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Identification problems for coupled damped sine-Gordon systems

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1 Introduction

The damped sine-Gordon equation described by

$$\frac{\partial^2 y}{\partial t}+\alpha\frac{\partial y}{\partial t}-\beta\Delta y+\gamma\sin y=f \quad \text{in} \quad (0,T)\times\Omega \tag{1.1}$$

is known as the dynamics of Josephson junctions driven by a current source f, where α,β,γ are physical constants. We refer to see the reference [9] for the physical modeling. In T[10], we can find the coupled damped sine-Gordon equations described by

$$\begin{cases} \dfrac{\partial^2 y_1}{\partial t}+\dfrac{\partial y_1}{\partial t}-\Delta y_1+\sin y_1+k(y_1-y_2)=f_1 \quad \text{in} \quad (0,T)\times\Omega, \\ \dfrac{\partial^2 y_2}{\partial t}+\dfrac{\partial y_2}{\partial t}-\Delta y_2+\sin y_2+k(y_2-y_1)=f_2 \quad \text{in} \quad (0,T)\times\Omega \end{cases} \tag{1.2}$$

and

$$\begin{cases} \dfrac{\partial^2 y_1}{\partial t}+\dfrac{\partial y_1}{\partial t}-\Delta y_1+\sin(y_1+y_2)=f_1 \quad \text{in} \quad (0,T)\times\Omega, \\ \dfrac{\partial^2 y_2}{\partial t}+\dfrac{\partial y_2}{\partial t}-\Delta y_2+\sin(y_1-y_2)=f_2 \quad \text{in} \quad (0,T)\times\Omega, \end{cases} \tag{1.3}$$

where k is a physical constant.

These equations (1.1)-(1.3) has become the target of their researches by many scientists for a long time. Indeed, we could find the studies as follows. In T[10], he has extensively studied the problems with respect to stability and existence of attractors. In BFL[2], L[3] and M[6], they verified numerically that these equations causes the special choice of the initial values and the forcing function to chaotic behaviors. The optimal control problems of regarding forcing functions as control variables were studied in HN[7] and NH[8]. Of course, there are many studies involved with the identification problems for linear systems (See A[1]). However we could not find the theoretical identification problems of the physical parameters being studied for (1.1)-(1.3). Hence in this paper we are devoted to study the identification problems of the coupled damped sine-Gordon equations described by

$$\begin{cases} \dfrac{\partial^2 y_1}{\partial t^2} + \alpha_{11}\dfrac{\partial y_1}{\partial t} + \alpha_{12}\dfrac{\partial y_2}{\partial t} - \beta_{11}\Delta y_1 + \gamma_{11}\sin(\delta_{11}y_1 + \delta_{12}y_2) \\ \qquad +k_{11}y_1 + k_{12}y_2 = f_1 \ \text{ in } \ (0,T)\times\Omega, \\ \dfrac{\partial^2 y_2}{\partial t^2} + \alpha_{21}\dfrac{\partial y_1}{\partial t} + \alpha_{22}\dfrac{\partial y_2}{\partial t} - \beta_{22}\Delta y_2 + \gamma_{22}\sin(\delta_{21}y_1 + \delta_{22}y_2) \\ \qquad +k_{21}y_1 + k_{22}y_2 = f_2 \ \text{ in } \ (0,T)\times\Omega, \end{cases} \tag{1.4}$$

where physical parameters $\beta_{ii} > 0, \alpha_{ij}, \gamma_{ii}, \delta_{ij}, k_{ij}$ are constants. Clearly (1.4) is a generalized form of (1.2) and (1.3). In our identification problems for (1.4) the parameters $\alpha_{ij}, \gamma_{ii}, \delta_{ij}$ and k_{ij} except β_{ii} are assumed to be unknown, and then we will deduce the necessary conditions on the optimal parameters minimizing a quadratic cost functional defined on an admissible set of parameters in the frame of the optimal control problems studied by L[4]. Whenever this method is introduced, we should estimate the first variation of the solution map between parameters and solution of (1.4), but sometimes it is not easy task. In particular, it is more difficult for the case where the diffusion parameters β_{ii} are unknown, and so let us study this case next time.

For studying the identification problems for (1.4) we need the fundamental results such as existence, uniqueness and regularity of weak solutions for (1.4) and we shall use those studied by NH[8]. We hope to refer to T[10] for more strong solutions of (1.2) and (1.3).

This paper is composed of three sections. In section 2 we explain the fundamental results of solutions for the coupled damped sine-Gordon equations. In section 3 we study existence and necessary conditions for the optimal parameters. Moreover we give an example deducing the bang-bang conditions from the necessary conditions on the optimal parameters.

2 Preliminaries

Let Ω be an open bounded set of the n dimensional Euclidean space R^n with a piecewise smooth boundary $\Gamma = \partial\Omega$. Set $Q = (0,T)\times\Omega$ and $\Sigma = (0,T)\times\Gamma$. We consider the coupled damped sine-Gordon equations described by

$$\begin{cases} \dfrac{\partial^2 y_1}{\partial t^2} + \alpha_{11}\dfrac{\partial y_1}{\partial t} + \alpha_{12}\dfrac{\partial y_2}{\partial t} - \beta_{11}\Delta y_1 + \gamma_{11}\sin(\delta_{11}y_1 + \delta_{12}y_2) \\ \qquad +k_{11}y_1 + k_{12}y_2 = f_1 \ \text{ in } \ Q, \\ \dfrac{\partial^2 y_2}{\partial t^2} + \alpha_{21}\dfrac{\partial y_1}{\partial t} + \alpha_{22}\dfrac{\partial y_2}{\partial t} - \beta_{22}\Delta y_2 + \gamma_{22}\sin(\delta_{21}y_1 + \delta_{22}y_2) \\ \qquad +k_{21}y_1 + k_{22}y_2 = f_2 \ \text{ in } \ Q \end{cases} \tag{2.1}$$

with the homogeneous Dirichlet boundary conditions

$$y_i = 0 \ \text{ on } \ \Sigma, \ \ i = 1,2, \tag{2.2}$$

where $\beta_{ii} > 0, \alpha_{ij}, \gamma_{ii}, \delta_{ij}, k_{ij} \in (-\infty,\infty), i,j = 1,2$ and Δ is the Laplacian and $f_i, i = 1,2$ are given functions. The initial values to (2.1) are given by

$$y_i(0,x) = y_0^i(x) \ \text{ in } \ \Omega \ \text{ and } \ \frac{\partial y_i}{\partial t}(0,x) = y_1^i(x) \ \text{ in } \ \Omega, \ \ i = 1,2. \tag{2.3}$$

For setting the partial differential equations (2.1) - (2.3) the ordinary ones, we introduce two Hilbert spaces H and V by $H = L^2(\Omega)$ and $V = H_0^1(\Omega)$, respectively. We endow these spaces with inner products and norms as follows:

$$(\psi, \phi) = \int_\Omega \psi(x)\phi(x)dx, \quad |\psi| = (\psi, \psi)^{1/2}, \quad \forall \ \phi, \psi \in L^2(\Omega); \tag{2.4}$$

$$(\!(\psi, \phi)\!) = \sum_{i=1}^{n} \int_\Omega \frac{\partial}{\partial x_i}\psi(x)\frac{\partial}{\partial x_i}\phi(x)dx, \quad \|\psi\| = (\!(\psi, \psi)\!)^{1/2}, \quad \forall \ \phi, \psi \in H_0^1(\Omega). \tag{2.5}$$

Then the pair (V, H) is a Gelfand triple space with the notation, $V \hookrightarrow H \equiv H' \hookrightarrow V'$ and $V' = H^{-1}(\Omega)$, which means that embeddings $V \subset H$ and $H \subset V'$ are continuous, dense and compact. By $\langle \cdot, \cdot \rangle$ denotes the dual pairing between V' and V.

