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CONVERGENCE OF FORMAL SOLUTIONS OF SINGULAR FIRST ORDER NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS OF TOTALLY CHARACTERISTIC TYPE

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1. Introduction

Let $(t,x)=(t_1,\ldots,t_d,x_1,\ldots,x_n)\in\mathbf{C}^d\times\mathbf{C}^n$ be $(d+n)$-dimensional complex variables $(d\geq 1,\ n\geq 1)$.

We consider the following first order nonlinear partial differential equation:

$$(1.1)\qquad \begin{cases} \displaystyle\sum_{i,j=1}^{d} a_{ij}(x)t_i\partial_{t_j}u+\sum_{k=1}^{n} b_k(x)\partial_{x_k}u+c(x)u \\ \qquad\qquad\displaystyle =\sum_{|l|=K} d_l(x)t^l+f_{K+1}(t,x,u,\{\partial_{t_j}u\},\{\partial_{x_k}u\}), \\ u(t,x)=O(|t|^K), \end{cases}$$

where $|t|=t_1+\cdots+t_d$, K is a fixed positive integer satisfying $K\geq 2$ and $a_{ij}(x)$, $b_k(x)$, $c(x)$ and $d_l(x)$ are holomorphic in a neighbourhood of the origin, and $f_{K+1}(t,x,u,\tau,\xi)$ $(\tau=(\tau_j)\in\mathbf{C}^d,\ \xi=(\xi_k)\in\mathbf{C}^n)$ is also holomorphic in a neighbourhood of the origin with the following Taylor expansion:

$$f_{K+1}(t,x,u,\tau,\xi)=\sum_{|p|+Kq+(K-1)|r|+K|s|\geq K+1} f_{pqrs}(x)t^pu^q\tau^r\xi^s,$$

where $q\in\mathbf{Z}_{\geq 0}=\{0,1,2,\ldots\}$, $p=(p_1,\ldots,p_d)\in(\mathbf{Z}_{\geq 0})^d$, $r=(r_1,\ldots,r_d)\in(\mathbf{Z}_{\geq 0})^d$, $s=(s_1,\ldots,s_n)\in(\mathbf{Z}_{\geq 0})^n$,

$$|p|=p_1+\cdots+p_d,\quad |r|=r_1+\cdots+r_d,\quad |s|=s_1+\cdots+s_n,$$

and

$$t^p=\prod_{j=1}^{d} t_j{}^{p_j},\quad \tau^r=\prod_{j=1}^{d}\tau_j{}^{r_j},\quad \xi^s=\prod_{k=1}^{n}\xi_k{}^{s_k}.$$

This eqauation seems to be a natural extension of totally characteristic type studied by Chen-Tahara ([CT]) to several time-space variables.

Here we remark that the assumption $K \geq 2$ implies $\partial_{t_j} u(0,0) = 0$ $(j = 1, 2, \dots, d)$ which assures that $(0, 0, u(0,0), \{\partial_{t_j} u(0,0)\}, \{\partial_{x_k} u(0,0)\})$ belongs to the domain of definition of $f_{K+1}(t, x, u, \tau, \xi)$.

Now our first theorem is stated as follows:

Theorem 1. *Let $\{\lambda_j\}_{j=1}^d$ be the eigenvalues of the matrix $(a_{ij}(0))$. We assume that $b_k(x) \not\equiv 0$ and $b_k(0) = 0$ for $k = 1, 2, \dots, n$, and let $\{\mu_k\}_{k=1}^n$ be the eigenvalues of Jacobi matrix of $(b_1(x), \dots, b_n(x))$ at $x = 0$. Then the formal power series solution of (1.1) exists uniquely and converges if the following conditions are satisfied:*

There exists a positive constant $\sigma_0 > 0$, such that

$$\left| \sum_{j=1}^{d} \lambda_j l_j + \sum_{k=1}^{n} \mu_k m_k \right| \geq \sigma_0(|l| + |m|) \qquad \text{(Poincaré condition)}, \tag{1.2}$$

and

$$\sum_{j=1}^{d} \lambda_j l_j + \sum_{k=1}^{n} \mu_k m_k + c(0) \neq 0 \qquad \text{(Non-resonance condition)} \tag{1.3}$$

hold for all $(l, m) \in (\mathbf{Z}_{\geq 0})^d \times (\mathbf{Z}_{\geq 0})^n$ with $|l| \geq K$ and $|m| \geq 0$.

Remark 1. It is easy to show the following proposition.

The conditions (1.2) and (1.3) imply that

$$\left| \sum_{j=1}^{d} \lambda_j l_j + \sum_{k=1}^{n} \mu_k m_k + c(0) \right| \geq \sigma(|l| + |m|) \tag{1.4}$$

holds by some positive constant $\sigma > 0$ for all $(l, m) \in (\mathbf{Z}_{\geq 0})^d \times (\mathbf{Z}_{\geq 0})^n$ with $|l| \geq K$ and $|m| \geq 0$. In the proof of Theorem 1, this condition will be used instead of (1.2) and (1.3). □

Next, we consider the following general equation:

$$\begin{cases} f(t, x, u(t,x), \{\partial_{t_j} u(t,x)\}, \{\partial_{x_k} u(t,x)\}) = 0, \\ u(0, x) \equiv 0. \end{cases} \tag{1.5}$$

Assumption 1. $f(t, x, u, \tau, \xi)$ $(\tau = (\tau_j) \in \mathbf{C}^d,\ \xi = (\xi_k) \in \mathbf{C}^n)$ is holomorphic in a neighbourhood of the origin, and is an entire function in τ variables for any fixed t, x, u and ξ. Moreover we assume that

$$f(0, x, 0, \tau, 0) \equiv 0 \tag{1.6}$$

for $x \in \mathbf{C}^n$ near the origin and $\tau \in \mathbf{C}^d$, which is a generalization of the definition of singular equations defined in [MS].

