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Order three symmetry of a vertex operator algebra

Hiromichi Yamada

Department of Mathematics
Hitotsubashi University
Kunitachi, Tokyo 186-8601, Japan
yamada@math.hit-u.ac.jp

In this note we shall report an attempt to find a vertex operator algebra which possesses an order three symmetry. We are actually interested in a subalgebra having this property of a vertex operator algebra associated with a lattice $L=\sqrt{2}A_2$. The work is not completed yet. We shall show some results so far obtained.

1 Notation and Setting

Let $\{\alpha_1, \alpha_2\}$ be the set of simple roots of type A_2, so that $\langle\alpha_i, \alpha_i\rangle = 2$ and $\langle\alpha_1, \alpha_2\rangle = -1$. We shall consider three automorphisms of the root lattice $\mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2$. First,

$$\sigma : \alpha_1 \longmapsto \alpha_2 \longmapsto -(\alpha_1+\alpha_2) \longmapsto \alpha_1$$

is an automorphism of order three. Exchange of α_1 and α_2 induces an order two automorphism ρ of the root lattice. Finally, let θ be the order two automorphism $\alpha \longmapsto -\alpha$ as usual. Note that $\rho\sigma\rho = \sigma^{-1}$. Let τ_i be the reflection with respect to α_i. Then $\tau_1\tau_2 = \sigma$ and $\tau_1\tau_2\tau_1 = \rho\theta$. Hence

$$\langle\tau_1, \tau_2, \theta\rangle = \langle\sigma, \rho, \theta\rangle \cong S_3 \times \mathbb{Z}_2$$

Let $L = \mathbb{Z}\sqrt{2}\alpha_1 + \mathbb{Z}\sqrt{2}\alpha_2$ be $\sqrt{2}$ times the root lattice of type A_2 and V_L be the vertex operator algebra associated with the lattice L as defined in [3]. The vertex operator algebra V_L was first studied in [5] and several applications were developed in [4], [5], [8]. We shall use the same notation as in [5]. The three automorphisms σ, ρ, θ can be extended to automorphisms of V_L. We shall denote these automorphisms of V_L by the same symbols.

We want to know the vertex operator subalgebra

$$(V_L)^{\sigma} = \{v \in V_L \mid \sigma v = v\}$$

and also its irreducible modules.

We need some other subalgebras. Let

$$\begin{aligned}V_L^{\pm} &= \{v \in V_L \mid \theta v = \pm v\},\\ V_L^{k} &= \{v \in V_L \mid \sigma v = \zeta^k v\},\\ V_L^{k,\pm} &= \{v \in V_L \mid \sigma v = \zeta^k v, \quad \theta v = \pm v\},\end{aligned}$$

where $\zeta = \exp(2\pi\sqrt{-1}/3)$ is a primitive cubic root of unity. Similar notations will be used for the homogeneous subspace

$$(V_L)_{(m)} = \{v \in V_L \mid \operatorname{wt} v = m\}$$

of weight m. For example,

$$(V_L^{k,\pm})_{(m)} = V_L^{k,\pm} \cap (V_L)_{(m)}.$$

Since $\rho\sigma\rho = \sigma^{-1}$ and since θ commutes with ρ and σ, it follows that $\rho(V_L^{0,\pm}) = V_L^{0,\pm}$ and $\rho(V_L^{1,\pm}) = V_L^{2,\pm}$.

By [1] it is known that there are three mutually orthogonal conformal vectors $\omega^1, \omega^2, \omega^3$ with central charges $\frac{1}{2}$, $\frac{7}{10}$, $\frac{4}{5}$ respectively in V_L. We shall recall their definition. For convenience, set

$$x(\alpha) = e^{\sqrt{2}\alpha} + e^{-\sqrt{2}\alpha}, \qquad w(\alpha) = \frac{1}{2}\alpha(-1)^2 - x(\alpha).$$

Now let

$$\begin{aligned}s^1 &= \frac{1}{4}w(\alpha_1),\\ s^2 &= \frac{1}{5}(w(\alpha_1) + w(\alpha_2) + w(\alpha_1+\alpha_2)),\\ \omega &= \frac{1}{6}(\alpha_1(-1)^2 + \alpha_2(-1)^2 + (\alpha_1+\alpha_2)(-1)^2).\end{aligned}$$

