数理解析研究所講究録 1219 巻 2001 年 206-220

Quantum Ergodicity of Eisenstein series for Arithmetic 3-Manifolds

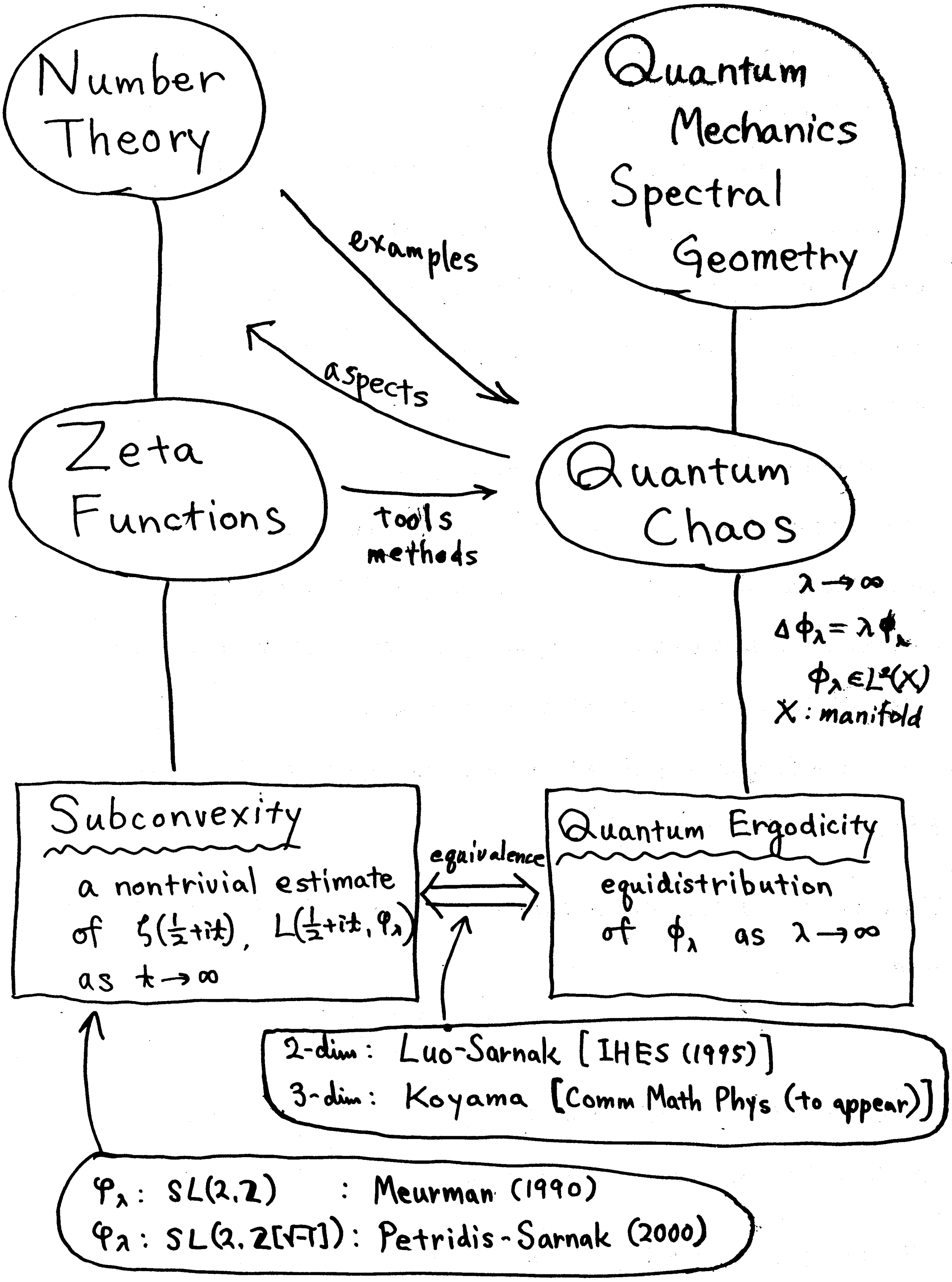
慶應大理工 小山信也 (Shin-ya Koyama)

Mathematics Department, Keio University

Abstract. We prove the quantum ergodicity for Eisenstein series for $PSL(2, O_K)$, where O_K is the integer ring of an imaginary quadratic field K of class number one.