For the equation (1.5), we do not know whether or not the equation has a formal solution in general. Therefore, we assume the following:

Assumption 2. The equation (1.5) has a formal solution of the form

$$u(t,x)=\sum_{j=1}^{d}\varphi_j(x)t_j+\sum_{|l|\geq 2,|m|\geq 0}u_{lm}t^l x^m\in \mathbf{C}[[t,x]]. \tag{1.7}$$

By the existence of a formal solution, $\{\varphi_j(x)\}$ satisfy the following system formally:

$$f(0,x,0,\{\varphi_j(x)\},0)\equiv 0 \quad \text{(trivial relation)}, \tag{1.8}$$

and

$$\begin{aligned}&\frac{\partial}{\partial t_i}f(t,x,u(t,x),\{\partial_{t_j}u(t,x)\},\{\partial_{x_k}u(t,x)\})\Big|_{t=0}\\&=\frac{\partial f}{\partial t_i}(0,x,0,\{\varphi_j(x)\},0)+\frac{\partial f}{\partial u}(0,x,0,\{\varphi_j(x)\},0)\varphi_i(x)\\&+\sum_{k=1}^{n}\frac{\partial f}{\partial \xi_k}(0,x,0,\{\varphi_j(x)\},0)\frac{\partial\varphi_i}{\partial x_k}(x)=0,\ \text{ for } i=1,2,\dots,d.\end{aligned} \tag{1.9}$$

The formal solution of this system is not convergent in general. Therefore, we assume

Assumption 3. The coefficients $\{\varphi_j(x)\}$ are all holomorphic in a neighbourhood of the origin of $\mathbf{C}^n$.

Remark 2. In the case $d=1$ (d is the dimension of t variables), a sufficient condition for the formal solution of (1.9) to converge has been already obtained by Miyake-Shirai [MS]. In the case $d\geq 2$, we give a sufficient condition for the formal solution of system (1.9) to be convergent, which will be given by Theorem 3 in Section 5, but for a while we consider the problem under Assumption 3 for simplicity of our arguments. □

Now we put $\mathbf{a}(x)=(0,x,0,\{\varphi_j(x)\},0)$ for simplicity, and define

(1.10)

$$A_{ij}(x):=\frac{\partial^2 f}{\partial t_i\partial\tau_j}(\mathbf{a}(x))+\frac{\partial^2 f}{\partial u\partial\tau_j}(\mathbf{a}(x))\varphi_i(x)+\sum_{k=1}^{n}\frac{\partial^2 f}{\partial\tau_j\partial\xi_k}(\mathbf{a}(x))\frac{\partial\varphi_i}{\partial x_k}(x),$$

for $i,j=1,2,\dots,d$. Moreover we define

$$B_k(x):=\frac{\partial f}{\partial\xi_k}(\mathbf{a}(x)),\quad \text{for } k=1,2,\dots,n. \tag{1.11}$$

Remark 3. The functions $A_{ij}(x)$ and $B_k(x)$ correspond to $a_{ij}(x)$ and $b_k(x)$ in Theorem 1, respectively (see (1.13) below). □

Here we assume that the equation is of totally characteristic type, that is,

Assumption 4. $B_k(x) \not\equiv 0$ and $B_k(0) = 0$, for $k = 1, 2, \dots, n$.

Now our second theorem which is our main result is stated as follows:

Theorem 2. *Suppose Assumptions 1, 2, 3 and 4. Let* $\{\lambda_j\}_{j=1}^d$ *be the eigenvalues of* $(A_{ij}(0))$*, and let* $\{\mu_k\}_{k=1}^n$ *be the eigenvalues of Jacobi matrix of the vector* $(B_k(x))$ *at* $x = 0$*. Then the formal solution (1.7) is convergent if the following condition is satisfied:*

There exists a positive constant $\sigma > 0$*, such that,*

$$\left|\sum_{j=1}^{d} \lambda_j l_j + \sum_{k=1}^{n} \mu_k m_k + \frac{\partial f}{\partial u}(\mathbf{a}(0))\right| \geq \sigma(|l| + |m|), \tag{1.12}$$

holds for all $(l, m) \in (\mathbf{Z}_{\geq 0})^d \times (\mathbf{Z}_{\geq 0})^n$ *with* $|l| \geq 2$, $|m| \geq 0$.

Remark 4. We put $v(t,x) = u(t,x) - \sum_{j=1}^{d} \varphi_j(x) t_j$ as a new unknown function. By Assumptions 1, 2, 3 and 4, we can easily see that $v(t,x)$ satisfies the equation of the following form:

$$\left\{\begin{array}{l} \displaystyle\sum_{i,j=1}^{d} A_{ij}(x) t_i \partial_{t_j} v + \sum_{k=1}^{n} B_k(x) \partial_{x_k} v + \frac{\partial f}{\partial u}(\mathbf{a}(x)) v \\ \qquad\qquad \displaystyle = \sum_{|l|=2} d_l(x) t^l + f_3(t, x, v, \{\partial_{t_j} v\}, \{\partial_{x_k} v\}), \\ v(t,x) = O(|t|^2). \end{array}\right. \tag{1.13}$$

This is an equation considered in Theorem 1 in the case $K = 2$. Therefore, it is sufficient to prove Theorem 1 in order to prove Theorem 2. □

2. Reduction of the equation

As is mentioned in Remark 4, it is sufficient to study the equation (1.1).

By the assumption of Theorem 1,

$$(a_{ij}(0)) \sim \begin{pmatrix} \lambda_1 & \delta_1 & & \\ & \lambda_2 & \ddots & \\ & & \ddots & \delta_{d-1} \\ & & & \lambda_d \end{pmatrix}, \quad \left.\frac{\partial(b_1, \dots, b_n)}{\partial(x_1, \dots, x_n)}\right|_{x=0} \sim \begin{pmatrix} \mu_1 & \nu_1 & & \\ & \mu_2 & \ddots & \\ & & \ddots & \nu_{n-1} \\ & & & \mu_n \end{pmatrix},$$

where δ_j, $\nu_k = 0$ or 1 $(1 \leq j \leq d-1, 1 \leq k \leq n-1)$.

