THE LATTICE VERTEX OPERATOR ALGEBRA $V_{\sqrt{2}D_l}$ AND SOME VERTEX OPERATOR ALGEBRAS CONSTRUCTED FROM $\mathbb{Z}_8$-CODES

CHING HUNG LAM

In this note, we shall discuss a construction of vertex operator algebra from $\mathbb{Z}_8$-codes and the lattice vertex operator algebra $V_{\sqrt{2}D_l}$. This construction is essentially a commutant or coset construction associated with certain lattice VOAs constructed from the lattice $V_{\sqrt{2}D_l}$. Most of the materials are already written in [3, 4, 9]. Please refer to the corresponding references for more details.

1. A GLUE LATTICE ASSOCIATED WITH $\sqrt{2}D_l$

We shall start by constructing some glue lattice $L_D$ from a $\mathbb{Z}_8$-code. First, let

$$D_l = \left\{ (x_1, \ldots, x_n) \in \mathbb{Z}^l \left| \sum_{i=1}^{l} x_i \text{ is even} \right. \right\}, \quad l = 3, 4, \ldots,$$

be the root lattice of type $D_l$. Then the dual lattice of $D_l$ is

$$D_l^* = \left\{ y \in \mathbb{Q} \otimes_{\mathbb{Z}} D_l \left| \langle x, y \rangle = \sum_{i=1}^{l} x_i y_i \in \mathbb{Z} \text{ for all } x \in D_l \right. \right\}$$

$$= \left\{ \frac{1}{2}(y_1, \ldots, y_n) \left| \text{all } y_i \text{'s are integers and have the same parity} \right. \right\}.$$

Note that $D_l^*/D_l \cong \mathbb{Z}_4$ if $l$ is odd and $D_l^*/D_l \cong \mathbb{Z}_2$ if $l$ is even.

Let $L$ be a lattice with basis $\{\alpha_1, \alpha_2, \ldots, \alpha_l\}$ such that $\langle \alpha_i, \alpha_j \rangle = 2\delta_{ij}$ and $N = \sum_{i,j=1}^{l} \mathbb{Z}(\alpha_i \pm \alpha_j)$. Then, $L$ is isomorphic to a direct sum of $l$ copies of the root lattice of type $A_1$ and $N \cong \sqrt{2}D_l = \left\{ \sum_{i=1}^{l} a_i \alpha_i | \sum_{i=1}^{l} a_i \equiv 0 \mod 2 \right\}$. Moreover, the dual lattice of $N$ is

$$N^* \cong \frac{1}{\sqrt{2}} D_l^* = \left\{ \frac{1}{4} \sum_{i=1}^{l} b_i \alpha_i \left| \text{all } b_i \text{'s are integers and have the same parity} \right. \right\}.$$
Note that $N^*/N \cong (\mathbb{Z}_2)^{l-1} \times \mathbb{Z}_8$ when $l$ is odd.

Actually, if we set $\gamma = \alpha_1 + \alpha_2 + \cdots + \alpha_l$. Then, $\gamma$ is a vector of square length $2l$ and the subgroup generated by the coset $\frac{\gamma}{4} + N$ is a cyclic group of order 8 in $N^*/N$.

From now on, we shall always assume $l$ is odd and $N \cong \sqrt{2}D_l$. First, let us consider the sublattice $R$ generated by the following eight cosets of $N$ in $L$

\[
\begin{array}{cccc}
N, & \frac{\gamma}{4} + N, & \frac{\gamma}{2} + N, & \frac{3\gamma}{4} + N, \\
\gamma + N, & \frac{5\gamma}{4} + N, & \frac{3\gamma}{2} + N, & \frac{7\gamma}{4} + N.
\end{array}
\]

Then we have

$R/N = \langle \frac{\gamma}{4} + N \rangle \cong \mathbb{Z}_8$.

For simplicity, we shall denote $L^i = i\gamma + N$ for any $i \in \mathbb{Z}_8$.

Let $R^n = R \oplus \cdots \oplus R$ be the orthogonal sum of $n$ copies of $R$. For any $\delta = (\delta_1, \ldots, \delta_n) \in \mathbb{Z}_8^n$, we define

$L_\delta = L^{\delta_1} + \cdots + L^{\delta_n} = \{(x_1, \ldots, x_n) \in R^n \mid x_i \in L^{\delta_i}, i = 0, \ldots, 7\}$.

For any subset $D \subset \mathbb{Z}_8^n$, we define

$L_D = \bigcup_{\delta \in D} L_\delta$.

**Definition 1.1.** Let $\delta = (\delta_1, \ldots, \delta_n) \in \mathbb{Z}_8^n$. The Euclidean weight of $\delta$ is defined to be

$w t \delta = \sum_{i=1}^n \min\{\delta_i^2, (8 - \delta_i)^2\} \in \mathbb{Z},$

where $\delta_i \in \{0, 1, \ldots, 7\}$ are considered as integers.

**Definition 1.2.** A linear $\mathbb{Z}_8$ code $D$ is said to be *doubly even* if

$w t \delta \equiv 0 \mod 16$

for any $\delta = (\delta_1, \ldots, \delta_n) \in D$. 
Theorem 1.3. Let $D \subset \mathbb{Z}_8$ be a doubly even $\mathbb{Z}_8$-code. Then $L_D$ is an even lattice.