For a variational formulation let us introduce a bilinear form

$$a(\phi, \varphi) = \int_\Omega \nabla\phi \cdot \nabla\varphi dx = (\!(\phi, \varphi)\!), \quad \forall \phi, \varphi \in H_0^1(\Omega).$$

This bilinear form $a(\cdot, \cdot)$ is symmetric, bounded on $V \times V$ and coercive on V, i.e.,

$$a(\phi, \phi) \geq \|\phi\|^2, \quad \forall \phi \in H_0^1(\Omega). \tag{2.6}$$

By the boundedness of $a(\cdot, \cdot)$ we can define the bounded linear operator $A \in \mathcal{L}(V, V')$, the space of bounded linear operators of V into V', by the relation $a(\phi, \psi) = \langle A\phi, \psi \rangle$. The operator A is an isomorphism from V onto V' and has a dense domain $D(A)$ in H, but it is not bounded in H.

With the operator A the equations (2.1)-(2.3) are written by the evolution forms in H as follows:

$$\begin{cases} \dfrac{d^2y_1}{dt^2} + \alpha_{11}\dfrac{dy_1}{dt} + \alpha_{12}\dfrac{dy_2}{dt} + \beta_{11}Ay_1 + \gamma_{11}\sin(\delta_{11}y_1 + \delta_{12}y_2) \\ \qquad +k_{11}y_1 + k_{12}y_2 = f_1 \ \text{ in } \ (0, T), \\ \dfrac{d^2y_2}{dt^2} + \alpha_{21}\dfrac{dy_1}{dt} + \alpha_{22}\dfrac{dy_2}{dt} + \beta_{22}Ay_2 + \gamma_{22}\sin(\delta_{21}y_1 + \delta_{22}y_2) \\ \qquad +k_{21}y_1 + k_{22}y_2 = f_2 \ \text{ in } \ (0, T), \\ y_i(0) = y_0^i \in V, \ \dfrac{dy_i}{dt}(0) = y_1^i \in H, \ \ i = 1, 2. \end{cases} \tag{2.7}$$

For defining (2.7) as a vectored evolution equation, we introduce the product Hilbert spaces $\mathcal{V} = V \times V$ and $\mathcal{H} = H \times H$ with the inner products defined by

$$(\!(\boldsymbol{\phi}, \boldsymbol{\psi})\!) = (\!(\phi_1, \psi_1)\!) + (\!(\phi_2, \psi_2)\!), \quad \forall \boldsymbol{\phi} = [\phi_1, \phi_2]^t, \quad \forall \boldsymbol{\psi} = [\psi_1, \psi_2]^t \in \mathcal{V},$$
$$(\boldsymbol{\phi}, \boldsymbol{\psi}) = (\phi_1, \psi_1) + (\phi_2, \psi_2), \quad \forall \boldsymbol{\phi} = [\phi_1, \phi_2]^t, \quad \forall \boldsymbol{\psi} = [\psi_1, \psi_2]^t \in \mathcal{H},$$

respectively. Here by $[\cdot, \cdot]^t$ denotes the transpose of the 1×2 vector $[\cdot, \cdot]$. Then the dual space of $\mathcal{V}$ is given by $\mathcal{V}' = V' \times V'$ and the dual pairing between $\mathcal{V}'$ and $\mathcal{V}$ is given by

$$\langle \boldsymbol{\phi}, \boldsymbol{\psi} \rangle = \langle \phi_1, \psi_1 \rangle + \langle \phi_2, \psi_2 \rangle, \quad \forall \boldsymbol{\phi} = [\phi_1, \phi_2]^t \in \mathcal{V}', \ \forall \boldsymbol{\psi} = [\psi_1, \psi_2]^t \in \mathcal{V}.$$

Since $V \hookrightarrow H \hookrightarrow V'$, the pair $(\mathcal{V}, \mathcal{H})$ is also a Gelfand triple space with the notation $\mathcal{V} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{V}'$. The norms of $\mathcal{V}$ and $\mathcal{H}$ are denoted simply by $\|\boldsymbol{\psi}\|$ and $|\boldsymbol{\psi}|$, respectively.

We denote by $M_2(K)$ the set of 2×2 matrices on K, $M_2^d(K)$ the set of diagonal matrices of $M_2(K)$. We set $R = (-\infty, +\infty)$, $R^+ = [0, +\infty)$ and $\dot{R}^+ = (0, +\infty)$. Let us define a norm on $M_2(R)$ as follows:

$$|\boldsymbol{\alpha}| = \sum_{i,j=1,2} |\alpha_{ij}| \text{ for } \boldsymbol{\alpha} = (\alpha_{ij}) \in M_2(R).$$

Then it is obvious that $M_2(R^+), M_2^d(R^+), M_2^d(R)$ are closed subsets of $M_2(R)$ and for all $\boldsymbol{\alpha} \in M_2(R)$ $\|\boldsymbol{\alpha\phi}\| \le |\boldsymbol{\alpha}|\ \|\boldsymbol{\phi}\|$, $\forall \boldsymbol{\phi} \in \mathcal{V}$, $|\boldsymbol{\alpha\phi}| \le |\boldsymbol{\alpha}|\ |\boldsymbol{\phi}|$, $\forall \boldsymbol{\phi} \in \mathcal{H}$ and $\|\boldsymbol{\alpha\phi}\|_{\mathcal{V}'} \le |\boldsymbol{\alpha}|\ \|\boldsymbol{\phi}\|_{\mathcal{V}'}$, $\forall \boldsymbol{\phi} \in \mathcal{V}'$.

As using notations of matrices and vectors we obtain the Cauchy problem in $\mathcal{H}$ for (2.7) :

$$\begin{cases} \mathbf{y}'' + \boldsymbol{\alpha}\mathbf{y}' + \boldsymbol{\beta}\mathbf{A}\mathbf{y} + \mathbf{k}\mathbf{y} + \boldsymbol{\gamma}\sin\boldsymbol{\delta}\mathbf{y} = \mathbf{f} \text{ in } (0,T), \\ \mathbf{y}(0) = \mathbf{y}_0, \quad \mathbf{y}'(0) = \mathbf{y}_1, \end{cases} \tag{2.8}$$

where $\boldsymbol{\alpha}, \boldsymbol{\delta}, \mathbf{k} \in M_2(R), \boldsymbol{\beta} \in M_2^d(\dot{R}^+), \boldsymbol{\gamma} \in M_2^d(R)$, $\mathbf{A}\mathbf{y} = [Ay_1, Ay_2]^t$ and $\sin\boldsymbol{\phi} = [\sin\phi_1, \sin\phi_2]^t$.