1. Introduction. We first explain the whole picture around the two fields, number theory and quantum chaos. Number theorists study the theory of zeta functions, where one of their chief concerns is to estimate the size of zeta functions such as $|\zeta(\frac{1}{2}+it)|$ along the critical line. A trivial estimate can be obtained from the convexity principle in the general theory of complex functions. We call it the convexity bound. Any estimate breaking the convexity bound is called a subconvexity bound. To obtain any subconvexity estimate is significant in number theory.

On the other hand in quantum mechanics or spectral geometry, there is a field called quantum chaos, where they study various problems as $\lambda \to \infty$, where λ is an eigenvalue of some self-adjoint operator Δ. Typically Δ is the Laplacian on $L^2(X)$ with X a Riemannian manifold. In such settings one of their interests is the asymptotic behavior of the eigenfunctions ϕ_λ. Quantum ergodicity means that they become equidistributed as $\lambda \to \infty$. We call ϕ_λ the Maass cusp form, especiall when X is an arithmetic manifold. If X is noncompact, there also appear continuous spectra, and we also regard ϕ_λ as the real analytic Eisenstein series. Maass cusp forms and Eisenstein series are central objects in number theory. In this manner quantum chaos presents a new aspect to number theory, as well as number theory


Number Theory
Quantum Mechanics Spectral Geometry
examples
aspects
Zeta Functions
tools methods
Quantum Chaos
λ → ∞
Δφ_λ = λφ_λ
φ_λ ∈ L²(X)
X : manifold
Subconvexity
a nontrivial estimate of ζ(1/2+it), L(1/2+it, φ_λ) as t → ∞
equivalence
Quantum Ergodicity
equidistribution of φ_λ as λ → ∞
2-dim : Luo-Sarnak [IHES (1995)]
3-dim : Koyama [Comm Math Phys (to appear)]
φ_λ : SL(2,Z) : Meurman (1990)
φ_λ : SL(2,Z[√-1]) : Petridis-Sarnak (2000)

gives good examples, strong tools and methods to the theory of quantum chaos.

One of remarkable facts connecting these two areas is the equivalence of subconvexity and quantum ergodicity. More precisely, the quantum ergodicity of real analytic Eisenstein series is equivalent to a subconvexity of the automorphic L-function for Maass cusp forms for the arithmetic manifold. This equivalence was discovered by Luo and Sarnak [LS] for $PSL(2,\mathbf{Z})$. The main theorem in this article is its generalization to $PSL(2,O)$ where O is the integer ring of an imaginary quadratic field.

In the case of $PSL(2,\mathbf{Z})$, a subconvexity bound is obtained by Meurman [M]. It has been considered to be a hard problem to generalize it to higher dimensional cases, but Sarnak and Petridis [SP] recently did it successfully. By using their remarkable result, the quantum ergodicity is proved for three dimensional cases.

In what follows we will describe more precisely.

Luo and Sarnak [LS] proved the quantum ergodicity of Eisenstein series for $PSL(2,\mathbf{Z})$. It is stated as follows:

Theorem 1.1. *Let A, B be compact Jordan measurable subsets of $PSL(2,\mathbf{Z})\backslash H^2$, then*

$$\lim_{t\to\infty}\frac{\mu_t(A)}{\mu_t(B)}=\frac{\mathrm{Vol}(A)}{\mathrm{Vol}(B)},$$

where $\mu_t=|E(z,\frac{1}{2}+it)|^2dV$ with $E(z,s)$ being the Eisenstein series for $PSL(2,\mathbf{Z})$, and dV is the volume element of the upper half plane H^2.

In this paper we will generalize Theorem 1.1 to three dimensional cases $X=PSL(2,O_K)\backslash H^3$, where O_K is the integer ring of an imaginary quadratic field K of class number one, and H^3 is the three dimensional upper half space. Our main theorem is analogously described as follows:

Theorem 1.2. *Let A, B be compact Jordan measurable subsets of X, then*

$$\lim_{t\to\infty}\frac{\mu_t(A)}{\mu_t(B)}=\frac{\mathrm{Vol}(A)}{\mathrm{Vol}(B)},$$

where $\mu_t=|E(v,1+it)|^2dV$ with $E(v,s)$ being the Eisenstein series for X, and dV is the volume element of H^3.