Proof. Since $D$ is a linear code, it is clear that $L_D$ is closed under addition. Thus, it is a lattice.

For any $a \in L^i$, $i \in \mathbb{Z}_8$,

$$\langle a, a \rangle \equiv \left( \frac{i \gamma}{4}, \frac{i \gamma}{4} \right) \equiv \frac{l}{8} i^2 \equiv \frac{l}{8} \min\{i^2, (8 - i)^2\} \mod 2.$$ 

Thus, for any $\delta \in D$ and $x = (x_1, \ldots, x_n) \in L_\delta$,

$$\langle x, x \rangle = \sum_{i=1}^{n} \langle x_i, x_i \rangle \equiv \frac{l}{8} \sum_{i=1}^{n} \min\{\delta_i^2, (8 - \delta_i)^2\} \equiv 0 \mod 2.$$ 

Hence, $L_D$ is even. \qed

Corollary 1.4. Let $D$ be a doubly even $\mathbb{Z}_8$ code. Then the Fock space $V_{L_D} = S(\hat{h}_\mathbb{Z}) \otimes \mathbb{C}\{L_D\}$ is a vertex operator algebra. Moreover,

$$V_{L_D} = \bigoplus_{\delta \in D} \left( \bigotimes_{i=1}^{n} V_{L^i} \right)$$ as a vector space.

2. Conformal vectors

In this section, we shall recall the construction of certain conformal vectors in $V_{\sqrt{2}D_l}$ [4].

Let

$$N = \sum_{i,j=1}^{l} \mathbb{Z}(\alpha_i \pm \alpha_j)$$

be a sublattice of $L$, which is isomorphic to the root lattice of type $\sqrt{2}D_l$. We choose the following elements as the simple roots of type $D_l$:

$$\beta_1 = (\alpha_1 + \alpha_2)/\sqrt{2}, \quad \beta_2 = (-\alpha_2 + \alpha_3)/\sqrt{2}, \quad \beta_3 = (-\alpha_1 + \alpha_2)/\sqrt{2},$$

$$\beta_i = (-\alpha_i + \alpha_{i+1})/\sqrt{2} \quad \text{for} \quad 3 \leq i \leq l - 1.$$
\[ \Phi^+_l = \{ (\alpha_i + \alpha_j)/\sqrt{2}, (-\alpha_i + \alpha_j)/\sqrt{2} | 1 \leq i < j \leq l \} \]

is the set of positive roots. Let

\[ w^\pm(\beta) = \frac{1}{2} \beta(-1)^2 \pm (e^{\sqrt{2}\beta} + e^{-\sqrt{2}\beta}) \]

and set

\[ s^1 = \frac{1}{4} w^-(\beta_1), \]
\[ s^2 = \frac{1}{5} (w^-(\beta_1) + w^-(\beta_2) + w^-(\beta_1 + \beta_2)), \]
\[ s^r = \frac{1}{2r} \sum_{1 \leq i < j \leq r} \left( w^- \left( (\alpha_i + \alpha_j)/\sqrt{2} \right) + w^- \left( (-\alpha_i + \alpha_j)/\sqrt{2} \right) \right), \quad 3 \leq r \leq l, \]
\[ \omega = \frac{1}{4(l-1)} \sum_{\beta \in \Phi^+_l} \beta(-1)^2. \]

It was shown by [5] that the elements

\[ \omega^1 = s^1, \quad \omega^i = s^i - s^{i-1}, 2 \leq i \leq l, \quad \omega^{l+1} = \omega - s^l \]

are mutually orthogonal conformal vectors. Their central charges \( c(\omega^i) \) are as follows:

\[ c(\omega^1) = 1/2, \quad c(\omega^2) = 7/10, \quad c(\omega^3) = 4/5, \quad \text{and} \quad c(\omega^i) = 1 \text{ for } 4 \leq i \leq l+1. \]

The subalgebra \( \text{Vir}(\omega^i) \) of the vertex operator algebra \( V_N \) generated by \( \omega^i \) is isomorphic to the Virasoro vertex operator algebra \( L(c(\omega^i), 0) \) which is the irreducible highest weight module for the Virasoro algebra with central charge \( c(\omega^i) \) and highest weight 0. Moreover, the subalgebra \( T \) of \( V_N \) generated by these conformal vectors is a tensor product of \( \text{Vir}(\omega^i) \)'s, namely,

\[ T = \text{Vir}(\omega^1) \otimes \cdots \otimes \text{Vir}(\omega^{l+1}) \]
\[ \cong L(c(\omega^1), 0) \otimes \cdots \otimes L(c(\omega^{l+1}), 0) \]
and $V_N$ is completely reducible as a $T$-module.