Let us define the solution Hilbert space $\mathbf{W}(0,T)$ by

$$\mathbf{W}(0,T) = \{\mathbf{g} | \mathbf{g} \in L^2(0,T;\mathcal{V}), \mathbf{g}' \in L^2(0,T;\mathcal{H}), \mathbf{g}'' \in L^2(0,T;\mathcal{V}')\}$$

with the inner product

$$(\mathbf{f},\mathbf{g})_{\mathbf{W}(0,T)} = \int_0^T ((\!(\mathbf{f}(t),\mathbf{g}(t))\!) + (\mathbf{f}'(t),\mathbf{g}'(t)) + (\mathbf{f}''(t),\mathbf{g}''(t))_{\mathcal{V}'})\, dt, \quad \mathbf{f},\mathbf{g} \in \mathbf{W}(0,T),$$

where $(\cdot,\cdot)_{\mathcal{V}'}$ is the inner product of $\mathcal{V}'$.

Now we give the definition of weak solutions of the coupled damped sine-Gordon equations.

Definition 2.1 A function $\mathbf{y}$ is said to be a weak solution of (2.8) if $\mathbf{y} \in \mathbf{W}(0,T)$ and $\mathbf{y}$ satisfies

$$\begin{aligned} &\langle \mathbf{y}''(\cdot), \boldsymbol{\phi}\rangle + (\boldsymbol{\alpha}\mathbf{y}'(\cdot), \boldsymbol{\phi}) + (\!(\boldsymbol{\beta}\mathbf{y}(\cdot), \boldsymbol{\phi})\!) + (\mathbf{k}\mathbf{y}(\cdot), \boldsymbol{\phi}) + (\boldsymbol{\gamma}\sin\boldsymbol{\delta}\mathbf{y}(\cdot), \boldsymbol{\phi}) = (\mathbf{f}(\cdot), \boldsymbol{\phi}) \\ &\qquad\qquad \text{for all } \boldsymbol{\phi} \in \mathcal{V} \text{ in the sense of } \mathcal{D}'(0,T), \\ &\mathbf{y}(0) = \mathbf{y}_0, \quad \mathbf{y}'(0) = \mathbf{y}_1, \end{aligned} \tag{2.9}$$

where $\mathcal{D}'(0,T)$ denotes the space of distributions on $(0,T)$.

For the existence and uniqueness of weak solutions for (2.8), we can state the following theorem.

Theorem 2.1 Let $\boldsymbol{\alpha}, \boldsymbol{\delta}, \mathbf{k} \in M_2(R), \boldsymbol{\beta} \in M_2^d(\dot{R}^+), \boldsymbol{\gamma} \in M_2^d(R)$ and $\mathbf{f}$, $\mathbf{y}_0$, $\mathbf{y}_1$ be given satisfying

$$\mathbf{f} \in L^2(0,T;\mathcal{H}), \quad \mathbf{y}_0 \in \mathcal{V}, \quad \mathbf{y}_1 \in \mathcal{H}. \tag{2.10}$$

Then the problem (2.8) has a unique weak solution $\mathbf{y}$ in $\mathbf{W}(0,T)$ and $\mathbf{y}$ has regularities

$$\mathbf{y} \in C([0,T];\mathcal{V}), \quad \mathbf{y}' \in C([0,T];\mathcal{H}). \tag{2.11}$$

Furthermore, we have the energy inequality

$$|\mathbf{y}'(t)|^2 + \|\mathbf{y}(t)\|^2 \le C(\|\mathbf{y}_0\|^2 + |\mathbf{y}_1|^2 + \|\mathbf{f}\|^2_{L^2(0,T;\mathcal{H})}), \quad t \in [0,T], \tag{2.12}$$

where C is a constant depending continuously on $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta}$ and $\mathbf{k}$, and $\sqrt{\boldsymbol{\beta}} \in M_2^d(\dot{R}^+)$ with elements $\sqrt{\beta_{ii}}, i = 1,2$.

We remark that for $\gamma = 0$ and $k = k(\cdot)$ in (2.9) we have the similar results as in Theorem 2.1 provided with $k(\cdot) \in L^\infty(0,T;M_2(R))$.

3 Identification problems for CSG

In this section we study the identification problems for the coupled damped sine-Gordon equations described by

$$\begin{cases} \mathbf{y}'' + \boldsymbol{\alpha}\mathbf{y}' + \boldsymbol{\beta}\mathbf{A}\mathbf{y} + \mathbf{k}\mathbf{y} + \boldsymbol{\gamma}\sin\boldsymbol{\delta}\mathbf{y} = \mathbf{f} \quad \text{in} \quad (0,T), \\ \mathbf{y}(0) = \mathbf{y}_0, \quad \mathbf{y}'(0) = \mathbf{y}_1. \end{cases} \tag{3.1}$$

Let us assume that the parameters $\boldsymbol{\alpha}, \boldsymbol{\gamma}, \boldsymbol{\delta}, \mathbf{k}$ appeared in (3.1) are unknowns. By $\mathcal{M} = (M_2(R))^4$ denotes the four times cartension product space and give the product norm on it. We define a set of parameters $\mathcal{P} \triangleq M_2(R) \times M_2^d(R) \times M_2(R) \times M_2(R)$. It is obvious that $\mathcal{P}$ is the closed subset of $\mathcal{M}$. Set $\mathbf{q} = (\boldsymbol{\alpha}, \boldsymbol{\gamma}, \boldsymbol{\delta}, \mathbf{k}) \in \mathcal{P}$. Since for each $\mathbf{q} \in \mathcal{P}$ there exists a unique weak solution $\mathbf{y} = \mathbf{y}(\mathbf{q}) \in \mathbf{W}(0,T)$ of (3.1), we can define uniquely the solution map $\mathbf{q} \to \mathbf{y}(\mathbf{q})$ of $\mathcal{P}$ into $\mathbf{W}(0,T)$. We will call $\mathbf{y}(\mathbf{q})$ the state of (3.1) depending on $\mathbf{q}$.

The cost functional attached to (3.1) is given by

$$J(\mathbf{q}) = \|\mathbf{C}\mathbf{y}(\mathbf{q}) - \mathbf{z}_d\|_{\mathcal{K}}^2 \quad \text{for} \quad \mathbf{q} \in \mathcal{P}, \tag{3.2}$$

where $\mathbf{z}_d \in \mathcal{K}$ is a desired value of $\mathbf{y}(\mathbf{q})$ and $\mathbf{C}$ is a bounded linear observation operator of $\mathbf{W}(0,T)$ into $\mathcal{K}$, an observation space.

Let $\mathcal{P}_{ad}$ be a convex closed subset of $\mathcal{P}$, which is called the admissible set. The quadratic cost identification problems (QCIP) subject to (3.2) and (3.1) are usually divided into existence and characterization problems. The detailed descriptions of them are as follows:

(i) The problem of finding an element $\mathbf{q}^* \in \mathcal{P}_{ad}$ such that

$$\inf_{\mathbf{q} \in \mathcal{P}_{ad}} J(\mathbf{q}) = J(\mathbf{q}^*); \tag{3.3}$$

(ii) The problem of giving a characterization to such the $\mathbf{q}^*$.