Then by transforming the variables, (1.1) is reduced to the following form:

$$\begin{aligned}(\Lambda + \Delta) v(t,x) \;&=\; \sum_{|l|=K} \alpha_l(x) t^l + \sum_{i,j=1}^{d} \beta_{ij}(x) t_i \partial_{t_j} v + \gamma(x) v \\ &\quad + \sum_{k=1}^{n} \varphi_k(x) \partial_{x_k} v + \widetilde{f}_{K+1}(t, x, v, \{\partial_{t_j} v\}, \{\partial_{x_k} v\}),\end{aligned} \tag{2.1}$$

with $v(t,x)=O(|t|^K)$, where

$$\Lambda=\sum_{j=1}^{d}\lambda_j t_j\partial_{t_j}+\sum_{k=1}^{n}\mu_k x_k\partial_{x_k}+c(0),$$

$$\Delta=\sum_{j=1}^{d-1}\delta_j t_j\partial_{t_{j+1}}+\sum_{k=1}^{n-1}\nu_k x_k\partial_{x_{k+1}},$$

and $\alpha_l(x)$, $\beta_{ij}(x)$, $\gamma(x)$ and $\varphi_k(x)$ are holomorphic in a neighbourhood of the origin, and satisfy $\beta_{ij}(x)=O(|x|)$, $\gamma(x)=O(|x|)$ and $\varphi_k(x)=O(|x|^2)$, and $\widetilde{f}_{K+1}(t,x,u,\tau,\xi)$ is a holomorphic function which has a similar Taylor expansion with $f_{K+1}(t,x,u,\tau,\xi)$.

In the following sections, we shall prove the existence and convergence of the unique formal solution of (2.1).

3. Preparation to prove Theorem 1

Let $\mathbf{C}[t,x]_{L,M}$ be the set of homogeneous polynomial of degree L in t variables and of degree M in x variables, that is,

$$\mathbf{C}[t,x]_{L,M}=\left\{ f_{LM}(t,x)=\sum_{|l|=L,\ |m|=M} f_{lm}t^l x^m \,\middle|\, f_{lm}\in\mathbf{C}\right\}.$$

For the operator $\Lambda+\Delta$, the following lemma holds:

Lemma 1. *For all $L\geq K$ and $M\geq 0$, the operator*

$$\Lambda+\Delta:\ \mathbf{C}[t,x]_{L,M}\longrightarrow\mathbf{C}[t,x]_{L,M}$$

is invertible. Moreover, if the majorant relation $f_{LM}(t,x)\ll F\times(t_1+\cdots+t_d)^L(x_1+\cdots+x_n)^M$ ($f_{LM}(x)\in\mathbf{C}[t,x]_{L,M}$, $F>0$) holds, then we obtain the following majorant relation:

$$(\Lambda+\Delta)^{-1}f_{LM}(t,x)\ll\frac{C}{L+M}F\times(t_1+\cdots+t_d)^L(x_1+\cdots+x_n)^M, \tag{3.1}$$

where $C>0$ is a positive constant independent of L and M.

Proof. We define a norm of $u_{LM}(t,x)\in\mathbf{C}[t,x]_{L,M}$ by

$$||u_{LM}||:=\inf\left\{C>0\mid u_{LM}(t,x)\ll C(t_1+\cdots+t_d)^L(x_1+\cdots+x_n)^M\right\}.$$

We remark that $\mathbf{C}[t,x]_{L,M}$ becomes a Banach space by this norm.

First, by (1.4) it is easily checked that Λ is invertible on $\mathbf{C}[t,x]_{L,M}$ and

$$||\Lambda^{-1}||\leq\frac{1}{\sigma(L+M)} \tag{3.2}$$

holds for the operator norm of Λ^{-1} on $\mathbf{C}[t,x]_{L,M}$.

Next, since $u_{LM}(t,x) \ll ||u_{LM}||(t_1+\cdots+t_d)^L(x_1+\cdots+x_n)^M$, we have

$$\begin{aligned}\Delta u_{LM}(t,x) &\ll \sum_{j=1}^{d-1} L|\delta_j|\cdot||u_{LM}||(t_1+\cdots+t_d)^L(x_1+\cdots+x_n)^M \\ &\quad+\sum_{k=1}^{n-1} M|\nu_k|\cdot||u_{LM}||(t_1+\cdots+t_d)^L(x_1+\cdots+x_n)^M \\ &\ll \left\{L(d-1)\max_{j=1,\dots,d-1}|\delta_j|+M(n-1)\max_{k=1,\dots,n-1}|\nu_k|\right\}\times \\ &\quad\times||u_{LM}||(t_1+\cdots+t_d)^L(x_1+\cdots+x_n)^M.\end{aligned}$$

Here we make a change of variables by $t_j=\varepsilon^{j-1}\tau_j$, $x_k=\varepsilon^{k-1}y_k$, then δ_j and ν_k (the components of nilpotent part of Jordan canonical form) turns to $\varepsilon\delta_j$ and $\varepsilon\nu_k$, respectively. Therefore, by choosing ε sufficiently small, we may assume that the components of nilpotent part are small enough. Hence we may assume that

$$\max_{j=1,\dots,d-1}|\delta_j|<\frac{\sigma}{2(d-1)},\quad \max_{k=1,\dots,n-1}|\nu_k|<\frac{\sigma}{2(n-1)}. \tag{3.3}$$

Then

$$\Delta u_{LM}(t,x) \ll \frac{\sigma(L+M)}{2}\,||u_{LM}||(t_1+\cdots+t_d)^L(x_1+\cdots+x_n)^M$$

holds, and we obtain

$$||\Delta|| \le \frac{\sigma(L+M)}{2}.$$

Therefore, the operator norm of $\Delta\Lambda^{-1}$ is estimated by

$$||\Delta\Lambda^{-1}|| \le \frac{1}{\sigma(L+M)}\frac{\sigma(L+M)}{2}=\frac{1}{2}<1.$$

By using the Neumann's series, we can see that $\Lambda+\Delta$ is invertible and the norm of the inverse operator is estimated by

$$||(\Lambda+\Delta)^{-1}|| \le \frac{2}{\sigma}\frac{1}{L+M},$$

which we want to prove since $C=2/\sigma$ is independent of L and M. □

Now, we define some notations, which are used in the proof of Theorem 1.