Next, we shall consider three automorphisms $\theta_1$, $\theta_2$, $\sigma$ of order two of the vertex operator algebra $V_{\mathbb{Z}\alpha}$ associated with a rank one lattice $\mathbb{Z}\alpha$, where $\langle \alpha, \alpha \rangle = 2$ (cf. [3, 4]). They are determined by

$$\begin{align*}
\theta_1 &: \alpha(-1) \mapsto \alpha(-1), & e^\alpha \mapsto -e^\alpha, & e^{-\alpha} \mapsto -e^{-\alpha}, \\
\theta_2 &: \alpha(-1) \mapsto -\alpha(-1), & e^\alpha \mapsto e^{-\alpha}, & e^{-\alpha} \mapsto e^\alpha, \\
\sigma &: \alpha(-1) \mapsto e^\alpha + e^{-\alpha}, & e^\alpha + e^{-\alpha} \mapsto \alpha(-1), & e^\alpha - e^{-\alpha} \mapsto -(e^\alpha - e^{-\alpha}).
\end{align*}$$

The automorphism $\theta_1$ maps $u \otimes e^\beta$ to $(-1)^{\langle \alpha_1 + \alpha_2 + \cdots + \alpha_l, \beta \rangle / 2} u \otimes e^\beta$ for $u \in M(1)$ and $\beta \in \mathbb{Z}\alpha$ and $\theta_2$ is the automorphism induced from the isometry $\beta \mapsto -\beta$ of $\mathbb{Z}\alpha$. Note also that

$$\sigma\theta_1\sigma = \theta_2, \quad \sigma(\alpha(-1)^2) = (\alpha(-1))^2 \quad \text{and} \quad \sigma(e^{\pm\alpha}) = (\alpha(-1) \mp (e^\alpha - e^{-\alpha}))/2.$$

Let $L$ be a lattice with basis $\{\alpha_1, \alpha_2, \ldots, \alpha_l\}$ such that $\langle \alpha_i, \alpha_j \rangle = 2\delta_{ij}$. Then, the vertex operator algebra $V_L$ is a tensor product $V_L = V_{\mathbb{Z}\alpha_1} \otimes \cdots \otimes V_{\mathbb{Z}\alpha_l}$ of $V_{\mathbb{Z}\alpha_i}$'s. Using the automorphisms $\theta_1$, $\theta_2$, and $\sigma$ of $V_{\mathbb{Z}\alpha_i}$ described above, we can define three automorphisms $\psi_1$, $\psi_2$, and $\tau$ of $V_L$ of order two by

$$\psi_1 = \theta_1 \otimes \cdots \otimes \theta_1, \quad \psi_2 = \theta_2 \otimes \cdots \otimes \theta_2, \quad \tau = \sigma \otimes \cdots \otimes \sigma.$$

Then

$$\psi_1(u \otimes e^\beta) = (-1)^{\langle \alpha_1 + \alpha_2 + \cdots + \alpha_i, \beta \rangle / 2} u \otimes e^\beta$$

for $u \in M(1)$ and $\beta \in L$, $\psi_2$ is the automorphism induced from the isometry $\beta \mapsto -\beta$ of $L$, and $\tau\psi_1\tau = \psi_2$.

Let $\varphi : V_L \to V_L$ be an automorphism defined by

$$\varphi : u \otimes e^\beta \mapsto (-1)^{\langle \alpha_2 + \alpha_3, \beta \rangle / 2} u \otimes e^\beta,$$

where $u \in M(1)$ and $\beta \in L$. The automorphism $\varphi$ acts as $\theta_2$ on $V_{\mathbb{Z}\alpha_2}$ and $V_{\mathbb{Z}\alpha_3}$ and acts as the identity on $V_{\mathbb{Z}\alpha_i}$ for $i \neq 2, 3$. Set $\rho = \varphi\tau$. Then we have
Lemma 2.1 (Dong at. el. [4]).

\[
\rho(s^1) = \frac{1}{4}w^{-}(\beta_3), \quad \rho(s^2) = \frac{1}{3}(w^{-}(\beta_3) + w^{-}(\beta_2) + w^{-}(\beta_2 + \beta_3)), \\
\rho(s^r) = \tau(s^r), \quad 3 \leq r \leq l, \quad \rho(\omega) = \omega.
\]

Let \( \tilde{\omega}^i = \rho(\omega^i) \) and set

\[
\gamma_r = \alpha_1 + \alpha_2 + \cdots + \alpha_r - r\alpha_{r+1}, \quad 1 \leq r \leq l - 1,
\]

\[
(2.5) \quad \gamma_l = \alpha_1 + \alpha_2 + \cdots + \alpha_l.
\]

Lemma 2.2 (cf. [4]).

1. The vectors \( \tilde{\omega}^1, \tilde{\omega}^2, \) and \( \tilde{\omega}^3 \) are the mutually orthonormal conformal vectors of \( V_{\mathbb{Z}(\alpha_1-\alpha_2)+\mathbb{Z}(\alpha_2-\alpha_3)} \cong V_{\sqrt{2}A_2} \) defined in [5].

2. \( \tilde{\omega}^{r+1} = \frac{1}{4r(r+1)} \gamma_r (-1)^2 \) for \( 3 \leq r \leq l - 1. \)

3. \( \tilde{\omega}^{l+1} = \frac{1}{4l} \gamma_l (-1)^2. \)

Note that for \( 3 \leq r \leq l, \) the element \( \tilde{\omega}^{r+1} \) is the Virasoro element of the vertex operator algebra \( V_{\mathbb{Z}^{\gamma_r}} \) associated with a rank one lattice \( \mathbb{Z}^{\gamma_r}. \)

Set \( U^\pm = \{ v \in U \mid \psi_2(v) = \pm v \} \) for any \( \psi_2 \)-invariant subspace \( U \) of \( V_L. \)

Lemma 2.3 (cf. [4]).