Then ω is the Virasoro element of V_L. The conformal vectors ω^i are defined by

$$\omega^1 = s^1, \qquad \omega^2 = s^2 - s^1, \qquad \omega^3 = \omega - s^2.$$

Denote by $\mathrm{Vir}(\omega^i)$ the subalgebra of V_L generated by ω^i. Then

$$\mathrm{Vir}(\omega^i) \cong L(c_i, 0), \qquad i = 1, 2, 3,$$

with $c_1 = \frac{1}{2}$, $c_2 = \frac{7}{10}$, $c_3 = \frac{4}{5}$. Since ω^i's are mutually orthogonal, the subalgebra T generated by ω^1, ω^2, and ω^3 are isomorphic to a tensor product of $\mathrm{Vir}(\omega^i)$'s:

$$\begin{aligned}T &\cong \mathrm{Vir}(\omega^1) \otimes \mathrm{Vir}(\omega^2) \otimes \mathrm{Vir}(\omega^3)\\ &\cong L(\tfrac{1}{2}, 0) \otimes L(\tfrac{7}{10}, 0) \otimes L(\tfrac{4}{5}, 0).\end{aligned}$$

As a T-module V_L is completely reducible and in fact it is a direct sum of 8 irreducible T-modules. Each irreducible T-module is of the form (see [2])

$$L(\frac{1}{2},\, h_1) \otimes L(\frac{7}{10},\, h_2) \otimes L(\frac{4}{5},\, h_3).$$

The following is the list of (h_1, h_2, h_3) of the irreducible direct summands in V_L:

$$(0,\ 0,\ 0), \qquad (0,\ \tfrac{3}{5},\ \tfrac{2}{5}), \qquad (\tfrac{1}{2},\ \tfrac{1}{10},\ \tfrac{2}{5}), \qquad (0,\ \tfrac{3}{5},\ \tfrac{7}{5}),$$

$$(\tfrac{1}{2},\ \tfrac{1}{10},\ \tfrac{7}{5}), \qquad (\tfrac{1}{2},\ \tfrac{3}{2},\ 0), \qquad (0,\ 0,\ 3), \qquad (\tfrac{1}{2},\ \tfrac{3}{2},\ 3).$$

More precisely, $(0,\ 0,\ 0)$, $(0,\ \frac{3}{5},\ \frac{7}{5})$, $(\frac{1}{2},\ \frac{1}{10},\ \frac{7}{5})$, $(\frac{1}{2},\ \frac{3}{2},\ 0)$ are the direct summands in V_L^+ and the remaining four are those in V_L^-.

We note that ω^i's are θ-invariant and that $s^2 = \omega^1 + \omega^2$ and ω^3 are σ-invariant. However, ω^1 is not σ-invariant. Although T is not invariant under σ, it contains a subalgebra

$$\mathrm{Vir}(s^2) \otimes \mathrm{Vir}(\omega^3) \cong L(\frac{6}{5},\, 0) \otimes L(\frac{4}{5},\, 0),$$

which is fixed by $\langle \sigma, \theta \rangle$, and $V_L^{k,\pm}$ is a module for $\mathrm{Vir}(s^2) \otimes \mathrm{Vir}(\omega^3)$. As a $\mathrm{Vir}(s^2) \otimes \mathrm{Vir}(\omega^3)$-module $V_L^{k,\pm}$ is completely reducible and each irreducible direct summand is of the form $L(\frac{6}{5},\, h) \otimes L(\frac{4}{5},\, h')$. Hence it is natural to ask:

Problem Determine the decomposition of $V_L^{k,\pm}$ into a direct sum of irreducible $\mathrm{Vir}(s^2) \otimes \mathrm{Vir}(\omega^3)$-modules.