Definition (1) Let $(t,x)\in \mathbf{C}^d\times\mathbf{C}^n$ $(d\ge 0,\ n\ge 0)$ be complex variables. For formal power series $f(t,x)=\sum_{|l|\ge 0,\ |m|\ge 0} f_{l,m}t^lx^m$, we define

$$|f|(t,x)=\sum_{|l|\ge 0,\ |m|\ge 0}|f_{l,m}|t^lx^m.$$

(2) Let $(t, X) \in \mathbf{C}^d \times \mathbf{C}$ $(d \geq 0)$ be complex variables. For formal power series $f(t, X) = \sum_{|l|\geq 0,\ M\geq 0} f_{l,M} t^l X^M$, we define the shift operator S by

$$S(f)(t, X) = \sum_{|l|\geq 0,\ M\geq 0} f_{l,M+1} t^l X^M = \frac{f(t, X) - f(t, 0)}{X}.$$

Remark 5. The following facts are easily shown:

- $f(t, x) \ll |f|(t, x)$;
- If $f(t, x)$ and $g(t, X)$ are convergent power series, then $|f|(t, x)$ and $S(g)(t, X)$ are also convergent. □

4. Proof of Theorem 1

First, we prove a unique existence of formal power series solution.

Let

$$u(t, x) = \sum_{|l|\geq K, |m|\geq 0} u_{lm} t^l x^m = \sum_{L\geq K} u_L(t, x) = \sum_{L\geq K, M\geq 0} u_{LM}(t, x)$$

be a formal solution of (2.1), where

$$u_{LM}(t, x) = \sum_{|l|=L, |m|=M} u_{lm} t^l x^m \in \mathbf{C}[t, x]_{L,M},$$

$$u_L(t, x) = \sum_{|l|=L} u_l(x) t^l = \sum_{M\geq 0} u_{LM}(t, x).$$

We put $P = \Lambda + \Delta$ for simplicity. We substitute $u(t, x) = \sum_{L\geq K} u_L(t, x)$ into (2.1), then we have the following recursion formula:

$$\begin{cases} Pu_K(t, x) = \sum_{|l|=K} \alpha_l(x) t^l + \sum_{i,j=1}^{d} \beta_{ij}(x) t_i \partial_{t_j} u_K(t, x) \\ \qquad\qquad + \gamma(x) u_K(t, x) + \sum_{k=1}^{n} \varphi_k(x) \partial_{x_k} u_K(t, x), \\ Pu_L(t, x) = \sum_{i,j=1}^{d} \beta_{ij}(x) t_i \partial_{t_j} u_L(t, x) + \gamma(x) u_L(t, x) + \sum_{k=1}^{n} \varphi_k(x) \partial_{x_k} u_L(t, x) \\ \qquad\qquad + G_L(t, x, \{u_p\}_{K\leq p<L}, \{\partial_{t_j} u_p\}_{K\leq p<L}, \{\partial_{x_k} u_p\}_{K\leq p<L}), \text{ for } L > K, \end{cases}$$

where $G_L(t, x, \zeta, \tau, \xi)$ is a polynomial of (t, ζ, τ, ξ).

First, we consider the case $L = K$. We substitute $u_K(t,x) = \sum_{M\geq 0} u_{KM}(t,x)$ into the above recursion formula, we have

$$\begin{cases} Pu_{K0}(t,x) = \sum_{|l|=K} \alpha_l(0)t^l, \\ Pu_{KM}(t,x) = \sum_{|l|=K} \alpha_l^M(x)t^l + \sum_{i,j=1}^{d}\sum_{p=1}^{M} \beta_{ij}^p(x)t_i\partial_{t_j} u_{K,M-p}(t,x) \\ \qquad + \sum_{p=1}^{M} \gamma^p(x)u_{K,M-p}(t,x) + \sum_{k=1}^{n}\sum_{p=2}^{M} \varphi_k^p(x)\partial_{x_k} u_{K,M-p+1}(t,x), \end{cases}$$

where we put

$$\alpha_l(x) = \sum_{M\geq 0} \alpha_l^M(x), \ \alpha_l^M(x) = \sum_{|m|=M} \alpha_{lm}x^m,$$

$$\beta_{ij}(x) = \sum_{M\geq 1} \beta_{ij}^M(x), \ \beta_{ij}^M(x) = \sum_{|m|=M} \beta_{ijm}x^m,$$

$$\gamma(x) = \sum_{M\geq 1} \gamma^M(x), \ \gamma^M(x) = \sum_{|m|=M} \gamma_m x^m,$$

$$\varphi_k(x) = \sum_{M\geq 2} \varphi_k^M(x), \ \varphi_k^M(x) = \sum_{|m|=M} \varphi_{km}x^m.$$

By Lemma 1, we know that the solution sequence $\{u_{KM}(t,x)\}_{M\geq 0}$ exists uniquely. Moreover, by the same argument, we see that $\{u_{LM}(t,x)\}$ $(L > K)$ exist uniquely. These show that the formal solution exists uniquely.