1. \( N = \{ \beta \in L \mid \langle \alpha_1 + \cdots + \alpha_l, \beta \rangle \equiv 0 \pmod{4} \}. \)

2. \( V_N = \{ v \in V_L \mid \psi_1(v) = v \}. \)

3. \( \rho(V_N) = V_L^+ . \)

The last assertion of the above lemma implies that the decomposition of \( V_N \) into a direct sum of irreducible \( T \)-modules is equivalent to that of \( V_L^+ \) as a \( \tilde{T} \)-module, where \( \tilde{T} = \rho(T) \) is of the form

\[
\tilde{T} = \text{Vir}(\tilde{\omega}^1) \otimes \cdots \otimes \text{Vir}(\tilde{\omega}^{l+1}) \cong L(\frac{1}{2},0) \otimes L(\frac{7}{10},0) \otimes L(\frac{4}{5},0) \otimes L(1,0) \otimes \cdots \otimes L(1,0).
\]
Next, we shall study the decomposition of $V_N \cong V_L^+$. Details are again written in [3, 4]. As in [3, 4], we set
\[ E = \mathbb{Z}(\alpha_1 - \alpha_2) + \mathbb{Z}(\alpha_2 - \alpha_3) \quad \text{and} \quad D = E + \mathbb{Z}\gamma_3 + \cdots + \mathbb{Z}\gamma_l. \]
The elements $\alpha_1 - \alpha_2, \alpha_2 - \alpha_3, \gamma_3, \ldots, \gamma_l$ form a basis of the lattice $D$. Since $\gamma_r = \sum_{i=1}^{r} i(\alpha_i - \alpha_{i+1})$, we can take
\[ \{\alpha_1 - \alpha_2, \alpha_2 - \alpha_3, 3(\alpha_3 - \alpha_4), \ldots, (l-2)(\alpha_{l-2} - \alpha_{l-1}), \gamma_{l-1}, \gamma_l\} \]
as another basis. The lattices $E$, $\mathbb{Z}\gamma_3$, $\mathbb{Z}\gamma_l$ are mutually orthogonal, so the vertex operator algebra $V_D$ associated with the lattice $D$ is a tensor product $V_D = V_E \otimes V_{\mathbb{Z}\gamma_3} \otimes \cdots \otimes V_{\mathbb{Z}\gamma_l}$.

Next, we want to describe the cosets of $D$ in $L$. Set
\[ \xi_r = \frac{1}{r(r+1)} \gamma_r, \quad 1 \leq r \leq l-1, \quad \text{and} \quad \xi_l = \frac{1}{l} \gamma_l. \]
Then we have
\[ -\xi_1 + \xi_2 = \frac{1}{3}(- (\alpha_1 - \alpha_2) + (\alpha_2 - \alpha_3)). \]
and
\[ -\xi_1 + \xi_2 + \cdots + \xi_l = \alpha_2. \]

To simplify the notation, we set $\eta = -\xi_1 + \xi_2$.

**Lemma 2.4.** $|D + \mathbb{Z}\alpha_2 : D|$ is equal to the least common multiple of $3, 4, \ldots, l$.

Note that $D + \mathbb{Z}\alpha_2 = L$ for $3 \leq l \leq 5$. Indeed, the coset $D + \alpha_2$ contains $\alpha_1, \alpha_2,$ and $\alpha_3$. Moreover, $\alpha_4 \in D + 9\alpha_2$ if $l = 4$, and $\alpha_4 \in D + 21\alpha_2$ and $\alpha_5 \in D + 36\alpha_2$ if $l = 5$. Hence $n\alpha_2, 0 \leq n \leq d - 1$, where $d$ denotes the least common multiple of $3, 4, \ldots, l$, form a complete system of representatives of the cosets of $D$ in $L$ in these three cases. However, $D + \mathbb{Z}\alpha_2 \neq L$ for $l \geq 6$. 

We shall use the following elements to describe all the cosets of $D$ in $L$. For $m_3, \ldots, m_{l-2}, n \in \mathbb{Z}$ we let
\[
\lambda = \lambda(m_3, m_4, \ldots, m_{l-2}, n) = m_3(\alpha_3 - \alpha_4) + m_4(\alpha_4 - \alpha_5) + \cdots + m_{l-2}(\alpha_{l-2} - \alpha_{l-1}) + n\alpha_2
\]
(2.8)
\[
\equiv (m_3 + n)\eta + \sum_{r=3}^{l-3}((r+1)m_r - rm_{r+1} + n)\xi_r + ((l-1)m_{l-2} + n)\xi_{l-2} + n\xi_{l-1} + n\xi_l \quad \text{(mod } D\text{).}
\]
The last congruence modulo $D$ comes from (4.1), (4.2), and the fact that $\alpha_r - \alpha_{r+1} = -(r-1)\xi_{r-1} + (r+1)\xi_r$.

Lemma 2.5. (1) $\{\lambda = \lambda(m_3, \ldots, m_{l-2}, n)| 0 \leq m_r \leq r - 1, 0 \leq n \leq l(l-1)-1\}$ forms a complete system of representatives of the cosets of $D$ in $L$.
(2) Every element in the coset $D + \lambda$ can be uniquely written in the form
\[
(\nu + (m_3 + n)\eta) + \sum_{i=3}^{l-3}(\mu_i + ((i+1)m_i - im_{i+1} + n)\xi_i)
\]
\[
+ (\mu_{l-2} + ((l-1)m_{l-2} + n)\xi_{l-2}) + (\mu_{l-1} + n\xi_{l-1}) + (\mu_l + n\xi_l)
\]
for $\nu \in E$ and $\mu_i \in \mathbb{Z}\gamma_i$.

