3 (∂_f-Poincaré Lemma) *The following sequence is exact:*

$$\mathcal{T}_Z^{0,q} \xrightarrow{\partial_f} \mathcal{T}_Z^{1,q} \xrightarrow{\partial_f} \cdots \xrightarrow{\partial_f} \mathcal{T}_Z^{n,q} \longrightarrow 0. \tag{11}$$

These lemmas are incorporated into the commutative diagram in Figure 1, in which all the horizontal sequences except the top one, as well as all the vertical sequences, are exact. In view of these exactness, a general theorem in homological algebra implies that there is a natural isomorphism of cohomology groups $H^{\cdot}(\Omega_Z^{\cdot}, d_f) \xrightarrow{\sim} H^{\cdot}(\mathcal{T}_Z^{\cdot}, d_f)$, since the total complex of the double complex $(\mathcal{T}_Z^{\cdot\cdot}, \partial_f, \bar{\partial})$ is given by $(\mathcal{T}_Z^{\cdot}, d_f)$. One can easily check that this isomorphism is actually induced from the inclusion $(\Omega_Z^{\cdot}, d_f) \hookrightarrow (\mathcal{T}_Z^{\cdot}, d_f)$. Hence Theorem 2.1 is proved if Lemmas 3.1, 3.2 and 3.2 are established.

Lemma 3.1 is quite standard in several complex variables. The proofs of Lemmas 3.2 and 3.3 are reduced to showing the exactness of the sequence:

$$\mathcal{T}_Z^{0,q} \xrightarrow{\partial} \mathcal{T}_Z^{1,q} \xrightarrow{\partial} \cdots \xrightarrow{\partial} \mathcal{T}_Z^{n,q} \longrightarrow 0. \tag{12}$$

Indeed, (10) is transformed into (12) by just taking complex conjugate. As for (11), we notice that there is a commutative diagram:

$$\begin{array}{ccccccccc}
\mathcal{T}_Z^{0,q} & \xrightarrow{\partial_f} & \mathcal{T}_Z^{1,q} & \xrightarrow{\partial_f} & \cdots & \xrightarrow{\partial_f} & \mathcal{T}_Z^{n,q} & \longrightarrow & 0 \\
\big\downarrow{\scriptstyle e^{2i\,\mathrm{Im}\,f}} & & \big\downarrow{\scriptstyle e^{2i\,\mathrm{Im}\,f}} & & & & \big\downarrow{\scriptstyle e^{2i\,\mathrm{Im}\,f}} & & \\
\mathcal{T}_Z^{0,q} & \xrightarrow{\partial} & \mathcal{T}_Z^{1,q} & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & \mathcal{T}_Z^{n,q} & \longrightarrow & 0,
\end{array} \tag{13}$$

where the vertical arrows are the multiplications by the function $e^{2i\,\mathrm{Im}\,f}$. In view of (9), they should be the multiplication by the function e^f. However, since f is a complex polynomial, the function $e^{-\bar{f}}$ is a constant with respect to the differential ∂, namely, it is an anti-holomorphic function, and so e^f can be replaced by $e^f e^{-\bar{f}} = e^{f-\bar{f}} = e^{2i\,\mathrm{Im}\,f}$. This replacement is a crucial trick in our argument. Since the function $e^{2i\,\mathrm{Im}\,f}$ and all its derivatives are at most of polynomial growth as $|z| \to \infty$, each vertical arrow in (13) is an isomorphism, yielding an isomorphism between (11) and (12). In this manner (11) has been untwisted and transformed into (12). Finally the exactness of (12) is obtained from Lemma 3.4 below by dualizing the sequence (14).

Lemma 3.4 *Let $\mathcal{S}_Z^{p,q}$ denote the space of smooth (p,q)-forms of Schwartz class on Z. The following sequence is exact:*

$$0 \longrightarrow \mathcal{S}_Z^{0,q} \xrightarrow{\partial} \mathcal{S}_Z^{1,q} \xrightarrow{\partial} \cdots \xrightarrow{\partial} \mathcal{S}_Z^{n,q}. \tag{14}$$

This lemma is established by using the Fourier transform, the Borel-Litt theorem, a division theorem for smooth functions, the flatness of the formal power series ring over the polynomial ring and the exactness of an algebraic Koszul complex.