2 Some Calculations

An irreducible direct summand isomorphic to $L(\frac{6}{5},\, h) \otimes L(\frac{4}{5},\, h')$ is generated by a highest weight vector $v = v(h, h')$ as a module for $\mathrm{Vir}(s^2) \otimes \mathrm{Vir}(\omega^3)$. Recall that a highest weight vector $v = v(h, h')$ is a vector which satisfies the conditions

$$(s^2)_1 v = hv, \qquad (\omega^3)_1 v = h'v,$$
$$(s^2)_n v = (\omega^3)_n v = 0 \quad \text{for} \quad n \geq 2.$$

Here we denote by u_n the component operator of the vertex operator $Y(u, z) = \sum_{n \in \mathbb{Z}} u_n z^{-n-1}$

It seems that $V_L^{k\pm}$ is a direct sum of infinitely many irreducible $\mathrm{Vir}(s^2) \otimes \mathrm{Vir}(\omega^3)$-modules. However, only a few irreducible direct summands are known. In fact, by a direct calculation we can determine the highest weight vectors $v(h, h')$ such that $h + h' \leq 3$. From this result we have

Lemma 2.1 (1) $V_L^{0,+}$ *contains three irreducible direct summands whose highest weights are at most* 3. *They are isomorphic to*

$$L(\frac{6}{5},0)\otimes L(\frac{4}{5},0),\qquad L(\frac{6}{5},\frac{8}{5})\otimes L(\frac{4}{5},\frac{7}{5}),\qquad L(\frac{6}{5},3)\otimes L(\frac{4}{5},0).$$

The automorphism ρ *acts as* 1 *on the first one and* -1 *on the other two irreducible direct summands.*

(2) *For* $k=1,2$, $V_L^{k,+}$ *contains two irreducible direct summands whose highest weights are at most* 3. *They are isomorphic to*

$$L(\frac{6}{5},\frac{3}{5})\otimes L(\frac{4}{5},\frac{7}{5}),\qquad L(\frac{6}{5},2)\otimes L(\frac{4}{5},0).$$

(3) $V_L^{0,-}$ *contains three irreducible direct summands whose highest weights are at most* 3. *They are isomorphic to*

$$L(\frac{6}{5},\frac{8}{5})\otimes L(\frac{4}{5},\frac{2}{5}),\qquad L(\frac{6}{5},\frac{13}{5})\otimes L(\frac{4}{5},\frac{2}{5}),\qquad L(\frac{6}{5},0)\otimes L(\frac{4}{5},3).$$

The automorphism ρ *acts as* 1 *on the first one and* -1 *on the other two irreducible direct summands.*

(4) *For* $k=1,2$, $V_L^{k,-}$ *contains only one irreducible direct summand whose highest weight is at most* 3. *It is isomorphic to*

$$L(\frac{6}{5},\frac{3}{5})\otimes L(\frac{4}{5},\frac{2}{5}).$$

Next, we shall consider the character of $V_L^{k,\pm}$. As a vector space $V_L=M(1)\otimes\mathbb{C}[L]$, where $M(1)$ is the free boson part and $\mathbb{C}[L]$ is the group algebra of the additive group L. Thus $\mathbb{C}[L]$ has a basis $\{e^\alpha\,|\,\alpha\in L\}$ with multiplication $e^\alpha e^\beta=e^{\alpha+\beta}$.

Let $\mathcal{H}_{n,j}$ be the space of homogeneous polynomials in two variables $\alpha_1(-n)$ and $\alpha_2(-n)$ of degree j. It is of dimention $j+1$ and

$$\{\alpha_1(-n)^{j-i}\alpha_2(-n)^i\,|\,0\le i\le j\}$$

forms its basis. Moreover, $\mathcal{H}_{n,j}$ is invariant under σ and ρ, and θ acts as $(-1)^j$ on $\mathcal{H}_{n,j}$. Note that

$$M(1)=\otimes_{n=1}^{\infty}(\oplus_{j=0}^{\infty}\mathcal{H}_{n,j})$$

as vector spaces. Set

$$\mathcal{H}_{n,j}(k)=\{v\in\mathcal{H}_{n,j}\,|\,\sigma v=\zeta^k v\},\quad k=0,1,2.$$

Then $\mathcal{H}_{n,j}=\mathcal{H}_{n,j}(0)\oplus\mathcal{H}_{n,j}(1)\oplus\mathcal{H}_{n,j}(2)$. Note that $\rho(\mathcal{H}_{n,j}(0))=\mathcal{H}_{n,j}(0)$ and $\rho(\mathcal{H}_{n,j}(1))=\mathcal{H}_{n,j}(2)$. The dimension of $\mathcal{H}_{n,j}(k)$ is as follows.