As usual we shall call $\mathbf{q}^*$ the optimal parameter for (QCIP) and $\mathbf{y}(\mathbf{q}^*)$ the optimal state of (3.1). It is well-known that there are not general methods for solving (i) and stronger conditions on (3.1) are required for solving it. For example, $\mathcal{P}_{ad}$ is a compact subset of $\mathcal{P}$. We solve the problem (i) under this assumption. It is also well-known that we can characterize $\mathbf{q}^*$ if we can derive the necessary conditions on $\mathbf{q}^*$. As one effective method for deriving the necessary conditions we are to consider the Gâteaux derivatives of the given cost function $J(\mathbf{q})$. Hence if we act the Gâteaux derivatives on $J(\mathbf{q})$, then we have a formal inequality, which is a necessary condition, given by

$$DJ(\mathbf{q}^*)(\mathbf{q} - \mathbf{q}^*) \geq 0 \quad \text{for all} \quad \mathbf{q} \in \mathcal{P}_{ad}, \tag{3.4}$$

where $DJ(\mathbf{q}^*)$ denotes the Gâteaux derivative of $J(\mathbf{q})$ at $\mathbf{p} = \mathbf{q}^*$ in the direction $\mathbf{q} - \mathbf{q}^*$. We analyze the inequality (3.4) by introducing the adjoint state equations with respect to the state equations and give a characterization to $\mathbf{q}^*$. Since the Gâteaux differentiability of $J(\mathbf{q})$ depends on $\mathbf{y}(\mathbf{q})$ only, it is enough to study that of $\mathbf{y}(\mathbf{q})$.

3.1 Existence of optimal parameters

Here we assume that $\mathcal{P}_{ad}$ is a compact subset of $\mathcal{P}$ and we show the existence of the optimal parameter $\mathbf{q}^*$. The following theorem is essential to solve the problem (i).

Theorem 3.1 The map $\mathbf{q} \to \mathbf{y}(\mathbf{q}) : \mathcal{P} \to \mathbf{W}(0,T)$ is weakly continuous. That is, $\mathbf{y}(\mathbf{q}_n) \to \mathbf{y}(\mathbf{q})$ weakly in $\mathbf{W}(0,T)$ as $\mathbf{q}_n \to \mathbf{q}$ strongly in $\mathcal{M}$.

The following theorem is immediately obtained by Theorem 3.1.

Theorem 3.2 If $\mathcal{P}_{ad} \subset \mathcal{P}$ is a compact subset of $\mathcal{M}$, then there exists at least one optimal parameter $\mathbf{q}^* \in \mathcal{P}_{ad}$.

3.2 Necessary conditions

We begin to show that the map $\mathbf{q} \to \mathbf{y}(\mathbf{q})$ of $\mathcal{P}$ into $\mathbf{W}(0,T)$ is Gâteaux differentiable at $\mathbf{q}^*$ in the direction $\mathbf{q} - \mathbf{q}^*$.

Theorem 3.3 The map $\mathbf{q} \to \mathbf{y}(\mathbf{q})$ of $\mathcal{P}$ into $\mathbf{W}(0,T)$ is weakly Gâteaux differentiable. That is, for fixed $\mathbf{q}^* = (\boldsymbol{\alpha}^*, \boldsymbol{\gamma}^*, \boldsymbol{\delta}^*, \mathbf{k}^*) \in \mathcal{P}_{ad}$ the weak Gâteaux derivative $\mathbf{z} = D\mathbf{y}(\mathbf{q}^*)(\mathbf{q}-\mathbf{q}^*)$ of $\mathbf{y}(\mathbf{q})$ at $\mathbf{q} = \mathbf{q}^*$ in the direction $\mathbf{q}-\mathbf{q}^*$ exists in $\mathbf{W}(0,T)$ and it is a unique weak solution of the evolution equations

$$\begin{cases} \mathbf{z}'' + \boldsymbol{\alpha}^*\mathbf{z}' + \beta\mathbf{A}\mathbf{z} + \mathbf{k}^*\mathbf{z} + \boldsymbol{\gamma}^*\cos(\boldsymbol{\delta}^*\mathbf{y}^*)\boldsymbol{\delta}^*\mathbf{z} \\ = (\boldsymbol{\gamma}^* - \boldsymbol{\gamma})\sin\boldsymbol{\delta}^*\mathbf{y}^* + \boldsymbol{\gamma}^*\cos(\boldsymbol{\delta}^*\mathbf{y}^*)(\boldsymbol{\delta}^* - \boldsymbol{\delta})\mathbf{y}^* + (\boldsymbol{\alpha}^* - \boldsymbol{\alpha})\mathbf{y}^{*\prime} + (\mathbf{k}^* - \mathbf{k})\mathbf{y}^* \quad \text{in} \quad (0,T), \\ \mathbf{z}(0) = \mathbf{z}'(0) = \mathbf{0}, \end{cases} \tag{3.1}$$

where

$$\cos\boldsymbol{\phi} = \begin{bmatrix} \cos\phi_1 & 0 \\ 0 & \cos\phi_2 \end{bmatrix}.$$

Since the map $\mathbf{q} \to \mathbf{y}(\mathbf{q}) : \mathcal{P} \to \mathbf{W}(0,T)$ is Gâteaux differentiable at $\mathbf{q}^*$ in the direction $\mathbf{q}-\mathbf{q}^*$, the inequality (3.4) is equivalent to

$$\langle \mathbf{C}\mathbf{y}(\mathbf{q}^*) - \mathbf{z}_d, \mathbf{C}\mathbf{z} \rangle_{\mathcal{K}',\mathcal{K}} \geq 0, \quad \forall \mathbf{q} \in \mathcal{P}_{ad}, \tag{3.2}$$

where $\mathbf{z}$ is the solution of (3.1). To avoid QCIP to be complicated we study it according to four types of very simple observations as follows:

1. Observe the distributed state $\mathbf{C}\mathbf{y}(\mathbf{q}) = \mathbf{y}(\mathbf{q}) \in L^2(0,T;\mathcal{H})$ and take $\mathcal{K} = L^2(0,T;\mathcal{H})$;
2. Observe the distributed velocity $\mathbf{C}\mathbf{y}(\mathbf{q}) = \mathbf{y}'(\mathbf{q}) \in L^2(0,T;\mathcal{H})$ and take $\mathcal{K} = L^2(0,T;\mathcal{H})$;
3. Observe the time terminal state $\mathbf{C}\mathbf{y}(\mathbf{q}) = \mathbf{y}(\mathbf{q};T) \in \mathcal{H}$ and take $\mathcal{K} = \mathcal{H}$;
4. Observe the time terminal velocity $\mathbf{C}\mathbf{y}(\mathbf{q}) = \mathbf{y}'(\mathbf{q};T) \in \mathcal{H}$ and take $\mathcal{K} = \mathcal{H}$.