Lemma 2.6. For $\lambda = \lambda(m_3, \ldots, m_{l-2}, n), 0 \leq m_r \leq r - 1, 0 \leq n \leq l(l-1)-1, we have $D + \lambda = D - \lambda$ if and only if $m_3, \ldots, m_{l-2}$, and $n$ satisfy one of the following conditions.
(1) $n = 0$, and $m_r = 0$ if $r$ is odd and $m_r = 0$ or $r/2$ if $r$ is even.
(2) $n = l(l-1)/2$ and $2m_r + l(l-1) \equiv 0 \pmod{r}$. Such an $m_r$ is unique if $r$ is odd and there are exactly two such $m_r$ if $r$ is even.

The automorphism $\psi_2$ fixes the conformal vectors $\tilde{\omega}^1, \ldots, \tilde{\omega}^{l+1}$, and so $\tilde{T} \subset V_D^+$. In particular, $\psi_2$ is a $\tilde{T}$-module isomorphism. We have $\psi_2(V_{D+\lambda}) = V_{D-\lambda}$, and thus $V_{D-\lambda}$ is isomorphic to $V_{D+\lambda}$ as a $\tilde{T}$-module. If $D+\lambda \neq D-\lambda$, the fixed
point subspace \((V_{D+\lambda} \oplus V_{D-\lambda})^+\) in \(V_{D+\lambda} \oplus V_{D-\lambda}\) is equal to \(\{v + \psi_2(v) \mid v \in V_{D+\lambda}\}\) and it is isomorphic to \(V_{D+\lambda}\).

If \(D + \lambda = D - \lambda\) for \(\lambda = (m_3, \ldots, m_{l-2}, n)\), then \(m_3, \ldots, m_{l-2},\) and \(n\) satisfy the conditions in Lemma 4.3. In this case \((4.5)\) is in the following form:

\[
V_{D+\lambda} = V_E \otimes V_{Z\gamma_3 + b_3\gamma_3} \otimes \cdots \otimes V_{Z\gamma_l + b_l\gamma_l},
\]

with \(b_i \in \{0, \frac{1}{2}\}\) for \(3 \leq i \leq l - 1\) and \(b_l = 0\) or \(b_l \in \{0, \frac{1}{2}\}\) depending on whether \(l\) is odd or even.

For \(\epsilon = (\epsilon_0, \epsilon_3, \ldots, \epsilon_l)\) with \(\epsilon_i = +\) or \(-\), set

\[
(2.9) \quad V^\epsilon_{D+\lambda} = V^{\epsilon_0}_E \otimes V^{\epsilon_3}_{Z\gamma_3 + b_3\gamma_3} \otimes \cdots \otimes V^{\epsilon_l}_{Z\gamma_l + b_l\gamma_l}.
\]

Then

\[
(2.10) \quad V^+_{D+\lambda} = \oplus_{\epsilon} V^\epsilon_{D+\lambda},
\]

where \(\epsilon\) runs over all \(\epsilon = (\epsilon_0, \epsilon_3, \ldots, \epsilon_l)\) such that even number of \(\epsilon_i\)'s are \(-\).

We divide a complete system of representatives of the cosets of \(D\) in \(L\) into three subsets \(\Lambda_1, \Lambda_2,\) and \(-\Lambda_2\) so that \(D + \lambda = D - \lambda\) if and only if \(\lambda \in \Lambda_1\).

Then

\[
V_L = \left(\oplus_{\lambda \in \Lambda_1} V_{D+\lambda}\right) \oplus \left(\oplus_{\lambda \in \Lambda_2} V_{D+\lambda}\right) \oplus \left(\oplus_{\lambda \in \Lambda_2} V_{D-\lambda}\right).
\]

By the above argument, we conclude that

**Theorem 2.7** (Dong at.el. [4]). As \(\tilde{T}\)-modules, \(V^+_L \cong \left(\oplus_{\lambda \in \Lambda_1} V^+_{D+\lambda}\right) \oplus \left(\oplus_{\lambda \in \Lambda_2} V_{D+\lambda}\right)\). Furthermore, the decomposition of \(V^+_{D+\lambda}, \lambda \in \Lambda_1,\) and \(V_{D+\lambda}, \lambda \in \Lambda_2,\) into a direct sum of irreducible \(\tilde{T}\)-modules is given by \((4.5)\) through \((4.10)\) and \((5.1)\) through \((5.7)\).

By the same method, we also have the decomposition of \(V^-_L \cong \left(\oplus_{\lambda \in \Lambda_1} V^-_{D+\lambda}\right) \oplus \left(\oplus_{\lambda \in \Lambda_2} V_{D+\lambda}\right)\) into a direct sum of irreducible \(\tilde{T}\)-modules.
3. Coset construction of vertex operator algebra

In this section, we shall use the decomposition obtained in the last section and the coset construction to construct some vertex operator algebras. First, we shall recall the definition of commutant (or coset) subalgebras of a vertex operator algebra (cf. [7]).