Lemma 2.2 (1) *If* $j \equiv 0 \pmod 3$, *then*

$$\dim \mathcal{H}_{n,j}(0) = j/3 + 1, \qquad \dim \mathcal{H}_{n,j}(k) = j/3, \quad k = 1, 2.$$

(2) *If* $j \equiv 1 \pmod 3$, *then*

$$\dim \mathcal{H}_{n,j}(0) = (j-1)/3, \qquad \dim \mathcal{H}_{n,j}(k) = (j+2)/3, \quad k = 1, 2.$$

(3) *If* $j \equiv 2 \pmod 3$, *then*

$$\dim \mathcal{H}_{n,j}(k) = (j+1)/3, \quad k = 0, 1, 2.$$

From this lemma we know the character of $V_L^{k,\pm}$.

Since $L = \sqrt{2}A_2$ and $\mathrm{wt}\, e^\alpha = \langle \alpha, \alpha \rangle /2$, the character of $\mathbb{C}[L]$ is nothing but the theta series of the root lattice of type A_2:

$$\begin{aligned} \mathrm{ch}\, \mathbb{C}[L] &= \sum_{\alpha \in L} q^{\mathrm{wt}\, e^\alpha} \\ &= \sum_{m,n \in \mathbb{Z}} q^{2m^2 - 2mn + 2n^2} \\ &= \theta_2(2\tau)\theta_2(6\tau) + \theta_3(2\tau)\theta_3(6\tau), \end{aligned}$$

where $q = \exp(\pi\sqrt{-1}\tau)$ and

$$\theta_2(\tau) = \sum_{n \in \mathbb{Z}} q^{(n+1/2)^2}, \qquad \theta_3(\tau) = \sum_{n \in \mathbb{Z}} q^{n^2}$$

are Jacobi theta series. Set

$$\mathbb{C}[L](k) = \{v \in \mathbb{C}[L] \mid \sigma v = \zeta^k v\}, \quad k = 0, 1, 2.$$

The automorphism σ acts fixed point freely on $L - \{0\}$. Hence

$$\begin{aligned} \mathbb{C}[L](0) &= \mathrm{span}\{e^\alpha + \sigma e^\alpha + \sigma^2 e^\alpha \mid 0 \neq \alpha \in L\} \cup \mathbb{C}e^0, \\ \mathbb{C}[L](1) &= \mathrm{span}\{e^\alpha + \zeta^2 \sigma e^\alpha + \zeta \sigma^2 e^\alpha \mid 0 \neq \alpha \in L\}, \\ \mathbb{C}[L](2) &= \mathrm{span}\{e^\alpha + \zeta \sigma e^\alpha + \zeta^2 \sigma^2 e^\alpha \mid 0 \neq \alpha \in L\}, \end{aligned}$$

and we have

Lemma 2.3 *The characters of* $\mathbb{C}[L](k)$ *are*

$$\begin{aligned} \mathrm{ch}\, \mathbb{C}[L](0) &= \frac{1}{3}\mathrm{ch}\, \mathbb{C}[L] + \frac{2}{3}, \\ \mathrm{ch}\, \mathbb{C}[L](k) &= \frac{1}{3}\mathrm{ch}\, \mathbb{C}[L] - \frac{1}{3}, \quad k = 1, 2. \end{aligned}$$

The character of $V_L^{k,\pm}$ follows from the above calculations.

3 Subalgebra W

The subalgebra

$$\{v \in V_L \mid (s^2)_1 v = 0\} \cong \mathbf{1}_{L(\frac{1}{2},0)} \otimes \mathbf{1}_{L(\frac{7}{10},0)} \otimes (L(\frac{4}{5}, 0) \oplus L(\frac{4}{5}, 3))$$

is contained in V_L^0 (see [5]). Here $\mathbf{1}_{L(c,0)}$ denotes the vacuum vector of $L(c, 0)$. We are interested in the counter part, namely,

$$\begin{aligned} W &= \{v \in V_L \mid (\omega^3)_1 v = 0\} \\ &= (L(\frac{1}{2}, 0) \otimes L(\frac{7}{10}, 0) \otimes \mathbf{1}_{L(\frac{4}{5},0)}) \oplus (L(\frac{1}{2}, \frac{1}{2}) \otimes L(\frac{7}{10}, \frac{3}{2}) \otimes \mathbf{1}_{L(\frac{4}{5},0)}). \end{aligned}$$