3.2.1 Case of $\mathbf{Cy}(\mathbf{q}) = \mathbf{y}(\mathbf{q}) \in L^2(0,T;\mathcal{H})$

In this case the cost functional is given by

$$J(\mathbf{q}) \;=\; \|\mathbf{y}(\mathbf{q}) - \mathbf{z}_d\|^2_{L^2(0,T;\mathcal{H})}, \tag{3.3}$$

where $\mathbf{z}_d \in L^2(0,T;\mathcal{H})$. Then the necessary condition (3.2) with respect to (3.3) is written by

$$(\mathbf{y}(\mathbf{q}^*) - \mathbf{z}_d, \mathbf{z})_{L^2(0,T;\mathcal{H})} \geq 0, \quad \forall \mathbf{q} \in \mathcal{P}_{ad}. \tag{3.4}$$

We introduce the adjoint state $\mathbf{p}$ given by evolution equations

$$\begin{cases} \mathbf{p}'' - \boldsymbol{\alpha}^{*t}\mathbf{p}' + \boldsymbol{\beta}\mathbf{A}\mathbf{p} + \mathbf{k}^{*t}\mathbf{p} + \boldsymbol{\delta}^{*t}\boldsymbol{\gamma}^* \cos(\boldsymbol{\delta}^*\mathbf{y}^*)\mathbf{p} = \mathbf{y}(\mathbf{q}^*) - \mathbf{z}_d \;\text{ in }\; (0,T), \\ \mathbf{p}(T) = \mathbf{p}'(T) = \mathbf{0}. \end{cases} \tag{3.5}$$

We can easily show existence and uniqueness of weak solutions for (3.5) if we take k defined by

$$\mathbf{k}(t) = \mathbf{k}^{*t} + \boldsymbol{\delta}^{*t}\boldsymbol{\gamma}^* \cos(\boldsymbol{\delta}^*\mathbf{y}^*(t)) \in L^\infty(0,T;M_2(R)).$$

Multiplying (3.5) by $\mathbf{z}$ and integrating it over $[0,T]$ by parts we have

$$\begin{aligned} &\int_0^T (\mathbf{y}(\mathbf{q}^*) - \mathbf{z}_d, \mathbf{z})\,dt \\ =\;& \int_0^T (\mathbf{p}'' - \boldsymbol{\alpha}^{*t}\mathbf{p}' + \boldsymbol{\beta}\mathbf{A}\mathbf{p} + \mathbf{k}^{*t}\mathbf{p} + \boldsymbol{\delta}^{*t}\boldsymbol{\gamma}^* \cos(\boldsymbol{\delta}^*\mathbf{y}^*)\mathbf{p}, \mathbf{z})\,dt \\ =\;& \int_0^T (\mathbf{p}, \mathbf{z}'' + \boldsymbol{\alpha}^*\mathbf{z}' + \boldsymbol{\beta}\mathbf{A}\mathbf{z} + \mathbf{k}^*\mathbf{z} + \boldsymbol{\gamma}^* \cos(\boldsymbol{\delta}^*\mathbf{y}^*)\boldsymbol{\delta}^*\mathbf{z})\,dt. \end{aligned}$$

Applying (3.1) to the the last equation we have

$$\begin{aligned} &\int_0^T (\mathbf{p}, (\boldsymbol{\gamma}^* - \boldsymbol{\gamma}) \sin\boldsymbol{\delta}^*\mathbf{y}^* + \boldsymbol{\gamma}^* \cos(\boldsymbol{\delta}^*\mathbf{y}^*)(\boldsymbol{\delta}^* - \boldsymbol{\delta})\mathbf{y}^* + (\boldsymbol{\alpha}^* - \boldsymbol{\alpha})\mathbf{y}^{*'} + (\mathbf{k}^* - \mathbf{k})\mathbf{y}^*)\,dt \\ =\;& \int_0^T (\mathbf{y}(\mathbf{q}^*) - \mathbf{z}_d, \mathbf{z})\,dt. \end{aligned}$$

Finally by (3.4) we have an necessary condition given by

$$\int_0^T (\mathbf{p}, (\boldsymbol{\gamma}^* - \boldsymbol{\gamma}) \sin\boldsymbol{\delta}^*\mathbf{y}^* + \boldsymbol{\gamma}^* \cos(\boldsymbol{\delta}^*\mathbf{y}^*)(\boldsymbol{\delta}^* - \boldsymbol{\delta})\mathbf{y}^* + (\boldsymbol{\alpha}^* - \boldsymbol{\alpha})\mathbf{y}^{*'} + (\mathbf{k}^* - \mathbf{k})\mathbf{y}^*)\,dt \geq 0$$

for all $\mathbf{q} \in \mathcal{P}_{ad}$.

Summarizing these we have the following theorem.

Theorem 3.4 The optimal parameter $\mathbf{q}^*$ for the cost (3.3) is characterized by the two states $\mathbf{y} = \mathbf{y}(\mathbf{q}^*), \mathbf{p} = \mathbf{p}(\mathbf{q}^*)$ of equations

$$\begin{cases} \mathbf{y}'' + \boldsymbol{\alpha}^*\mathbf{y}' + \boldsymbol{\beta}\mathbf{A}\mathbf{y} + \mathbf{k}^*\mathbf{y} + \boldsymbol{\gamma}^* \sin\boldsymbol{\delta}^*\mathbf{y} = \mathbf{f} \;\text{ in }\; (0,T), \\ \mathbf{y}(0) = \mathbf{y}_0, \;\; \mathbf{y}'(0) = \mathbf{y}_1, \end{cases} \tag{3.6}$$

$$\begin{cases} \mathbf{p}'' - \alpha^{*t}\mathbf{p}' + \beta A\mathbf{p} + \mathbf{k}^{*t}\mathbf{p} + \delta^{*t}\gamma^* \cos(\delta^*\mathbf{y})\mathbf{p} = \mathbf{y} - \mathbf{z}_d \text{ in } (0,T), \\ \mathbf{p}(T) = \mathbf{p}'(T) = \mathbf{0} \end{cases} \tag{3.7}$$

and one inequality

$$\int_0^T (\mathbf{p}, (\gamma^* - \gamma)\sin\delta^*\mathbf{y} + \gamma^*\cos(\delta^*\mathbf{y})(\delta^* - \delta)\mathbf{y} + (\alpha^* - \alpha)\mathbf{y}' + (\mathbf{k}^* - \mathbf{k})\mathbf{y})\, dt \geq 0 \tag{3.8}$$

for all $\mathbf{q} \in \mathcal{P}_{ad}$.

3.2.2 Case of $\mathbf{Cy}(\mathbf{q}) = \mathbf{y}'(\mathbf{q}) \in L^2(0,T;\mathcal{H})$

In this case the cost functional is given by

$$J(\mathbf{q}) = \|\mathbf{y}'(\mathbf{q}) - \mathbf{z}_d\|^2_{L^2(0,T;\mathcal{H})}, \tag{3.9}$$

where $\mathbf{z}_d \in L^2(0,T;\mathcal{H})$. Then the necessary condition (3.2) with respect to (3.3) is written by

$$(\mathbf{y}'(\mathbf{q}^*) - \mathbf{z}_d, \mathbf{z}')_{L^2(0,T;\mathcal{H})} \geq 0, \quad \forall \mathbf{q} \in \mathcal{P}_{ad}. \tag{3.10}$$

We introduce the adjoint state $\mathbf{p}$ defined by evolution equations equations

$$\begin{cases} \mathbf{p}'' - \alpha^{*t}\mathbf{p}' + \beta A\mathbf{p} + \mathbf{k}^{*t}\mathbf{p} + \int_t^T \delta^{*t}\gamma^* \cos(\delta^*\mathbf{y}^*)\mathbf{p}\, ds = \mathbf{y}'(\mathbf{q}^*) - \mathbf{z}_d \text{ in } (0,T), \\ \mathbf{p}(T) = \mathbf{p}'(T) = \mathbf{0}. \end{cases} \tag{3.11}$$