Indeed we show that as $t\to\infty$,

$$\mu_t(A)\sim\frac{2\mathrm{Vol}(A)}{\zeta_K(2)}\log t,$$

where $\zeta_K(s)$ is the Dedekind zeta function.

In two dimensional cases numerical examples [HR] suggested that the quantum ergodicity would hold. For higher dimensional cases no numerical examples are known. Theorem 1.2 is the first result along this direction.

The author would like to express his thanks to Professor Peter Sarnak, who introduced the author to the subject.

2. Three-Dimensional Settings. In this section we introduce some notation on the three-dimensional hyperbolic space.

A point in the hyperbolic three-dimensional space H^3 is denoted by $v=z+yj$, $z=x_1+x_2i\in\mathbf{C}$, $y>0$. We fix an imaginary quadratic field K whose class number is one. Denote its discriminant by D_K and integer ring $O=O_K$. Put $D=|D_K|$. We often regard O as a lattice in $\mathbf{R}^2$, which is denoted by L with the fundamental domain $F_L\subset\mathbf{R}^2$. Also put $\omega=\omega_K=D^{-1/2}$, the inverse different of K. The group $\Gamma=PSL(2,O)$ acts on H^3 and the quotient space $X=\Gamma\backslash H^3$ is a three dimensional arithmetic hyperbolic orbifold. The Laplacian on X is defined by

$$\Delta=-y^2\left(\frac{d^2}{dx_1^2}+\frac{d^2}{dx_2^2}+\frac{d^2}{dy^2}\right)+y\frac{d}{dy}.$$

It has a self-adjoint extension on $L^2(X)$. It is known that the spectra of Δ is composed of both discrete and continuous ones. The eigenfunction for a discrete spectrum is called a cusp form. We denote it by $\phi_j(v)$ with eigenvalue λ_j $(0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots)$. We put $\lambda_j = 1 + r_j^2$. We shall assume the $\phi_j(v)$'s to be chosen so that they are eigenfunctions of the ring of Hecke operators and are L^2-normalized. The Fourier development of $\phi_j(v)$ is given in [S] (2.20):

$$\phi_j(v) = \sum_{n \in O^*/\sim} \rho_j(n) y K_{ir_j}(2\pi |n| y) e(\langle n, z \rangle), \tag{2.1}$$

where $n \sim m$ means that they generate the same ideal in O, and $\langle n, z \rangle$ is the standard inner product in $\mathbf{R}^2$ with K_ν being the K-Bessel function.

For a Maass-Hecke cusp form $\phi_j(v)$ with its Fourier development given by (2.1), we have the Rankin-Selberg convolution L-function $L(s, \phi_j \times \phi_j)$ and the second symmetric power L-function $L^{(2)}(s, \phi_j)$ which satisfy the following:

$$L(s, \phi_j \times \phi_j) = \zeta_K(2s) \sum_{n \in O^*/\sim} \frac{|\lambda_j(n)|^2}{N(n)^s}$$

$$L^{(2)}(s, \phi_j) = \sum_{n \in O^*/\sim} \frac{c_j(n)}{N(n)^s} = \zeta_K(s)^{-1} L(s, \phi_j \times \phi_j),$$

with $\rho_j(n) = \sqrt{\frac{\sinh \pi r_j}{r_j}} v_j(n)$, $v_j(n) = v_j(1)\lambda_j(n)$ and $c_j(n) = \sum_{l^2 k = n} \lambda_j(k^2)$. It is known that the both functions converge in $\mathrm{Re}(s) > 1$. The functional equation of $L(s, \phi_j \times \phi_j)$ is inherited from the Eisenstein series by our unfolding the integral. We compute that

$$\int_X |\phi_j(v)|^2 E(v, 2s) dv = |\rho_j(1)|^2 \frac{L(s, \phi_j \times \phi_j)}{\zeta_K(2s)} \frac{\Gamma(s + ir_j)\Gamma(s - ir_j)\Gamma(s)^2}{8\pi^{2s}\Gamma(2s)}$$

is invariant under changing the variable s to $1-s$. We normalize such that $\|\phi_j\| = 1$ with respect to the Petersson inner product

$$\langle f, g \rangle = \frac{1}{\mathrm{vol}(X)} \int_X f(v)\overline{g(v)} dv.$$