4 Twisted Hodge-Kodaira Decomposition

The strategy for proving Theorem 2.2 is to develop a twisted version of Hodge-Kodaira theory and a crucial role is played by the twisted Laplacian:

$$\Delta_f = d_f d_f^* + d_f^* d_f,$$

where d_f^* is the formal Hermitian adjoint of d_f. A simple inspection shows that Δ_f is unitarily equivalent to the real twisted Laplacian $\Delta_g = d_g d_g^* + d_g^* d_g$ associated with the real part g of f, namely,

$$\Delta_f = e^{i\,\mathrm{Im}\,f}\,\Delta_g\,e^{-i\,\mathrm{Im}\,f} \qquad \text{with} \quad g = \mathrm{Re}\ f. \tag{15}$$

It should be mentioned here that Witten [24] used the twisted Laplacian Δ_{tg} on a Riemannian manifold to develop his Morse theory as a supersymmetric quantum mechanics, where g is a Morse function and t is a large parameter. Afterwards, Helffer and Sjöstrand [8] worked out to justify some of Witten's speculations mathematically. On the other hand, we use the twisted Laplacian rather differently; we are not concerned with Morse theory but with Hodge-Kodaira theory. Moreover our function $g = \mathrm{Re}\ f$ is not necessarily a Morse function.

We need an explicit formula for the twisted Laplacian Δ_g. To state it we introduce some notation. Setting $m = 2n$ and $z_j = x_{2j-1} + i\,x_{2j}$ ($j = 1, \dots, n$), we think of $Z = \mathbb{C}^n$ as the real Euclidean space $\mathbb{R}^m$ with real coordinates $x = (x_1, \dots, x_m)$. For an ordered subset $I = (i_1, \dots, i_p) \subset \{1, \dots, m\}$, we write $dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_p}$ and $|I| = p$. We employ a formula of Witten [24, formula (13)]. In our case this formula reads:

Proposition 4.1 *The twisted Laplacian Δ_g acting on p-forms is written*

$$\Delta_g \ : \quad \sum_{|I|=p} \phi_I\, dx_I \ \mapsto \ \sum_{|I|=p} \sum_{|J|=p} A_{IJ}\, \phi_J\, dx_I,$$

where the matrix elements A_{IJ} are described as follows:

(i) *Diagonal: if $I = J$, then*

$$A_{II} = \Delta + |dg|^2 + \sum_{i=1}^{m} \varepsilon_i(I)\, \frac{\partial^2 g}{\partial x_i^2},$$

where $\Delta = -(\partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_m^2)$ *is the usual Laplacian and* $\varepsilon_i(I)$ *is a sign function defined by*

$$\varepsilon_i(I) = \begin{cases} +1 & (i \in I), \\ -1 & (i \notin I). \end{cases}$$

(ii) *First off-diagonals: if* $|I \cap J| = p - 1$, *then*

$$A_{IJ} = 2\,(-1)^{\ell_i(I)+\ell_j(J)} \frac{\partial^2 g}{\partial x_i \partial x_j},$$

where $i \in I \setminus J$ *and* $j \in J \setminus I$, *and* $\ell_i(I) = k$ *if* $i \in I$ *is the* k*-th smallest element of* I.

(iii) *Remaining entries: if* $|I \cap J| < p - 1$, *then* $A_{IJ} = 0$.

This proposition shows that the twisted Laplacian Δ_g may be thought of as a Schrödinger operator with matrix-valued potential: $\Delta_g = \Delta + |dg|^2 + V$, where the potential term consists of the scalar part $|dg|^2$ and the tri-diagonal matrix part V. Since f is a complex polynomial having g as its real part, the Cauchy-Riemann equation for f yields $|dg|^2 = |df|^2$. Moreover, since f is a polynomial of degree N having f_0 as its top homogeneous component, $|dg|^2$ is expressed as the sum of the real homogeneous polynomial $|df_0|^2$ of degree $2(N-1)$ with a real polynomial of degree less than $2(N-1)$. On the other hand, Proposition 4.1 implies that each entry of V is a real polynomial of degree less than $N-1$. Hence Δ_g is written in the form:

$$\Delta_g = \Delta + |df_0|^2 + U,$$

where U is a tri-diagonal symmetric matrix whose entries are real polynomials of degree less than $2(N-1)$. Assumption 1.1 then assures that the principal term $|df_0|^2$ of the potential has a *uniform* polynomial growth of degree $2(N-1) \geq 2$ as $|x| \to \infty$. This fact leads us to a twisted version of Hodge-Kodaira decomposition on the open manifold $Z = \mathbb{R}^m$.