Through the approach as similar as we do in Theorem 2.1, we can prove existence, uniqueness and regularity of weak solutions of (3.11). Since $\mathbf{z}' \notin L^2(0,T;\mathcal{V})$, the following calculations are done formally. We can refer to [7] for the justice. Let us multiply $\mathbf{z}'$ on the both side hands of (3.11) and integrate it on $[0,T]$ by parts. Then we have

$$\begin{aligned} &\int_0^T (\mathbf{y}'(\mathbf{q}^*) - \mathbf{z}_d, \mathbf{z}')\, dt \\ = &\int_0^T (\mathbf{p}'' - \alpha^{*t}\mathbf{p}' + \beta A\mathbf{p} + \mathbf{k}^{*t}\mathbf{p} + \int_t^T \delta^{*t}\gamma^*\cos(\delta^*\mathbf{y}^*)\mathbf{p}\, ds, \mathbf{z}')\, dt \\ = &-\int_0^T (\mathbf{p}', \mathbf{z}'' + \alpha^*\mathbf{z}' + \beta A\mathbf{z} + \mathbf{k}^*\mathbf{z} + \gamma^*\cos(\delta^*\mathbf{y}^*)\delta^*\mathbf{z})\, dt. \end{aligned}$$

Since $\mathbf{z}$ is a unique solution of (3.1), we have

$$\begin{aligned} &\int_0^T (\mathbf{y}'(\mathbf{q}^*) - \mathbf{z}_d, \mathbf{z}')\, dt \\ = &-\int_0^T (\mathbf{p}', (\gamma^* - \gamma)\sin\delta^*\mathbf{y}^* + \gamma^*\cos(\delta^*\mathbf{y}^*)(\delta^* - \delta)\mathbf{y}^* + (\alpha^* - \alpha)\mathbf{y}^{*'} + (\mathbf{k}^* - \mathbf{k})\mathbf{y}^*)\, dt. \end{aligned}$$

Hence by (3.10) we have an necessary condition on $\mathbf{q}^*$ given by

$$\int_0^T (\mathbf{p}', (\gamma^* - \gamma)\sin\delta^*\mathbf{y}^* + \gamma^*\cos(\delta^*\mathbf{y}^*)(\delta^* - \delta)\mathbf{y}^* + (\alpha^* - \alpha)\mathbf{y}^{*'} + (\mathbf{k}^* - \mathbf{k})\mathbf{y}^*)\, dt \leq 0$$

for all $\mathbf{q} \in \mathcal{P}_{ad}$.

Summarizing these we have the following theorem.

Theorem 3.5 The optimal parameter $\mathbf{q}^*$ for the cost (3.9) is characterized by the two states $\mathbf{y} = \mathbf{y}(\mathbf{q}^*), \mathbf{p} = \mathbf{p}(\mathbf{q}^*)$ of equations

$$\begin{cases} \mathbf{y}'' + \boldsymbol{\alpha}^*\mathbf{y}' + \boldsymbol{\beta}\mathbf{A}\mathbf{y} + \mathbf{k}^*\mathbf{y} + \boldsymbol{\gamma}^* \sin\boldsymbol{\delta}^*\mathbf{y} = \mathbf{f} \text{ in } (0,T), \\ \mathbf{y}(0) = \mathbf{y}_0, \quad \mathbf{y}'(0) = \mathbf{y}_1, \end{cases} \tag{3.12}$$

$$\begin{cases} \mathbf{p}'' - \boldsymbol{\alpha}^{*t}\mathbf{p}' + \boldsymbol{\beta}\mathbf{A}\mathbf{p} + \mathbf{k}^{*t}\mathbf{p} + \displaystyle\int_t^T \boldsymbol{\delta}^{*t}\boldsymbol{\gamma}^* \cos(\boldsymbol{\delta}^*\mathbf{y})\mathbf{p}\, ds = \mathbf{y}' - \mathbf{z}_d \text{ in } (0,T), \\ \mathbf{p}(T) = \mathbf{p}'(T) = \mathbf{0} \end{cases} \tag{3.13}$$

and one inequality

$$\int_0^T (\mathbf{p}', (\boldsymbol{\gamma}^* - \boldsymbol{\gamma})\sin\boldsymbol{\delta}^*\mathbf{y} + \boldsymbol{\gamma}^*\cos(\boldsymbol{\delta}^*\mathbf{y})(\boldsymbol{\delta}^* - \boldsymbol{\delta})\mathbf{y} + (\boldsymbol{\alpha}^* - \boldsymbol{\alpha})\mathbf{y}' + (\mathbf{k}^* - \mathbf{k})\mathbf{y})\, dt \le 0 \tag{3.14}$$

for all $\mathbf{q} \in \mathcal{P}_{ad}$.

3.2.3 Case of $\mathbf{C}\mathbf{y}(\mathbf{q}) = \mathbf{y}(\mathbf{q}; T) \in \mathcal{H}$

In this case the cost functional is given by

$$J(\mathbf{q}) = |\mathbf{y}(\mathbf{q}; T) - \mathbf{z}_d|^2, \tag{3.15}$$

where $\mathbf{z}_d \in \mathcal{H}$. Then the necessary condition (3.2) with respect to (3.15) is written by

$$(\mathbf{y}(\mathbf{q}^*; T) - \mathbf{z}_d, \mathbf{z}(T)) \ge 0, \quad \forall \mathbf{q} \in \mathcal{P}_{ad}. \tag{3.16}$$

We introduce the adjoint state $\mathbf{p}$ given by evolution equations

$$\begin{cases} \mathbf{p}'' - \boldsymbol{\alpha}^{*t}\mathbf{p}' + \boldsymbol{\beta}\mathbf{A}\mathbf{p} + \mathbf{k}^{*t}\mathbf{p} + \boldsymbol{\delta}^{*t}\boldsymbol{\gamma}^*\cos(\boldsymbol{\delta}^*\mathbf{y}^*)\mathbf{p} = \mathbf{0} \text{ in } (0,T), \\ \mathbf{p}(T) = \mathbf{0}, \quad \mathbf{p}'(T) = \mathbf{y}(\mathbf{q}^*; T) - \mathbf{z}_d. \end{cases} \tag{3.17}$$

If we take $\mathbf{y}(\mathbf{q}^*; T) - \mathbf{z}_d \in \mathcal{H}$ and $k(t) = \mathbf{k}^{*t} + \boldsymbol{\delta}^{*t}\boldsymbol{\gamma}^*\cos(\boldsymbol{\delta}^*\mathbf{y}^*(t))$, then there is an unique weak solution $\mathbf{p} \in \mathbf{W}(0,T)$ of (3.17). Let us multiply $\mathbf{z}$ on the both sides of the first equation of (3.17) and integrate it on $[0,T]$ by parts. Then we have

$$\begin{aligned} 0 &= \int_0^T (\mathbf{p}'' - \boldsymbol{\alpha}^{*t}\mathbf{p}' + \boldsymbol{\beta}\mathbf{A}\mathbf{p} + \mathbf{k}^{*t}\mathbf{p} + \boldsymbol{\delta}^{*t}\boldsymbol{\gamma}^*\cos(\boldsymbol{\delta}^*\mathbf{y}^*)\mathbf{p}, \mathbf{z})\, dt \\ 0 &= (\mathbf{p}'(T), \mathbf{z}(T)) + \int_0^T (\mathbf{p}, \mathbf{z}'' + \boldsymbol{\alpha}^*\mathbf{z}' + \boldsymbol{\beta}\mathbf{A}\mathbf{z} + \mathbf{k}^*\mathbf{z} + \boldsymbol{\gamma}^*\cos(\boldsymbol{\delta}^*\mathbf{y}^*)\boldsymbol{\delta}^*\mathbf{z})\, dt. \end{aligned}$$