The residue R_j of $L(s, \phi_j \times \phi_j)$ at its unique simple pole $s = 1$ is equal to

$$\frac{8\pi\zeta_K(2)}{|v_j(1)|^2}\mathrm{Res}_{s=2}E(v,s) = \frac{8\pi\zeta_K(2)\mathrm{Vol}(F_L)}{|v_j(1)|^2\mathrm{Vol}(X)}, \tag{2.2}$$

where $\mathrm{Res}_{s=2}E(v,s) = \mathrm{Vol}(F_L)/\mathrm{Vol}(X)$ is known by Sarnak [S] Lemma 2.15.

3. Proofs. In this section we prove Theorem 1.2. We first define the Eisenstein series by

$$E(v,s) = \sum_{\Gamma_\infty\backslash\Gamma} y(\gamma v)^s, \tag{3.1}$$

where $y(v) = y$ for $v = z + jy \in H^3$ and $\mathrm{Re}(s) > 2$. Here the group Γ_∞ is given by

$$\Gamma_\infty = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in O \right\}.$$

The Fourier development of $E(v,s)$ is known by Asai [A] and Elstrodt et al. [E]:

$$\begin{aligned} E(v,s) = y^s &+ y^{2-s}\frac{\xi_K(s-1)}{\xi_K(s)} \\ &+ \frac{2}{\xi_K(s)} \sum_{n\in O^*/\sim} |n|^{s-1}\sigma_{2(1-s)}(n)e^{4\pi i\mathrm{Re}(n\omega z)}K_{s-1}(4\pi|n\omega|y)y, \end{aligned} \tag{3.2}$$

where $\sigma_s(n) = \sum_{d|n} |d|^s$ and $\xi_K(s) = (\frac{\sqrt{D}}{2\pi})^s\Gamma(s)\zeta_K(s)$.

Our goal is to prove the equidistribution of the measure $\mu_t = |E(v, 1+it)|^2 dV(v)$, where $dV(v) = \frac{dx_1dx_2dy}{y^3}$. We consider its inner product with various functions spanning $L^2(X)$. We begin with inner products with Maass cusp forms ϕ_j.

Proposition 3.1. *For any fixed* ϕ_j,

$$\lim_{t\to\infty}\int_X \phi_j d\mu_t = 0$$

Proof. Set

$$J_j(t) = \int_X \phi_j d\mu_t = \int_X \phi_j(v)E(v,1+it)E(v,1-it)\frac{dx_1dx_2dy}{y^3} \tag{3.3}$$

with $z = x_1 + x_2 i$. To investigate this we first consider

$$I_j(s) = \int_X \phi_j(v) E(v, 1+it) E(v, s) \frac{dx_1 dx_2 dy}{y^3}. \tag{3.4}$$

All of the above integrals converge since ϕ_j is a cusp form. We unfold the integral (3.4) to get

$$I_j(s) = \int_0^\infty \int_{F_L} \phi_j(v) E(v, 1+it) y^s \frac{dx_1 dx_2 dy}{y^3}. \tag{3.5}$$

Denote the conjugate of $v = z + yj \in H^3$ by $\overline{v} = z - yj$. As is well-known in the two dimensional case, the space of the Maass cusp forms is expressed as a direct sum of spaces of even and odd cusp forms. Here even (resp. odd) cusp forms are ones satisfying $\phi_j(1-\overline{v}) = \epsilon\phi_j(v)$ with $\epsilon = 1$ (resp. -1). Since $E(v,s) = E(1-\overline{v}, s)$, it follows that $I_j(s) \equiv 0$ if ϕ_j odd. So we may assume that ϕ_j is even. In this case the Fourier development (2.1) is written as

$$\phi_j(v) = y \sum_{n \in O^*/\sim} \rho_j(n) K_{ir_j}(2\pi|n|y) \cos(2\pi i \langle n, z\rangle), \tag{3.6}$$

where $1 + r_j^2 = \lambda_j$. Normalizing the coefficients by $\rho_j(n) = \rho_j(1)\lambda_j(n)$, the multiplicative relations are satisfied by $\lambda_j(n)$. These amount to

$$L(\phi_j, s) := \sum_{n \in O^*/\sim} \frac{\lambda_j(n)}{N(n)^s} = \prod_{(p):\text{prime ideal}} \left(1 - \frac{\lambda_j(p)}{N(p)^s} + \frac{1}{N(p)^{2s}}\right)^{-1}. \tag{3.7}$$