Next, we prove the convergence of the formal solution. We put $U(t,x) = Pu(t,x)$ as a new unknown function. By Lemma 1, the equation (2.1) is reduced to the following equation:

$$\begin{aligned} (4.1) \qquad U(t,x) &= \sum_{|l|=K} \alpha_l(x)t^l + \sum_{i,j=1}^{d} \beta_{ij}(x)t_i\partial_{t_j} P^{-1}U(t,x) \\ &\quad + \gamma(x)P^{-1}U(t,x) + \sum_{k=1}^{n} \varphi_k(x)\partial_{x_k} P^{-1}U(t,x) \\ &\quad + \tilde{f}_{K+1}(t,x,P^{-1}U(t,x),\{\partial_{t_j}P^{-1}U(t,x)\},\{\partial_{x_k}P^{-1}U(t,x)\}). \end{aligned}$$

We know that (4.1) has a unique formal solution of the form

$$U(t,x) = \sum_{|l|\geq K, |m|\geq 0} U_{lm}t^l x^m = \sum_{L\geq K} U_L(t,x) = \sum_{L\geq K, M\geq 0} U_{LM}(t,x).$$

In order to get a majorant series of $U(t,x)$, we prepare some majorant series for the coefficients of (4.1). We put $T = t_1 + \cdots + t_d$, $X = x_1 + \cdots + x_n$, and choose

$$\sum_{|l|=K} \alpha_l(x)t^l \ll A(X)T^K, \quad \beta_{ij}(x) \ll |\beta_{ij}|(X,\ldots,X) =: XB_{ij}(X),$$

$$\gamma(x) \ll |\gamma|(X,\ldots,X) =: XG(X), \quad \varphi_k(x) \ll |\varphi_k|(X,\ldots,X) =: X^2\Phi_k(X),$$

$$\begin{aligned} \tilde{f}_{K+1}(t,x,u,\tau,\xi) &\ll |\tilde{f}_{K+1}|(T,\ldots,T,X,\ldots,X,u,\tau,\xi) \\ &=: F_{K+1}(T,X,u,\tau,\xi) \\ &= \sum_{|p|+Kq+(K-1)|r|+K|s|\geq K+1} F_{pqrs}(X)T^{|p|}u^q\tau^r\xi^s, \end{aligned}$$

where $A(X)$, $B_{ij}(X)$, $G(X)$ and $\Phi_k(X)$ are holomorphic in a neighbourhood of $X = 0$, and $F_{K+1}(T,X,u,\tau,\xi)$ is also holomorphic near $(T,X,u,\tau,\xi) = (0,0,0,0,0)$.

Now, we consider the following equation:

$$\begin{aligned} (4.2) \qquad w(T,X) &= A(X)T^K + C\sum_{i,j=1}^{d} XB_{ij}(X)w(T,X) \\ &\quad + CXG(X)w(T,X) + C\sum_{k=1}^{n} X^2\Phi_k(x)(t,x)S(w)(T,X) \\ &\quad + F_{K+1}\left(T,X,Cw,\left\{\frac{Cw}{T}\right\},\{CS(w)\}\right), \end{aligned}$$

where C is a positive constant appeared in Lemma 1.

Let $w(T,X) = \sum_{L\geq K, M\geq 0} w_{LM}(T,X)$ be the formal solution of (4.2). By the construction of (4.2), we can easily check that $U(t,x) \ll w(T,X)$ by the next lemma.

Lemma 2. *For two formal power series $U(t,x)$ and $w(T,X)$ satisfying*

$$U(t,x) = \sum_{L\geq K, M\geq 0} U_{LM}(t,x) \ll w(T,X) = \sum_{L\geq K, M\geq 0} w_{LM}T^LX^M,$$

the following majorant relations hold:

(1) $P^{-1}U(t,x) \ll Cw(T,X)$,

(2) $t_i\partial_{t_j}P^{-1}U(t,x) \ll Cw(T,X)$,

(3) $\partial_{t_j}P^{-1}U(t,x) \ll \dfrac{Cw(T,X)}{T}$,

(4) $\partial_{x_k}P^{-1}U(t,x) \ll CS(w)(T,X)$.

Proof. By using Lemma 1, we can prove this lemma easily. First, (1) is proved as follows:

$$P^{-1}U(t,x) = \sum_{L\geq K, M\geq 0} P^{-1}U_{LM}(t,x) \ll \sum_{L\geq K, M\geq 0} \frac{C}{L+M} w_{LM}T^LX^M \ll Cw(T,X).$$

Secondly, (2) and (3) is proved as follows:

$$\begin{aligned} t_i\partial_{t_j}P^{-1}U(t,x) &= \sum_{L\geq K, M\geq 0} t_i\partial_{t_j}P^{-1}U_{LM}(t,x) \\ &\ll \sum_{L\geq K, M\geq 0} \frac{CL}{L+M}\, w_{LM}T^LX^M \ll Cw(T,X); \end{aligned}$$

$$\begin{aligned} \partial_{t_j}P^{-1}U(t,x) &= \sum_{L\geq K, M\geq 0} \partial_{t_j}P^{-1}U_{LM}(t,x) \\ &\ll \sum_{L\geq K, M\geq 0} \frac{CL}{L+M}\, w_{LM}T^{L-1}X^M \ll \frac{Cw(T,X)}{T}. \end{aligned}$$

Finally, (4) is proved as follows:

$$\begin{aligned} \partial_{x_k}P^{-1}U(t,x) &= \sum_{L\geq K, M\geq 0} \partial_{x_k}P^{-1}U_{LM}(t,x) \\ &\ll \sum_{L\geq K, M\geq 1} \frac{CM}{L+M}\, w_{LM}T^LX^{M-1} \ll CS(w)(T,X). \end{aligned}$$

This completes the proof. □

Since $w(T,X) \gg 0$, we have

$$XS(w)(T,X) = w(T,X) - w(T,0) \ll w(T,X). \tag{4.3}$$

Let us consider the following equation:

$$\begin{aligned} v(T,X) &= A(X)T^K + CXh(X)v(T,X) \\ &\quad + F_{K+1}\left(T, X, Cv, \left\{\frac{Cv}{T}\right\}, \{CS(v)\}\right), \end{aligned} \tag{4.4}$$

with $v(T,X) = O(T^K)$, where $h(X) = \sum_{i,j=1}^{d} B_{ij}(X) + G(X) + \sum_{k=1}^{n} \Phi_k(X)$. Then the following majorant relation is obvious:

$$w(T,X) \ll v(T,X).$$

We put $y(T,X) = v(T,X)/T$ as a new unknown function. By substituting this into (4.4), we see that $y(T,X)$ satisfies

$$\begin{aligned} y(T,X) &= A(X)T^{K-1} + CXh(X)y(T,X) \\ &\quad + \frac{1}{T}F_{K+1}\left(T, X, CTy, \{Cy\}, \{CTS(y)\}\right), \end{aligned} \tag{4.5}$$

with $y(T,X) = O(T^{K-1})$.