Since $\mathbf{z}$ is the weak solution of (3.1), we have

$$\begin{aligned} &-(\mathbf{y}(\mathbf{q}^*; T) - \mathbf{z}_d, \mathbf{z}(T)) \\ &= \int_0^T (\mathbf{p}, (\boldsymbol{\gamma}^* - \boldsymbol{\gamma})\sin\boldsymbol{\delta}^*\mathbf{y}^* + \boldsymbol{\gamma}^*\cos(\boldsymbol{\delta}^*\mathbf{y}^*)(\boldsymbol{\delta}^* - \boldsymbol{\delta})\mathbf{y}^* + (\boldsymbol{\alpha}^* - \boldsymbol{\alpha})\mathbf{y}^{*'} + (\mathbf{k}^* - \mathbf{k})\mathbf{y}^*)\, dt. \end{aligned}$$

Finally by (3.16) we have an necessary condition given by

$$\int_0^T (\mathbf{p}, (\boldsymbol{\gamma}^* - \boldsymbol{\gamma})\sin\boldsymbol{\delta}^*\mathbf{y}^* + \boldsymbol{\gamma}^*\cos(\boldsymbol{\delta}^*\mathbf{y}^*)(\boldsymbol{\delta}^* - \boldsymbol{\delta})\mathbf{y}^* + (\boldsymbol{\alpha}^* - \boldsymbol{\alpha})\mathbf{y}^{*'} + (\mathbf{k}^* - \mathbf{k})\mathbf{y}^*)\, dt \le 0$$

for all $\mathbf{q} \in \mathcal{P}_{ad}$.

Summarizing these we have the following theorem.

Theorem 3.6 The optimal parameter $\mathbf{q}^*$ for the cost (3.15) is characterized by the two states $\mathbf{y} = \mathbf{y}(\mathbf{q}^*), \mathbf{p} = \mathbf{p}(\mathbf{q}^*)$ of equations

$$\begin{cases} \mathbf{y}'' + \alpha^*\mathbf{y}' + \beta\mathbf{A}\mathbf{y} + \mathbf{k}^*\mathbf{y} + \gamma^* \sin\delta^*\mathbf{y} = \mathbf{f} \text{ in } (0,T), \\ \mathbf{y}(0) = \mathbf{y}_0, \quad \mathbf{y}'(0) = \mathbf{y}_1, \end{cases} \tag{3.18}$$

$$\begin{cases} \mathbf{p}'' - \alpha^{*t}\mathbf{p}' + \beta\mathbf{A}\mathbf{p} + \mathbf{k}^{*t}\mathbf{p} + \delta^{*t}\gamma^* \cos(\delta^*\mathbf{y})\mathbf{p} = \mathbf{0} \text{ in } (0,T), \\ \mathbf{p}(T) = \mathbf{0}, \quad \mathbf{p}'(T) = \mathbf{y}(T) - \mathbf{z}_d \end{cases} \tag{3.19}$$

and one inequality

$$\int_0^T (\mathbf{p}, (\gamma^* - \gamma)\sin\delta^*\mathbf{y} + \gamma^*\cos(\delta^*\mathbf{y})(\delta^* - \delta)\mathbf{y} + (\alpha^* - \alpha)\mathbf{y}' + (\mathbf{k}^* - \mathbf{k})\mathbf{y})\, dt \le 0 \tag{3.20}$$

for all $\mathbf{q} \in \mathcal{P}_{ad}$.

3.2.4 Case of $\mathbf{C}\mathbf{y}'(\mathbf{q}) = \mathbf{y}'(\mathbf{q}; T) \in \mathcal{H}$

In this case the cost functional is given by

$$J(\mathbf{q}) \;=\; |\mathbf{y}'(\mathbf{q}; T) - \mathbf{z}_d|^2, \tag{3.21}$$

where $\mathbf{z}_d \in \mathcal{H}$. Then the necessary condition (3.2) with respect to (3.21) is written by

$$(\mathbf{y}'(\mathbf{q}^*; T) - \mathbf{z}_d, \mathbf{z}'(T)) \ge 0, \quad \forall \mathbf{q} \in \mathcal{P}_{ad}. \tag{3.22}$$

We consider the adjoint state $\mathbf{p}$ given by evolution equations

$$\begin{cases} \mathbf{p}'' - \alpha^{*t}\mathbf{p}' + \beta\mathbf{A}\mathbf{p} + \mathbf{k}^{*t}\mathbf{p} + \delta^{*t}\gamma^* \cos(\delta^*\mathbf{y}^*)\mathbf{p} = \mathbf{0} \text{ in } (0,T), \\ \mathbf{p}(T) = \mathbf{y}'(\mathbf{q}^*; T) - \mathbf{z}_d, \quad \mathbf{p}(T) = \alpha^{*t}(\mathbf{y}'(\mathbf{q}^*; T) - \mathbf{z}_d). \end{cases} \tag{3.23}$$

Since $\mathbf{y}'(\mathbf{q}^*; T) - \mathbf{z}_d \notin \mathcal{V}$ in spite of $\alpha^{*t}(\mathbf{y}'(\mathbf{q}^*; T) - \mathbf{z}_d) \in \mathcal{H}$, we can not give any information of solutions for the equation (3.23). Hence the following calculations are completely formal. It is meaningful to deduce the necessary conditions on q^* in spite of formality. Let us multiply $\mathbf{z}$ on the both sides of the first equation of (3.17) and integrate it on $[0,T]$ by parts. Then we have

$$\begin{aligned} \mathbf{0} &= \int_0^T (\mathbf{p}'' - \alpha^{*t}\mathbf{p}' + \beta\mathbf{A}\mathbf{p} + \mathbf{k}^{*t}\mathbf{p} + \delta^{*t}\gamma^* \cos(\delta^*\mathbf{y}^*)\mathbf{p}, \mathbf{z})\, dt \\ (\mathbf{p}(T), \mathbf{z}'(T)) &= (\mathbf{p}'(T) - \alpha^{*t}\mathbf{p}(T), \mathbf{z}(T)) \\ &\quad + \int_0^T (\mathbf{p}, \mathbf{z}'' + \alpha^*\mathbf{z}' + \beta\mathbf{A}\mathbf{z} + \mathbf{k}^*\mathbf{z} + \gamma^* \cos(\delta^*\mathbf{y}^*)\delta^*\mathbf{z})\, dt. \end{aligned}$$

Since $\mathbf{p}'(T) - \alpha^{*t}\mathbf{p}(T) = \mathbf{0}$, $\mathbf{p}(T) = \mathbf{y}(\mathbf{q}^*; T) - \mathbf{z}_d$ and $\mathbf{z}$ is the weak solution of (3.1), by last equality above we have

$$\begin{aligned} &(\mathbf{y}(\mathbf{q}^*; T) - \mathbf{z}_d, \mathbf{z}'(T)) \\ &= \int_0^T (\mathbf{p}, (\gamma^* - \gamma)\sin\delta^*\mathbf{y}^* + \gamma^*\cos(\delta^*\mathbf{y}^*)(\delta^* - \delta)\mathbf{y}^* + (\alpha^* - \alpha)\mathbf{y}^{*'} + (\mathbf{k}^* - \mathbf{k})\mathbf{y}^*)\, dt. \end{aligned}$$

Finally by (3.22) we have an necessary condition given by

$$\int_0^T (\mathbf{p}, (\gamma^* - \gamma)\sin\boldsymbol{\delta}^*\mathbf{y}^* + \gamma^*\cos(\boldsymbol{\delta}^*\mathbf{y}^*)(\boldsymbol{\delta}^* - \boldsymbol{\delta})\mathbf{y}^* + (\boldsymbol{\alpha}^* - \boldsymbol{\alpha})\mathbf{y}^{*'} + (\mathbf{k}^* - \mathbf{k})\mathbf{y}^*)\,dt \geq 0$$

for all $\mathbf{q} \in \mathcal{P}_{ad}$.