Theorem 4.2 (Twisted Hodge-Kodaira Decomposition) *Every* Δ_g*-harmonic tempered current is necessarily a* Δ_g*-harmonic smooth differential form of Schwartz class, and vice versa, namely,*

$$\{\, \phi \in \mathcal{T}_Z^{\cdot} \,:\, \Delta_g \phi = 0 \,\} = \{\, \phi \in \mathcal{S}_Z^{\cdot} \,:\, \Delta_g \phi = 0 \,\}. \tag{16}$$

The linear space $\mathcal{H}_g^{\cdot}$ *defined by the both sides of* (16) *is finite dimensional. There exists a* W*-equivariant, continuous, linear operator* $G_g : \mathcal{T}_Z^{\cdot} \to \mathcal{T}_Z^{\cdot}$ *(Green operator), which restricts to a continuous operator* $G_g : \mathcal{S}_Z^{\cdot} \to \mathcal{S}_Z^{\cdot}$, *such that the following conditions are satisfied:*

(i) *There exist W-equivariant direct sum decompositions*:

$$\begin{aligned} \mathcal{T}_Z^{\cdot} &= \mathcal{H}_g^{\cdot} \oplus d_g d_g^* G_g \mathcal{T}_Z^{\cdot} \oplus d_g^* d_g G_g \mathcal{T}_Z^{\cdot}, &(17)\\ \mathcal{S}_Z^{\cdot} &= \mathcal{H}_g^{\cdot} \oplus d_g d_g^* G_g \mathcal{S}_Z^{\cdot} \oplus d_g^* d_g G_g \mathcal{S}_Z^{\cdot}. &(18)\end{aligned}$$

(ii) *Let $H_g : \mathcal{T}_Z^{\cdot} \to \mathcal{H}_g^{\cdot}$ be the projection relative to the decomposition* (17). *Then it restricts to the projection $H_g : \mathcal{S}_Z^{\cdot} \to \mathcal{H}_g^{\cdot}$ relative to the decomposition* (18). *Moreover, as operators acting on $\mathcal{T}_Z^{\cdot}$ or on $\mathcal{S}_Z^{\cdot}$, the following commutation relations are valid*:

(a) $I = H_g + \Delta_g G_g = H_g + G_g \Delta_g$,

(b) $H_g G_g = G_g H_g = H_g \Delta_g = \Delta_g H_g = 0$,

(c) $d_g G_g = G_g d_g$, $d_g^* G_g = G_g d_g^*$.

Theorem 4.2 implies that both of the cohomology groups $H^{\cdot}(\mathcal{T}_Z^{\cdot}, d_g)$ and $H^{\cdot}(\mathcal{S}_Z^{\cdot}, d_g)$ are W-equivariantly isomorphic to the harmonic space $\mathcal{H}_g^{\cdot}$. In view of (15), Theorem 4.2 clearly leads to the same type of decomposition theorem for the twisted Laplacian Δ_f. Thus we obtain the following corollary.

Corollary 4.3 *There exist W-equivariant isomorphisms*:

$$H^{\cdot}(\mathcal{S}_Z^{\cdot}, d_f) \xleftarrow{\sim} \mathcal{H}_f^{\cdot} \xrightarrow{\sim} H^{\cdot}(\mathcal{T}_Z^{\cdot}, d_f),$$

where $\mathcal{H}_f^{\cdot}$ is the space of Δ_f-harmonic tempered p-currents on Z, or equivalently the space of Δ_f-harmonic smooth forms of Schwartz class on Z.