We decompose the formal solution $y(T,X)$ as follows:

$$y(T,X) = y_1(X)T^{K-1} + y_2(X)T^K + T^Kz(T,X).$$

We remark that $y_1(X)$ and $y_2(X)$ are holomorphic functions in a neighbourhood of $X=0$. Indeed, $y_1(X)$ and $y_2(X)$ are given by

$$y_1(X) = \frac{A(X)}{1-CXh(X)},$$

$$y_2(X) = \frac{1}{1-CXh(X)} \sum_{|p|+Kq+(K-1)|r|+K|s|=K+1} F_{pqrs}(X)\{Cy_1(X)\}^{q+|r|}\{CS(y_1)(X)\}^{|s|}.$$

These are holomorphic functions in a neighbourhood of $X=0$.

In this case, $z(T,X)$ satisfies the following equation:

$$(4.6)\qquad \begin{cases} z(T,X) = CXh(X)z(T,X) + H(T,X,Tz(T,X),TS(z)(T,X)), \\ z(0,X) \equiv 0, \end{cases}$$

where

$$\begin{aligned} H(T,X,\eta_1,\eta_2) \;=\; & \frac{1}{T^{K+1}}\Big[F_{K+1}(T,X,Cy_1(X)T^K + Cy_2(X)T^{K+1} + CT^K\eta_1, \\ & \{Cy_1(X)T^{K-1} + Cy_2(X)T^K + CT^{K-1}\eta_1\}, \\ & \{CS(y_1)(X)T^K + CS(y_2)(X)T^{K+1} + CT^K\eta_2\})\Big] \\ & - \sum_{|p|+Kq+(K-1)|r|+K|s|=K+1} F_{pqrs}(X)(Cy_1(X))^{q+|r|}(CS(y_1)(X))^{|s|}. \end{aligned}$$

Remark 6. The order of zeros in T variable of $H(T,X,CTz(T,X),CTS(z)(T,X))$ is greater than or equal to 1. □

In order to prove the convergence of $z(T,X)$, it is sufficient to show the following:

Lemma 3. *There exists a small positive constant $\varepsilon > 0$ such that $z_\varepsilon(\rho) = z(\varepsilon\rho,\rho)$ is convergent in a neighbourhood of $\rho = 0$.*

Proof. We substitute $T=\varepsilon\rho$ and $X=\rho$ into (4.6) and by using the relation (4.3), we have

$$\rho S(z)(\varepsilon\rho,\rho) \ll z_\varepsilon(\rho).$$

By this relation, the following majorant relation also holds,

$$TS(z)(T,X)|_{T=\varepsilon\rho, X=\rho} = \varepsilon\rho S(z)(\varepsilon\rho,\rho) \ll \varepsilon z_\varepsilon(\rho).$$

Here we consider

$$(4.7)\qquad \psi(\rho) = C\rho h(\rho)\psi(\rho) + H(\varepsilon\rho,\rho,\varepsilon\rho\psi(\rho),\varepsilon\psi(\rho)).$$

In the right hand side of (4.7), we decompose $H(\varepsilon\rho, \rho, \varepsilon\rho\psi(\rho), \varepsilon\psi(\rho))$ into the term of $\psi(\rho)$ and otherwise as follows:

$$H(\varepsilon\rho, \rho, \varepsilon\rho\psi(\rho), \varepsilon\psi(\rho)) = \varepsilon\frac{\partial H}{\partial \eta_2}(0,0,0,0)\psi(\rho) + \widetilde{H}(\varepsilon\rho, \rho, \varepsilon\rho\psi(\rho), \varepsilon\psi(\rho)).$$

We remark that the following fact holds:

$$\frac{\partial \widetilde{H}}{\partial \psi}(\varepsilon\rho, \rho, \varepsilon\rho\psi, \varepsilon\psi)\bigg|_{\rho=0,\psi=0} = 0.$$

We put $(\partial H/\partial \eta_2)(0,0,0,0) = K_0 \geq 0$, then (4.7) is rewritten by

$$(4.8)\qquad (1-\varepsilon K_0)\psi(\rho) = C\rho h(\rho)\psi(\rho) + \widetilde{H}(\varepsilon\rho, \rho, \varepsilon\rho\psi(\rho), \varepsilon\psi(\rho)).$$

We choose $\varepsilon > 0$ with $1-\varepsilon K_0 > 0$. Then by using the implicit function theorem, we can see that (5.8) has a unique holomorphic solution $\psi(\rho)$ with $\psi(0) = 0$ in a neighbourhood of $\rho = 0$. Moreover we can see $z_\varepsilon(\rho) \ll \psi(\rho)$.

Thus we complete the proof of Lemma 3. □

5. Solvability of the System (1.9)

In this section, we give a sufficient condition for the formal solution of the system (1.9) to be convergent. Recall that (1.9) is

$$(1.9)\qquad \frac{\partial f}{\partial t_i}(0,x,0,\{\varphi_j(x)\},0) + \frac{\partial f}{\partial u}(0,x,0,\{\varphi_j(x)\},0)\varphi_i(x) + \sum_{k=1}^{n}\frac{\partial f}{\partial \xi_k}(0,x,0,\{\varphi_j(x)\},0)\frac{\partial \varphi_i(x)}{\partial x_k} = 0, \quad i = 1,2,\dots,d.$$

By Assumption 4 of Theorem 2, the condition

$$\frac{\partial f}{\partial \xi_k}(0,0,0,\{\varphi_j(0)\},0) = 0, \qquad k = 1,2,\dots,n$$

was assumed.