**Definition 3.1.** Let \((V, Y, \omega, 1)\) be a vertex operator algebra and \((W, Y, \omega', 1)\) be a vertex operator subalgebra of \(V\). Note that the Virasoro elements of \(V\) and \(W\) are different. The commutant of \(W\) in \(V\) is defined to be the subspace

\[
W^c = \{ v \in V | w_n v = 0, \text{ for all } w \in W \text{ and } n \geq 0 \}
\]

Similarly, for any \(V\)-module \(M\), the commutant of \(W\) in \(M\) is defined to be

\[
M^c = \{ u \in M | w_n u = 0, \text{ for all } w \in W \text{ and } n \geq 0 \}
\]

The following facts are well-known in the theory of vertex operator algebra (cf. [7], for example).

**Proposition 3.2.** \((W^c, Y, \omega'', 1)\) is a vertex operator algebra where \(\omega'' = \omega - \omega'\) and \(M^c\) is a \(W^c\)-module for any \(V\)-module \(M\).

Now, let \(L^i = \frac{i}{4} \gamma + N\) be defined as in Section 1. Denote

\[
M^i = \{ v \in V_{L^i} | (\omega^1)v = (\omega^2)v = \cdots = (\omega^i)v = 0 \},
\]

where \(\omega^i, i = 0, \ldots, 7\), are defined as in (2.3). Note that \(M^0\) is a VOA and \(M^i, i = 0, \ldots, 7\), are \(M^0\)-modules.

For any \(\delta = (\delta_1, \ldots, \delta_n) \in \mathbb{Z}^n\), we define

\[
M_\delta = \bigotimes_{i=1}^{n} M^{\delta_i}.
\]
For any $\mathbb{Z}_8$ code $D$, define

$$M_D = \bigoplus_{\delta \in D} M_\delta.$$ 

**Theorem 3.3.** If $D$ is a doubly even $\mathbb{Z}_8$ code, then $M_D$ is a vertex operator algebra.

**Proof.** Let $D$ be a doubly even $\mathbb{Z}_8$ code. Then $L_D$ is an even lattice and

$$V_{L_D} = \bigoplus_{\delta \in D} (V_{L_{\delta_1}} \otimes \cdots \otimes V_{L_{\delta_l}})$$

is a VOA (cf. Section 1). Note that

$$M_D = \{v \in V_{L_D} | (\hat{\omega}^1)_1 v = (\hat{\omega}^2)_1 v = \cdots (\hat{\omega}^l)_1 v = 0\},$$

where $\hat{\omega}^i = \omega^i \otimes 1 \otimes \cdots \otimes 1 + 1 \otimes \omega^i \otimes \cdots \otimes 1 + \cdots + 1 \otimes 1 \otimes \cdots \otimes \omega^i$, $i = 1, \ldots, l$.

Thus, $M_D$ is a vertex operator subalgebra of $V_{L_D}$. \hfill \square

**Remark 3.4.** As in [8, 11], one can define the so-called coordinate automorphisms for $M_D$ as follows:

For any $\alpha \in (\mathbb{Z}_8^*)^n$,

$$\sigma_\alpha (u) = \xi^{(\alpha, \beta)} u \quad \text{for } u \in M_\beta,$$

where $\xi$ is a primitive 8-th root of unity and $\mathbb{Z}_8^* = \{1, 3, 5, 7\}$ is the units of the ring $\mathbb{Z}_8$.

4. **The case for $l = 3$**

In this section, we shall explain the above construction for the case $l = 3$ in more details. The other cases, in principle, can be done in a similar way.

Let $L = \mathbb{Z}\alpha_1 \oplus \mathbb{Z}\alpha_2 \oplus \mathbb{Z}\alpha_3$ and $N = \sum_{i,j=1}^3 \mathbb{Z}(\alpha_i \pm \alpha_j)$. Moreover, we shall denote $E = \mathbb{Z}(\alpha_1 - \alpha_2) \oplus \mathbb{Z}(\alpha_2 - \alpha_3)$ and $F = \mathbb{Z}\gamma$, where $\gamma = \alpha_1 + \alpha_2 + \alpha_3$. 
Theorem 4.1 (Kitazume at.el. [9]). Let

$$N = \sum_{i,j=1}^{3} \mathbb{Z}(\alpha_i \pm \alpha_j) \cong \sqrt{2}D_3 \cong \sqrt{2}A_3.$$  

Then

$$V_N \cong (V_E^+ \otimes V_F^+) \oplus (V_E^- \otimes V_F^-) \oplus \left(V_{E+\sqrt{2}(\beta_1-\beta_2)/3}^+ \otimes V_{3,1}^F\right),$$

$$V_{\frac{1}{4}\gamma+N} \cong (V_E^{T,-} \otimes V_{F,T_1}^+) \oplus (V_E^{T,+} \otimes V_{F,T_1}^-),$$

$$V_{\frac{1}{2}\gamma+N} \cong (V_E^+ \otimes V_{2,+}^F) \oplus (V_E^- \otimes V_{2,-}^F) \oplus (V_{E+\sqrt{2}(\beta_1-\beta_2)/3}^+ \otimes V_{3,1}^F),$$

$$V_{\gamma+N} \cong (V_E^+ \otimes V_{F}^-) \oplus (V_E^- \otimes V_{F}^+) \oplus \left(V_{E+\sqrt{2}(\beta_1-\beta_2)/3}^+ \otimes V_{3,1}^F\right),$$

where $V_E^T = S(h_{Z+\frac{1}{2}}) \otimes T$ is a $\psi_2$-twisted module of $V_E$ and $T$ is an irreducible $\hat{E}/K$ module such that $e^a \cdot t = t$ for $a \in E$, $t \in T$, and $K = \{ \pm e^b \, | \, b \in 2E \}$ is a central extension of $2E$, and $V_{F,T_1}^+$ and $V_{F,T_2}^+$ are the two inequivalent irreducible $\psi_2$-twisted modules for $V_F$.