Once Theorem 4.2 is established, Theorem 2.2 readily follows from Corollary 4.3. Basically the proof of Theorem 4.2 follows the standard arguments as in Wells [23, Chap. 4]. An essential difference here is to make use of the pseudo-differential calculus of Kumano-go and Taniguchi [14]. They developed a global calculus which allows for weight functions of polynomial growth as $|x| \to \infty$ and permits a global treatment of operators like $-\Delta + |x|^{2k}$ and its inverse (see also Beals [5]).

5 Real Structure

There are natural real structures on $H^n(\Omega_Z^{\cdot}, d_{\pm f})$ compatible with the duality. Let us recall some terminology. A *real structure* on a complex vector space, say, V, is an anti-$\mathbb{C}$-linear automorphism $J : V \to V$ such that $J^2 = 1$. Given a real structure J on V, let $V_{\mathbb{R}} = \{ v \in V : Jv = v \}$. Then $V_{\mathbb{R}}$ is a real vector space such that $V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} = V$. We refer to $V_{\mathbb{R}}$ as *the real part* of V relative to J. When a group W acts on V, a real structure J on V is said to be W-*equivariant* if J commutes with the action of W on V. If J is W-equivariant, then $V_{\mathbb{R}}$ is acted on by W in a natural manner.

Theorem 5.1 *The cohomology groups $H^n(\Omega^{\cdot}_Z, d_{\pm f})$ admit W-equivariant real structures $J_{\pm f} : H^n(\Omega^{\cdot}_Z, d_{\pm f}) \to H^n(\Omega^{\cdot}_Z, d_{\pm f})$ such that the W-invariant complex duality* (4) *restricts to a W-invariant real duality:*

$$H^n_{\mathbb{R}}(\Omega^{\cdot}_Z, d_f) \times H^n_{\mathbb{R}}(\Omega^{\cdot}_Z, d_{-f}) \to \mathbb{R}, \tag{19}$$

where $H^n_{\mathbb{R}}(\Omega^{\cdot}_Z, d_{\pm f})$ denote the real parts of $H^n(\Omega^{\cdot}_Z, d_{\pm f})$ relative to $J_{\pm f}$.

The real structures mentioned in Theorem 5.1 are constructed as follows. There exists (well-defined) anti-$\mathbb{C}$-linear automorphisms:

$$J_{\pm f} : \mathcal{S}^p_Z \to \mathcal{S}^p_Z, \quad \psi \mapsto e^{\mp 2i \operatorname{Im} f}\, \bar{\psi}.$$

It is easy to see that $J_{\pm f}$ give real structures on $\mathcal{S}^p_Z$. A simple check shows that $J_{\pm f}$ commute with the twisted exterior differentials $d_{\pm f}$ and so $J_{\pm f}$ define real structures on the twisted de Rham complexes $(\mathcal{S}^{\cdot}_Z, d_{\pm f})$ of Schwartz class. Passing to cohomology, they induce W-equivariant real structures $J_{\pm f}$ on $H^n(\mathcal{S}^{\cdot}_Z, d_{\pm f})$. Then, through the W-equivariant isomorphism (8), one obtains W-equivariant real structures $J_{\pm f}$ on $H^n(\Omega^{\cdot}_Z, d_{\pm f})$. The complex duality (4) induces the real duality (19), when restricted to the real parts.

Corollary 5.2 *Let W be a finite unitary reflection group and set $T = Z/W$. Then there exist natural real structures on $H^n(\Omega^{\cdot}_T, d_{\pm f})$ such that the complex duality in Corollary 1.3 restricts to a real duality:*

$$H^n_{\mathbb{R}}(\Omega^{\cdot}_T, d_f) \times H^n_{\mathbb{R}}(\Omega^{\cdot}_T, d_{-f}) \to \mathbb{R}.$$

6 Super-symmetry

It is interesting to consider a transformation that permutes the $\pm$ cohomology groups $H^n(\Omega^{\cdot}_Z, d_{\pm f})$ under which the duality behaves naturally.

Definition 6.1 Given a finite subgroup W of $U(n)$ and a W-invariant polynomial f, a unitary transformation $\tau \in U(n)$ is called a *super-symmetry* relative to (W, f), if the following conditions are satisfied:

$$\tau^2 \in W \qquad \text{and} \qquad \tau^* f = -f.$$

Here is a simple example. Let W be the symmetric group S_n acting on $\mathbb{C}^n$ by permuting the coordinates, and f be a symmetric polynomial having only odd homogeneous components. Then $\tau = -1$ is a super-symmetry.