Let $\varphi(x) = {}^t(\varphi_1(x),\dots,\varphi_d(x))$ be the unknown functions. We put $\varphi(0) = {}^t(\varphi_1^0,\dots,\varphi_d^0) \in \mathbf{C}^d$ as the constant term of $\varphi(x)$. We substitute $\varphi_j(x) = \varphi_j^0 + \psi_j(x)$ into the system (1.9), and by restricting at $x = 0$, we see that $\{\varphi_j^0\}$ satisfies the following system:

$$(5.1)\qquad \frac{\partial f}{\partial t_i}(0,0,0,\{\varphi_j^0\},0) + \frac{\partial f}{\partial u}(0,0,0,\{\varphi_j^0\},0)\varphi_i^0 = 0, \qquad i = 1,2,\dots,d.$$

This system has some roots by the assumption of the existence of a formal solution, and we fix such $\{\varphi_j^0\}$.

For such fixed $\{\varphi_j^0\}$, we see that $\{\psi_j(x)\}$ satisfies the system of the followir

$$\begin{aligned}(5.2)\qquad &\sum_{k=1}^{n}\sum_{l=1}^{n}\frac{\partial^2 f}{\partial\xi_k\partial x_l}(0,0,0,\{\varphi_j^0\},0)x_l\frac{\partial\psi_i}{\partial x_k}(x)\\ &+\sum_{k=1}^{n}\sum_{p=1}^{d}\frac{\partial^2 f}{\partial\xi_k\partial\tau_p}(0,0,0,\{\varphi_j^0\},0)\psi_p(x)\frac{\partial\psi_i}{\partial x_k}(x)\\ &+\frac{\partial f}{\partial u}(0,0,0,\{\varphi_j^0\},0)\psi_i(x)\\ &+\sum_{p=1}^{d}\left\{\frac{\partial^2 f}{\partial t_i\partial\tau_p}(0,0,0,\{\varphi_j^0\},0)+\frac{\partial^2 f}{\partial u\partial\tau_p}(0,0,0,\{\varphi_j^0\},0)\varphi_i^0\right\}\psi_p(x)\\ &+\sum_{l=1}^{n}\left\{\frac{\partial^2 f}{\partial t_i\partial x_l}(0,0,0,\{\varphi_j^0\},0)+\frac{\partial^2 f}{\partial u\partial x_l}(0,0,0,\{\varphi_j^0\},0)\varphi_i^0\right\}x_l\\ &=(\text{degree in } x \text{ is greater than or equal to 2}),\quad i=1,2,\dots,d.\end{aligned}$$

This system is written as follows for simplicity,

$$\begin{aligned}(5.3)\qquad &\sum_{k=1}^{n}\sum_{l=1}^{n}a_{kl}x_l\frac{\partial\psi_i}{\partial x_k}(x)+\sum_{k=1}^{n}\sum_{p=1}^{d}b_{kp}\psi_p(x)\frac{\partial\psi_i}{\partial x_k}(x)\\ &+c\psi_i(x)+\sum_{p=1}^{d}d_{ip}\psi_p(x)+\sum_{l=1}^{n}e_{il}x_l\\ &=(\text{degree in } x \text{ is greater than or equal to 2}),\quad i=1,2,\dots,d,\end{aligned}$$

where

$$a_{kl}:=\frac{\partial^2 f}{\partial\xi_k\partial x_l}(0,0,0,\{\varphi_j^0\},0),\ b_{kp}:=\frac{\partial^2 f}{\partial\xi_k\partial\tau_p}(0,0,0,\{\varphi_j^0\},0),$$

$$c:=\frac{\partial f}{\partial u}(0,0,0,\{\varphi_j^0\},0),$$

$$d_{ip}:=\frac{\partial^2 f}{\partial t_i\partial\tau_p}(0,0,0,\{\varphi_j^0\},0)+\frac{\partial^2 f}{\partial u\partial\tau_p}(0,0,0,\{\varphi_j^0\},0)\varphi_i^0,$$

$$e_{il}:=\frac{\partial^2 f}{\partial t_i\partial x_l}(0,0,0,\{\varphi_j^0\},0)+\frac{\partial^2 f}{\partial u\partial x_l}(0,0,0,\{\varphi_j^0\},0)\varphi_i^0.$$

Here we decompose $\psi_i(x)$ into $\psi_i(x) = \widetilde{\psi_i}(x) + \eta_i(x)$ ($\widetilde{\psi_i}(x) = \sum_{k=1}^{n} \psi_{ik} x_k$, $\eta_i(x) = O(|x|^2)$). We substitute this into the system (5.3) and obtain

$$
\begin{aligned}
(5.4) \qquad & \sum_{k=1}^{n}\sum_{l=1}^{n} a_{kl} x_l \left(\frac{\partial \widetilde{\psi_i}}{\partial x_k}(x) + \frac{\partial \eta_i}{\partial x_k}(x) \right) \\
& + \sum_{k=1}^{n}\sum_{p=1}^{d} b_{kp} (\widetilde{\psi_p}(x) + \eta_p(x)) \left(\frac{\partial \widetilde{\psi_i}}{\partial x_k}(x) + \frac{\partial \eta_i}{\partial x_k}(x) \right) \\
& + c(\widetilde{\psi_i}(x) + \eta_i(x)) + \sum_{p=1}^{d} d_{ip} (\widetilde{\psi_p}(x) + \eta_p(x)) + \sum_{l=1}^{n} e_{il} x_l \\
= \; & (\text{degree in } x \text{ is greater than or equal to } 2), \quad i = 1, 2, \ldots, d.
\end{aligned}
$$

By picking up the degree 1 part on the both sides, we see that $\{\widetilde{\psi_i}(x)\}$ satisfy the following system:

$$
\begin{aligned}
(5.5) \qquad & \sum_{k=1}^{n}\sum_{l=1}^{n} a_{kl} x_l \frac{\partial \widetilde{\psi_i}}{\partial x_k}(x) + \sum_{k=1}^{n}\sum_{p=1}^{d} b_{kp} \widetilde{\psi_p}(x) \frac{\partial \widetilde{\psi_i}}{\partial x_k}(x) \\
& + c\widetilde{\psi_i}(x) + \sum_{p=1}^{d} d_{ip} \widetilde{\psi_p}(x) + \sum_{l=1}^{n} e_{il} x_l = 0,
\end{aligned}
$$

for $i = 1, 2, \ldots, d$.