The subalgebra W was studied in [8] to construct certain vertex operator algebras associated with $\mathbb{Z}_2 \times \mathbb{Z}_2$ codes. In this note we shall study W as a module for

$$T' = \mathrm{Vir}(s^2) = L(\frac{6}{5}, 0) \otimes \mathbf{1}_{L(\frac{4}{5},0)}.$$

As a T'-module, W is completely reducible and each irreducible direct summand is of the form $L(\frac{6}{5}, h)$ for some $h \geq 0$. The characters of irreducible unitary highest weight modules for Virasoro algebras are known (see for example [6], [7], [9]). The characters of those modules appeared above are

$$\begin{aligned} \mathrm{ch}\, L(\frac{1}{2}, 0) &= P(q) \sum_{j \in \mathbf{Z}} (q^{j(12j+1)} - q^{(3j+1)(4j+1)}), \\ \mathrm{ch}\, L(\frac{1}{2}, \frac{1}{2}) &= q^{\frac{1}{2}} P(q) \sum_{j \in \mathbf{Z}} (q^{j(12j+5)} - q^{(3j+2)(4j+1)}), \\ \mathrm{ch}\, L(\frac{7}{10}, 0) &= P(q) \sum_{j \in \mathbf{Z}} (q^{j(20j+1)} - q^{(4j+1)(5j+1)}), \\ \mathrm{ch}\, L(\frac{7}{10}, \frac{3}{2}) &= q^{\frac{3}{2}} P(q) \sum_{j \in \mathbf{Z}} (q^{j(20j+11)} - q^{(4j+3)(5j+1)}), \\ \mathrm{ch}\, L(\frac{6}{5}, 0) &= P(q)(1-q), \\ \mathrm{ch}\, L(\frac{7}{10}, h) &= q^h P(q), \quad \text{for} \quad h > 0, \end{aligned}$$

where

$$P(q) = \sum_{n \geq 0} p(n) q^n$$

is the generating function of the partition numbers. The decomposition of W into a direct sum of irreducible T'-modules $L(\frac{6}{5}, h)$ will follow if one writes

$$\mathrm{ch}\, W = \mathrm{ch}\, L(\frac{1}{2}, 0)\, \mathrm{ch}\, L(\frac{7}{10}, 0) + \mathrm{ch}\, L(\frac{1}{2}, \frac{1}{2})\, \mathrm{ch}\, L(\frac{7}{10}, \frac{3}{2})$$

as a linear combination of $\operatorname{ch} L(\frac{6}{5}, h)$'s. On the other hand, it seems difficult to compute the character of

$$\{v \in W \mid \sigma v = \zeta^k v\}, \quad k = 0, 1, 2.$$

The weight 2 subspace $W_{(2)}$ is of dimension 3 and spanned by $\{w(\alpha_1), w(\alpha_2), w(\alpha_1 + \alpha_2)\}$. Let

$$v_h = w(\alpha_2) - w(\alpha_1 + \alpha_2).$$

This vector is a highest weight vector in $L(\frac{1}{2}, \frac{1}{2}) \otimes L(\frac{7}{10}, \frac{3}{2}) \otimes \mathbf{1}_{L(\frac{4}{5},0)}$ for $L(\frac{1}{2}, 0) \otimes L(\frac{7}{10}, 0) \otimes \mathbf{1}_{L(\frac{4}{5},0)}$, that is,

$$(\omega^1)_1 v_h = \frac{1}{2} v_h, \qquad (\omega^2)_1 v_h = \frac{3}{2} v_h,$$
$$(\omega^1)_n v_h = (\omega^2)_n v_h = 0 \quad \text{for} \quad n \geq 2.$$

The vertex operator algebra W is generated by its weight 2 subspace $W_{(2)}$. The following property of W is suggested by Masahiko Miyamoto.

Proposition 3.1 *The vertex operator algebra W is generated by one vector v_h or u, where*

$$u = w(\alpha_1) + \zeta^2 w(\alpha_2) + \zeta w(\alpha_1 + \alpha_2)$$

and thus $\sigma u = \zeta u$. More precisely,

$$\begin{aligned} W_{(2)} &= \operatorname{span}\{v_h, (v_h)_1 v_h, ((v_h)_1 v_h)_1((v_h)_1 v_h)\} \\ &= \operatorname{span}\{u, u_1 u, (u_1 u)_1 u\}. \end{aligned}$$

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