Theorem 4.2 (cf. [9]). Let

$$M^i = \{ v \in V_{L,i} \mid (\omega^1)_1 v = (\omega^2)_1 v = 0 \}, \text{ and}$$

$$W^i = \{ v \in V_{L,i} \mid (\omega^1)_1 v = 0 \text{ and } (\omega^2)_1 v = \frac{3}{5} v \}.$$

Then, $M^0$ is a simple VOA and $M^i$ and $W^i$ are irreducible $M^0$-modules.
Moreover, by Theorem 4.1,

\[ M^0 = (L(\frac{4}{5},0) \otimes V_F^+) \oplus (L(\frac{4}{5},3) \otimes V_F^-) \oplus (L(\frac{4}{5},\frac{2}{3}) \otimes V_3+F), \]

\[ M^1 = (L(\frac{4}{5},\frac{1}{8}) \otimes V_F^{T_1-}) \oplus (L(\frac{4}{5},\frac{13}{8}) \otimes V_F^{T_1+}), \]

\[ M^2 = (L(\frac{4}{5},0) \otimes V_F^+) \oplus (L(\frac{4}{5},3) \otimes V_F^-) \oplus (L(\frac{4}{5},\frac{2}{3}) \otimes V_3+F), \]

\[ M^3 = (L(\frac{4}{5},\frac{1}{8}) \otimes V_F^{T_2-}) \oplus (L(\frac{4}{5},\frac{13}{8}) \otimes V_F^{T_2+}), \]

\[ M^4 = (L(\frac{4}{5},0) \otimes V_F^-) \oplus (L(\frac{4}{5},3) \otimes V_F^+) \oplus (L(\frac{4}{5},\frac{2}{3}) \otimes V_3+F), \]

\[ M^5 = (L(\frac{4}{5},\frac{1}{8}) \otimes V_F^{T_1-}) \oplus (L(\frac{4}{5},\frac{13}{8}) \otimes V_F^{T_1+}), \]

\[ M^6 = (L(\frac{4}{5},0) \otimes V_F^-) \oplus (L(\frac{4}{5},3) \otimes V_F^+) \oplus (L(\frac{4}{5},\frac{2}{3}) \otimes V_3+F), \]

\[ M^7 = (L(\frac{4}{5},\frac{1}{8}) \otimes V_F^{T_2-}) \oplus (L(\frac{4}{5},\frac{13}{8}) \otimes V_F^{T_2+}), \]

\[ M^0 = (L(\frac{4}{5},0) \otimes V_F^+) \oplus (L(\frac{4}{5},3) \otimes V_F^-) \oplus (L(\frac{4}{5},\frac{2}{3}) \otimes V_3+F), \]

\[ M^1 = (L(\frac{4}{5},\frac{1}{8}) \otimes V_F^{T_1-}) \oplus (L(\frac{4}{5},\frac{13}{8}) \otimes V_F^{T_1+}), \]

\[ M^2 = (L(\frac{4}{5},0) \otimes V_F^+) \oplus (L(\frac{4}{5},3) \otimes V_F^-) \oplus (L(\frac{4}{5},\frac{2}{3}) \otimes V_3+F), \]

\[ M^3 = (L(\frac{4}{5},\frac{1}{8}) \otimes V_F^{T_2-}) \oplus (L(\frac{4}{5},\frac{13}{8}) \otimes V_F^{T_2+}), \]

\[ M^4 = (L(\frac{4}{5},0) \otimes V_F^-) \oplus (L(\frac{4}{5},3) \otimes V_F^+) \oplus (L(\frac{4}{5},\frac{2}{3}) \otimes V_3+F), \]

\[ M^5 = (L(\frac{4}{5},\frac{1}{8}) \otimes V_F^{T_1-}) \oplus (L(\frac{4}{5},\frac{13}{8}) \otimes V_F^{T_1+}), \]

\[ M^6 = (L(\frac{4}{5},0) \otimes V_F^-) \oplus (L(\frac{4}{5},3) \otimes V_F^+) \oplus (L(\frac{4}{5},\frac{2}{3}) \otimes V_3+F), \]

\[ M^7 = (L(\frac{4}{5},\frac{1}{8}) \otimes V_F^{T_2-}) \oplus (L(\frac{4}{5},\frac{13}{8}) \otimes V_F^{T_2+}), \]

\[ W^0 = (L(\frac{4}{5},\frac{7}{5}) \otimes V_F^+) \oplus (L(\frac{4}{5},\frac{2}{5}) \otimes V_F^-) \oplus (L(\frac{4}{5},\frac{1}{15}) \otimes V_3+F), \]