By the existence of a formal solution, (5.5) has some solutions $\{\widetilde{\psi_i}(x)\}$ of linear functions, and we fix such $\{\widetilde{\psi_i}(x)\}$.

For fixed $\{\varphi_i^0\}$ and $\{\widetilde{\psi_i}(x)\}$, we see that $\{\eta_i(x)\}$ satisfy the following system:

$$
\begin{aligned}
(5.6) \qquad & \sum_{k=1}^{n}\sum_{l=1}^{n} \left(a_{kl} + \sum_{p=1}^{d} b_{kp} \psi_{pl} \right) x_l \frac{\partial \eta_i}{\partial x_k}(x) + c\eta_i(x) + \sum_{p=1}^{d} \left(d_{ip} + \sum_{k=1}^{n} b_{kp} \psi_{ik} \right) \eta_p(x) \\
& = (\text{degree in } x \text{ is greater than or equal to } 2.), \qquad i = 1, 2, \ldots, d.
\end{aligned}
$$

We remark that the degree 2 part in the right hand side of this system does not include $\{\eta_i(x)\}$.

The following theorem holds:

Theorem 3. *Let* $(A_{kl})_{k,l=1,2,\ldots,n}$ *be a matrix defined by*

$$
(A_{kl})_{k,l=1,2,\ldots,n} = \left(a_{kl} + \sum_{p=1}^{d} b_{kp} \psi_{pl} \right)_{k,l=1,2,\ldots,n}.
$$

Let $\{\kappa_k\}_{k=1}^n$ be the eigenvalues of $(A_{kl})_{k,l=1,2,\dots,n}$. If there exists a positive constant σ_0 such that the condition

$$\left|\sum_{k=1}^n \kappa_k m_k\right| \geq \sigma_0|m|, \quad \textit{(Poincaré condition)}$$

holds for all $m = (m_1, \dots, m_n) \in (\mathbf{Z}_{\geq 0})^n$ with $|m| \geq 2$, then the formal solution of the system (1.9) is convergent in a neighbourhood of the origin.

Remark 7. Let $(B_{ip})_{i,p=1,2,\dots,d}$ be a matrix defined by

$$(B_{ip})_{i,p=1,2,\dots,d} = \left(d_{ip} + \sum_{k=1}^n b_{kp}\psi_{ik}\right)_{i,p=1,2,\dots,d},$$

and let $\{\omega_j\}_{j=1}^d$ be the eigenvalues of $(B_{ip})_{i,p=1,2,\dots,d}$.

By the same argument in Remark 1, we have

$$\left|\sum_{k=1}^n \kappa_k m_k + c + \omega_j\right| \geq \sigma|m|, \text{ by some } \sigma > 0, \text{ and } j = 1, 2, \dots, d, \tag{5.7}$$

for large m, which will be used in the proof. □

6. Proof of Theorem 3

The proof of Theorem 3 is the same method of the proof of Theorem 1 in case that the unknown function is a vector values. However, there are some difference in the detail. Therefore, we introduce only the outline of the proof of Theorem 3 in this section.

Step 1. By taking a linear transformation of the independent variables and a linear transformation of the unknown functions, (5.6) is reduced to the following form:

$$\begin{aligned}(\Lambda + \Delta + \mathbf{B})\begin{pmatrix} w_1(x) \\ \vdots \\ w_d(x)\end{pmatrix} &:= \left\{\begin{pmatrix}\Lambda_1 & & \\ & \ddots & \\ & & \Lambda_d\end{pmatrix} + \begin{pmatrix}\Delta & & \\ & \ddots & \\ & & \Delta\end{pmatrix} + \mathbf{B}\right\}\begin{pmatrix} w_1(x) \\ \vdots \\ w_d(x)\end{pmatrix} \\ &= \begin{pmatrix}\sum_{|m|=2} a_{1,m}x^m + g_{3,1}(x, w(x), \partial_x w(x)) \\ \vdots \\ \sum_{|m|=2} a_{d,m}x^m + g_{3,d}(x, w(x), \partial_x w(x))\end{pmatrix},\end{aligned} \tag{6.1}$$

where $w_j(x)$ $(j = 1, 2, \dots, d)$ denote new unknown functions after linear transformations and

$$\Lambda_j = \sum_{k=1}^{n} \kappa_k x_k \partial_{x_k} + c + \omega_j, \quad \Delta = \sum_{k=1}^{n-1} \varepsilon_k x_k \partial_{x_{k+1}}, \quad \mathbf{B} = \begin{pmatrix} 0 & e_1 & & \\ & \ddots & \ddots & \\ & & \ddots & e_{d-1} \\ & & & 0 \end{pmatrix},$$

where ε_j and e_j denote the nilpotent components of the Jordan canonical forms of the matrices (A_{kl}) and (B_{ip}), respectively, and

$$g_{3,i}(x, \eta, \zeta) = \sum_{|\alpha|+2|\beta|+|\gamma|\geq 3} g^{(i)}_{\alpha\beta\gamma} x^\alpha \eta^\beta \zeta^\gamma.$$

Step 2. We define $\mathbf{C}[x]_M$ by $\mathbf{C}[x]_M = \{\sum_{|m|=M} u_m x^m \ ; \ u_m \in \mathbf{C}\}$, and define a norm of $u(x) = {}^t(u_1(x), \dots, u_d(x)) \in (\mathbf{C}[x]_M)^d$ by

$$||u|| := \inf\{C > 0 \ ; \ u_i(x) \ll C(x_1 + \cdots + x_n)^M, \ i = 1, 2, \dots, d\}.$$

By the same argument in the proof of Lemma 1 and by Remark 7, we can prove the same results of Lemma 1 for the operator $\Lambda + \Delta + \mathbf{B}$.

Step 3. By the same method in the previous sections, we can construct a majorant equation whose formal solution is a majorant function of the all unknown functions of the system. Finally, by the implicit function theorem, we prove the convergence of the formal solution of the majorant equation.

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