A super-symmetry τ induces W-equivariant isomorphisms of de Rham complexes $\tau : (\mathcal{F}_Z^{\cdot}, d_{\pm f}) \xrightarrow{\sim} (\mathcal{F}_Z^{\cdot}, d_{\mp f})$, where $\mathcal{F}_Z^{\cdot} = \Omega_Z^{\cdot}, \mathcal{S}_Z^{\cdot}, \mathcal{T}_Z^{\cdot}$, and then W-equivariant isomorphisms of de Rham cohomology groups:

$$\tau : H^n(\mathcal{F}_Z^{\cdot}, d_{\pm f}) \xrightarrow{\sim} H^n(\mathcal{F}_Z^{\cdot}, d_{\mp f}) \quad (\mathcal{F}_Z^{\cdot} = \Omega_Z^{\cdot}, \mathcal{S}_Z^{\cdot}, \mathcal{T}_Z^{\cdot}). \tag{20}$$

Note that they are compatible with the isomorphisms (8). Thus a super-symmetry permutes the $\pm$ cohomology groups. The duality (4), combined with the super symmetry (20), induces a W-invariant self-duality:

$$H^n(\Omega_Z^{\cdot}, d_f) \times H^n(\Omega_Z^{\cdot}, d_f) \to \mathbb{C}, \quad (\phi_1, \phi_2) \mapsto \langle\langle \phi_1, \phi_2 \rangle\rangle := \langle \phi_1, \tau\phi_2 \rangle. \tag{21}$$

Note that (21) is compatible with the real structure J_f. The self-duality (21) has a beautiful property, when restricted to the W-invariant components.

Theorem 6.2 *For a super-symmetry τ relative to (W, f), the associated self-duality* (21) *induces a nondegenerate bilinear form:*

$$H^n(\Omega_Z^{\cdot}, d_f)^W \times H^n(\Omega_Z^{\cdot}, d_f)^W \to \mathbb{C},$$

which is symmetric when n is even, and skew-symmetric when n is odd.

Let $H^n_{\mathbb{R}}(\Omega_Z^{\cdot}, d_f)^W$ denote the W-invariant component of the real cohomology group $H^n_{\mathbb{R}}(\Omega_Z^{\cdot}, d_f)$, or equivalently the real part of the W-invariant complex cohomology group $H^n(\Omega_Z^{\cdot}, d_f)^W$. The compatibility of the self-duality (21) with the real structure J_f leads to a refinement of Theorem 6.2.

Theorem 6.3 *For a super-symmetry τ relative to (W, f), the self-duality* (21) *induces a real nondegenerate bilinear form:*

$$H^n_{\mathbb{R}}(\Omega_Z^{\cdot}, d_f)^W \times H^n_{\mathbb{R}}(\Omega_Z^{\cdot}, d_f)^W \to \mathbb{R},$$

which is symmetric when n is even, and skew-symmetric when n is odd.

When W is a finite unitary reflection group, Theorem 6.3 yields:

Corollary 6.4 *Let W be a finite unitary reflection group and set $T = Z/W$. For a super-symmetry τ relative to (W, f), there exists a real nondegenerate bilinear form:*

$$H^n_{\mathbb{R}}(\Omega_T^{\cdot}, d_f) \times H^n_{\mathbb{R}}(\Omega_T^{\cdot}, d_f) \to \mathbb{R}. \tag{22}$$

which is symmetric when n is even, and skew-symmetric when n is odd.

We pose the problem of constructing a basis of $H^n_{\mathbb{R}}(\Omega^\cdot_T, d_f)$ and computing the intersection matrix of the self-duality (22) relative to that basis for various finite unitary reflection groups W and various W-invariant polynomials f satisfying Assumption 1.1. It is also an interesting problem to discuss the connection of our construction with the classical Picard-Lefschetz theory for isolated surface singularities, vanishing cycles, Hodge structures, and so on.