\[ W^1 = (L(\frac{4}{5},\frac{7}{5}) \otimes V_F^+) \oplus (L(\frac{4}{5},\frac{2}{5}) \otimes V_F^-) \oplus (L(\frac{4}{5},\frac{1}{15}) \otimes V_3+F), \]

\[ W^2 = (L(\frac{4}{5},\frac{7}{5}) \otimes V_F^+) \oplus (L(\frac{4}{5},\frac{2}{5}) \otimes V_F^-) \oplus (L(\frac{4}{5},\frac{1}{15}) \otimes V_3+F), \]

\[ W^3 = (L(\frac{4}{5},\frac{7}{5}) \otimes V_F^+) \oplus (L(\frac{4}{5},\frac{2}{5}) \otimes V_F^-) \oplus (L(\frac{4}{5},\frac{1}{15}) \otimes V_3+F), \]

\[ W^4 = (L(\frac{4}{5},\frac{7}{5}) \otimes V_F^+) \oplus (L(\frac{4}{5},\frac{2}{5}) \otimes V_F^-) \oplus (L(\frac{4}{5},\frac{1}{15}) \otimes V_3+F), \]

\[ W^5 = (L(\frac{4}{5},\frac{7}{5}) \otimes V_F^+) \oplus (L(\frac{4}{5},\frac{2}{5}) \otimes V_F^-) \oplus (L(\frac{4}{5},\frac{1}{15}) \otimes V_3+F), \]

\[ W^6 = (L(\frac{4}{5},\frac{7}{5}) \otimes V_F^+) \oplus (L(\frac{4}{5},\frac{2}{5}) \otimes V_F^-) \oplus (L(\frac{4}{5},\frac{1}{15}) \otimes V_3+F), \]

\[ W^7 = (L(\frac{4}{5},\frac{7}{5}) \otimes V_F^+) \oplus (L(\frac{4}{5},\frac{2}{5}) \otimes V_F^-) \oplus (L(\frac{4}{5},\frac{1}{15}) \otimes V_3+F), \]
as $L(4/5, 0) \otimes V^+_F$-modules.

**Remark 4.3.** In fact, one can show that $M^i$ and $W^i$, $i = 0, \ldots, 7$, are exactly all the inequivalent irreducible modules for $M^0$. Moreover, the fusion rules for $M^0$-modules are given as:

\[
M^i \times M^j = M^{i+j},
\]

\[
M^i \times W^j = W^{i+j},
\]

\[
W^i \times W^j = M^{i+j} + W^{i+j},
\]

where $i, j \in \mathbb{Z}_8$.

Now, let us discuss the construction some irreducible $M_D$-modules using induced modules.

Let $U = U^{\delta_1} \otimes \cdots \otimes U^{\delta_n}$ be an irreducible $(M^0)^{\otimes n}$-module such that $U^{\delta_i} = M^{\delta_i}$ or $W^{\delta_i}$, $\delta_i = 0, 1, \ldots, 7$.

Define

\[
\text{Ind}^D U = \bigoplus_{\alpha \in D} (U^{\alpha_1 + \delta_1} \otimes \cdots \otimes U^{\alpha_n + \delta_n}),
\]

where $U^{\alpha_1 + \delta_1} = M^{\alpha_1 + \delta_1}$ (or $W^{\alpha_1 + \delta_1}$ respectively) if $U^{\delta_i} = M^{\delta_i}$ (or $W^{\delta_i}$ respectively). $\text{Ind}^D U$ is called an induced module.

**Theorem 4.4.** If $(\delta, D) = 0$, then $\text{Ind}^D U$ is an $M_D$-module.

**Proof.** First, we shall note that $U = U^{\delta_1} \otimes \cdots \otimes U^{\delta_n}$ can be considered as a subset of $V_{L_\delta} \cong \otimes_{i=1}^n V_{L^{\delta_i}}$ for any $\delta \in \mathbb{Z}_8^n$. Therefore,

\[
\text{Ind}^D U = \bigoplus_{\alpha \in D} (U^{\alpha_1 + \delta_1} \otimes \cdots \otimes U^{\alpha_n + \delta_n}) \subset V_{L_{\delta+D}}.
\]

If $(\delta, D) = 0$, then $(L_D, L_{\delta+D}) \subset \mathbb{Z}$. Thus, $V_{L_{\delta+D}}$ is a $V_{L_D}$-module. Note that $M_D$ is a subVOA of $V_{L_D}$ and the action of $M_D$ on $\text{Ind}^D U$ is closed. Thus, $\text{Ind}^D U$ is a $M_D$-module. \qed
Remark 4.5. If $(\delta, D) \neq 0$, we believe that $\text{Ind}^D U$ will still define a $g$-twisted module of $M_D$, where $g$ is an automorphism of $M_D$ such that

$$g(u) = \xi^{(\delta, \alpha)} u, \quad \text{for } u \in M_\alpha \text{ and } \alpha \in D,$$

and $\xi$ is a primitive 8-th root of unity.

Remark 4.6. Suppose $V_F^+$ is rational. Then one can show that $\text{Ind}^D U$ is an irreducible $M_D$-module. Moreover, all irreducible $M_D$-modules are induced modules and $M_D$ is rational (cf. [9]).

REFERENCES


**DEPARTMENT OF MATHEMATICS, NATIONAL CHENG KUNG UNIVERSITY, TAINAN, TAIWAN 701**

*E-mail address: chlam@math.ncku.edu.tw*