Summarizing these we have the following theorem.

Theorem 3.7 Assume that

$$\mathbf{y}'(\mathbf{q}^*; T) - \mathbf{z}_d \in \mathcal{V}.$$

Then the optimal parameter $\mathbf{q}^*$ for the cost (3.21) is characterized by the two states $\mathbf{y} = \mathbf{y}(\mathbf{q}^*), \mathbf{p} = \mathbf{p}(\mathbf{q}^*)$ of equations

$$\begin{cases} \mathbf{y}'' + \boldsymbol{\alpha}^*\mathbf{y}' + \beta\mathbf{A}\mathbf{y} + \mathbf{k}^*\mathbf{y} + \gamma^*\sin\boldsymbol{\delta}^*\mathbf{y} = \mathbf{f} \ \text{ in } \ (0,T), \\ \mathbf{y}(0) = \mathbf{y}_0, \ \ \mathbf{y}'(0) = \mathbf{y}_1, \end{cases} \tag{3.24}$$

$$\begin{cases} \mathbf{p}'' - \boldsymbol{\alpha}^{*t}\mathbf{p}' + \beta\mathbf{A}\mathbf{p} + \mathbf{k}^{*t}\mathbf{p} + \boldsymbol{\delta}^{*t}\gamma^*\cos(\boldsymbol{\delta}^*\mathbf{y})\mathbf{p} = \mathbf{0} \ \text{ in } \ (0,T), \\ \mathbf{p}(T) = \mathbf{y}'(T) - \mathbf{z}_d, \ \ \mathbf{p}(T) = \boldsymbol{\alpha}^{*t}(\mathbf{y}'(T) - \mathbf{z}_d) \end{cases} \tag{3.25}$$

and one inequality

$$\int_0^T (\mathbf{p}, (\gamma^* - \gamma)\sin\boldsymbol{\delta}^*\mathbf{y} + \gamma^*\cos(\boldsymbol{\delta}^*\mathbf{y})(\boldsymbol{\delta}^* - \boldsymbol{\delta})\mathbf{y} + (\boldsymbol{\alpha}^* - \boldsymbol{\alpha})\mathbf{y}' + (\mathbf{k}^* - \mathbf{k})\mathbf{y})\,dt \geq 0 \tag{3.26}$$

for all $\mathbf{q} \in \mathcal{P}_{ad}$.

Proof. All calculations are true under the assumption $\mathbf{y}'(\mathbf{q}^*; T) - \mathbf{z}_d \in \mathcal{V}$.

Example 3.1 Let us deduce the bang-bang principle for the case of $\mathbf{C}\mathbf{y}(\mathbf{q}) = \mathbf{y}(\mathbf{q}) \in L^2(0,T;\mathcal{H})$. In this case the necessary condition (3.8) is equivalent to

$$\int_0^T ((\boldsymbol{\alpha}^* - \boldsymbol{\alpha})\mathbf{y}'(t), \mathbf{p}(t))\,dt \geq 0, \ \ \forall \boldsymbol{\alpha} \in M_2(R), \tag{3.27}$$

$$\int_0^T ((\mathbf{k}^* - \mathbf{k})\mathbf{y}(t), \mathbf{p}(t))\,dt \geq 0, \ \ \forall \mathbf{k} \in M_2(R), \tag{3.28}$$

$$\int_0^T ((\gamma^* - \gamma)\sin\boldsymbol{\delta}^*\mathbf{y}(t), \mathbf{p}(t))\,dt \geq 0, \ \ \forall \gamma \in M_2^d(R), \tag{3.29}$$

$$\int_0^T ((\boldsymbol{\delta}^* - \boldsymbol{\delta})\mathbf{y}(t), \gamma^*\cos(\boldsymbol{\delta}^*\mathbf{y}(t))\mathbf{p}(t))\,dt \geq 0, \ \ \forall \boldsymbol{\delta} \in M_2(R). \tag{3.30}$$

First let us characterize (3.27). For this we take the component sets for $\boldsymbol{\alpha}$ as follows:

$$\alpha_{ij} \in [\overline{\alpha}_{ij}^1, \overline{\alpha}_{ij}^2], \ \ i,j = 1,2.$$

Put $a_{ij} = \int_Q \frac{\partial y_j}{\partial t}(x,t)p_i(x,t)\,dxdt$ and assume that $a_{ij} \neq 0$ for all $i,j = 1,2$. Then (3.27) is also equivalent to the following four conditions

$$(\alpha_{ij}^* - \alpha_{ij})a_{ij} \geq 0, \ \ \forall \alpha_{ij} \in [\overline{\alpha}_{ij}^1, \overline{\alpha}_{ij}^2], \ \ i,j = 1,2.$$

Consequently it is easily verified by these conditions that

$$\alpha_{ij}^* = \frac{1}{2}\{\text{sign}(a_{ij})+1\}\overline{\alpha}_{ij}^2 - \frac{1}{2}\{\text{sign}(a_{ij})-1\}\overline{\alpha}_{ij}^1, \quad i,j=1,2.$$

Now for characterizing (3.28)-(3.30) we take the component sets for $\mathbf{k}, \gamma$ and δ as follows:

$$k_{ij} \in [\overline{k}_{ij}^1, \overline{k}_{ij}^2], \quad \gamma_{ii} \in [\overline{\gamma}_{ii}^1, \overline{\gamma}_{ii}^2], \quad \delta_{ij} \in [\overline{\delta}_{ij}^1, \overline{\delta}_{ij}^2], \quad i,j=1,2.$$

Assume that for $i,j=1,2$

$$\begin{aligned} c_{ij} &= \int_Q y_j(x,t)p_i(x,t)\,dxdt \neq 0, \\ d_i &= \int_Q \sin(\sum_{j=1}^{2} \delta_{ij} y_j(x,t))p_i(x,t)\,dxdt \neq 0 \\ e_{ij} &= \gamma_{ii}^* \int_Q y_j(x,t)\cos\left(\sum_{k=1}^{2} \delta_{ik}^* y_k(x,t)\right) p_i(x,t)\,dxdt \neq 0. \end{aligned}$$

Then we have for $i,j=1,2$

$$\begin{aligned} k_{ij}^* &= \frac{1}{2}\{\text{sign}(c_{ij})+1\}\overline{k}_{ij}^2 - \frac{1}{2}\{\text{sign}(c_{ij})-1\}\overline{k}_{ij}^1, \\ \gamma_{ii}^* &= \frac{1}{2}\{\text{sign}(d_i)+1\}\overline{\gamma}_{ii}^2 - \frac{1}{2}\{\text{sign}(d_i)-1\}\overline{\gamma}_{ii}^1, \\ \delta_{ij}^* &= \frac{1}{2}\{\text{sign}(e_{ij})+1\}\overline{\delta}_{ij}^2 - \frac{1}{2}\{\text{sign}(e_{ij})-1\}\overline{\delta}_{ij}^1. \end{aligned}$$

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