7 Generalized Airy functions

The classical Airy function in single-variable is defined by an integral:

$$\mathrm{Ai}(a) = \int_c e^{\frac{1}{3}t^3 + at}\, dt \qquad (a \in \mathbb{C}),$$

where c is a cycle chosen in such a way that the integrand is exponentially decreasing at infinity along c (see Figure 2). The Airy function is an important special function arising in mathematical optics (see Airy [1]). A generalization of the Airy function into several variables was introduced by Gel'fand, Retakh and Serganova [7] and was studied in some depth by Kimura [12, 13].

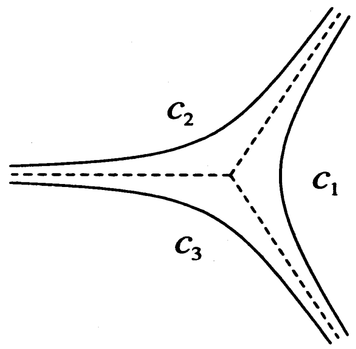


Figure 2: Cycles for the Airy integral

Following [7, 12], we recall the definition of a generalized Airy function. First, let $\theta_k(t)$ be the k-th coefficient of a generating series:

$$\log(1 + t_1 X + \cdots + t_n X^n) = \sum_{k=1}^{\infty} \theta_k(t)\, X^k.$$

Then $\theta_k(t)$ is a weighted homogeneous polynomial of degree k in variables $t = (t_1, \ldots, t_n)$, where t_j is assumed to be of degree j. For small values of k,

$\theta_1(t) = t_1$
$\theta_2(t) = t_2 - \frac{1}{2}t_1^2$
$\theta_3(t) = t_3 - t_2 t_1 + \frac{1}{3}t_1^3$
$\theta_4(t) = t_4 - t_3 t_1 - \frac{1}{2}t_2^2 + t_2 t_1^2 - \frac{1}{4}t_1^4$
$\theta_5(t) = t_5 - t_4 t_1 - t_3 t_2 + t_3 t_1^2 + t_2^2 t_1 - t_2 t_1^3 + \frac{1}{5}t_1^5$

Table 1: Polynomials $\theta_k(t)$

polynomials $\theta_k(t)$ are illustrated in Table 1. Set

$$f = f(a,t) = \sum_{k=0}^{N} (-1)^k \, e_k(a) \, \theta_{N-k+1}(t),$$

where $e_k(a)$ is the k-th elementary symmetric polynomial of $a = (a_1, \dots, a_N)$ with $N \geq n$. Let $T = \mathbb{C}^n$ be the complex n-space with coordinates $t = (t_1, \dots, t_n)$. Next we define a family Φ of supports in T as follows: an element of Φ is a closed subset c of T such that $\operatorname{Re}\theta_{N+1}(t)|_c \to -\infty$, quicker than $-||t||^q$ for some $q > 0$ as $||t|| := \sum_{j=1}^{n} |t_j|^{1/j} \to \infty$ (see Pham [18]). A generalized Airy function is now defined by an integral:

$$A(a) = \int_c e^{f(a,t)} \, \omega,$$

where $\omega \in \Omega_T^n$ is a d_f-closed polynomial n-form and c is an n-cycle with support in Φ. This integral depends only on the cohomology class $[\omega] \in H^n(\Omega_T^\cdot, d_f)$ and the homology class $[c] \in H_n^\Phi(T)$, where $H_n^\Phi(T)$ denote the integral n-th homology group of T with supports in Φ.

If we introduce new variables $z = (z_1, \dots, z_n)$ such that

$$t_j = (-1)^j e_j(z) \qquad (j = 1, \dots, n),$$

where $e_j(z)$ is the j-th elementary symmetric polynomial of z, then we can easily see that f, as a polynomial of z, satisfies Assumption 1.1. In this case the group W is the symmetric group S_n acting on $Z = \mathbb{C}^n$ by permuting the coordinates, having degrees $d = (1, 2, \dots, n)$. Thus we have

$$\dim H^n(\Omega_T^\cdot, d_{\pm f}) = \mu, \quad \text{where} \quad \mu = \binom{N}{n}.$$

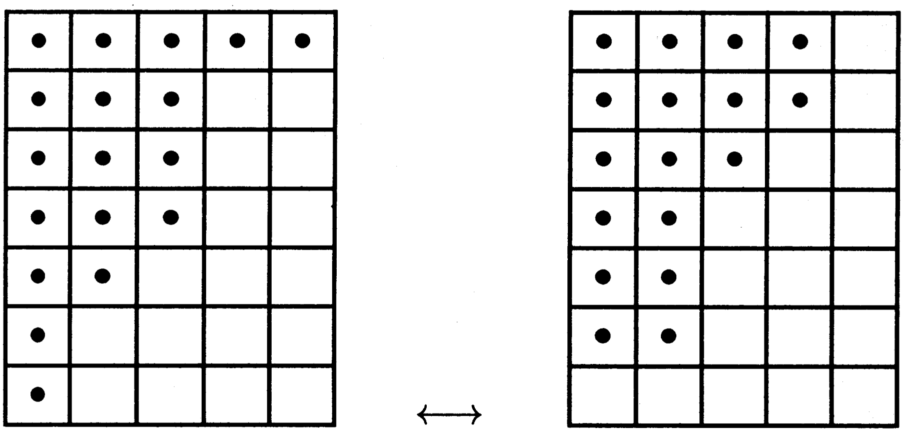
There are bases of the cohomology groups $H^n(\Omega_T^{\cdot}, d_{\pm f})$ written in terms of Schur polynomials. Given a Young diagram λ, let $s_\lambda(z)$ denote the Schur polynomial in z attached to the diagram λ (see Macdonald [15]). Let $R(p,q)$ be the rectangular Young diagram with p rows and q columns, and let $\mathcal{Y}(p,q)$ be the set of all Young subdiagrams of $R(p,q)$.

Theorem 7.1 *Denote by $\phi_\lambda^\pm$ the cohomology classes in $H^n(\Omega_T^{\cdot}, d_{\pm f})$ represented by the polynomial differential n-form $s_\lambda(z)\, dt_1 \wedge \cdots \wedge dt_n$. Then the sets $\{\phi_\lambda^\pm : \lambda \in \mathcal{Y}(n, N-n)\}$ form bases of the cohomology groups $H^n(\Omega_T^{\cdot}, d_{\pm f})$.*

We are interested in the intersection matrix of the duality (5) relative to the bases in Theorem 7.1. Iwasaki and Matsumoto [11] were able to calculate it explicitly in terms of skew-Schur polynomials. Let $s_{\lambda/\mu}(a)$ denote the skew-Schur polynomial of $a = (a_1, \ldots, a_N)$ attached to a pair (λ, μ) of Young diagrams (see Macdonald [15]). To state the result, we need the concept of complementary diagrams. Given a Young diagram $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p) \in \mathcal{Y}(p,q)$, its complementary diagram $\check{\lambda} \in \mathcal{Y}(p,q)$ is defined by

$$\check{\lambda} = (q - \lambda_p, q - \lambda_{p-1}, \ldots, q - \lambda_1).$$

Pictorially, $\check{\lambda}$ is obtained by rotating the rectangle $R(p,q)$, together with λ, around its center by 180° and then deleting λ from $R(p,q)$. For instance, a Young diagram $\lambda = (5,3,3,3,2,1,1)$ has the complementary diagram $\check{\lambda} = (4,4,3,2,2,2,0)$ in $\mathcal{Y}(7,5)$ (see Figure 3). Taking complementary diagrams $\lambda \mapsto \check{\lambda}$ defines an involution on the set $\mathcal{Y}(p,q)$.



$\lambda = (5,3,3,3,2,1,1)$ $\qquad\qquad$ $\check{\lambda} = (4,4,3,2,2,2,0)$

Figure 3: Complementary Diagrams

Theorem 7.2 *With respect to the bases $\{\phi_\lambda^\pm\}$ constructed in Theorem 7.1, the intersection pairing $H^n(\Omega_T^\cdot, d_f) \times H^n(\Omega_T^\cdot, d_{-f}) \to \mathbb{C}$ is represented as*

$$\langle \phi_\lambda^+, \phi_\mu^- \rangle = (-1)^{\frac{1}{2}n(n-1)}\, n!\, s_{\lambda/\tilde{\mu}}(a) \qquad \textit{for} \quad \lambda,\, \mu \in \mathcal{Y}(n, N-n).$$

This theorem provides us with a cohomological interpretation of skew-Schur polynomials by means of a twisted intersection theory.

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