By substituting (3.2) and (3.6) into (3.5) we have

$$\begin{aligned} I_j(s) = \int_0^\infty \int_{F_L} &\left(y \sum_{n \in O^*/\sim} \rho_j(n) K_{ir_j}(2\pi|n|y) \cos(2\pi\langle n, z\rangle) \right) \\ &\left(y^{1+it} + y^{1-it} \frac{\xi_K(it)}{\xi_K(1+it)} \right. \\ &\left. + \frac{2y}{\xi_K(1+it)} \sum_{m \in O^*/\sim} |m|^{it} \sigma_{-2it}(m) e^{4\pi i \mathrm{Re}(m\omega z)} K_{it}(4\pi|m|\omega y) \right) \\ &y^s \frac{dx_1 dx_2 dy}{y^3}. \end{aligned} \tag{3.8}$$

Now we have

$$\int_{F_L} \cos(2\pi i\langle n\omega, z\rangle)dv = \begin{cases} 0 & n \in O - \{0\} \\ 1 & n = 0 \end{cases}.$$

In the expansion of (3.8), we appeal to the formula $\cos x \cos y = \frac{1}{2}(\cos(x+y) + \cos(x-y))$. Only the terms with $n = m$ remain as follows:

$$\begin{aligned} I_j(s) &= \frac{2}{\xi_K(1+it)} \int_0^\infty \sum_{n \in O^*/\sim} |n|^{it}\sigma_{-2it}(n) K_{it}(2\pi|n|y)\rho_j(n) K_{ir_j}(2\pi|n|y) y^s \frac{dy}{y} \\ &= \frac{2}{\xi_K(1+it)} \sum_{n \in O^*/\sim} \frac{|n|^{it}\sigma_{-2it}(n)\rho_j(n)}{|n|^s} \int_0^\infty K_{it}(2\pi y) K_{ir_j}(2\pi y) y^s \frac{dy}{y}. \end{aligned}$$

An evaluation of the integral involving Bessel functions [GR] yields

$$I_j(s) = \frac{2\pi^{-s}}{\xi_K(1+it)} \frac{\Gamma(\frac{s+ir_j+it}{2})\Gamma(\frac{s+ir_j-it}{2})\Gamma(\frac{s-ir_j+it}{2})\Gamma(\frac{s-ir_j-it}{2})}{\Gamma(s)} R(s)$$

with

$$R(s) = \sum_{n \in O^*/\sim} \frac{|n|^{it}\sigma_{-2it}(n)\rho_j(n)}{|n|^s}.$$

We compute $R(s)$ as follows:

$$
\begin{aligned}
R(s) &= \frac{1}{\rho_j(1)} \prod_{(p):\text{prime ideal}} \sum_{k=0}^{\infty} \frac{\lambda_j(p^k)|p|^{ikt}\sigma_{-2it}(p^k)}{|p|^{ks}} \\
&= \frac{1}{\rho_j(1)} \prod_{(p)} \sum_{k=0}^{\infty} \frac{\lambda_j(p^k)|p|^{ikt}}{|p|^{ks}} \sum_{l=0}^{k} |p|^{-2itl} \\
&= \frac{1}{\rho_j(1)} \prod_{(p)} \sum_{k=0}^{\infty} \frac{\lambda_j(p^k)|p|^{ikt}}{|p|^{ks}} \frac{1-|p|^{-2it(k+1)}}{1-|p|^{-2it}} \\
&= \frac{1}{\rho_j(1)(1-|p|^{-2it})} \prod_{(p)} \left(\sum_{k=0}^{\infty} \lambda_j(p^k)|p|^{-k(s-it)} - |p|^{-2it} \sum_{k=0}^{\infty} \lambda_j(p^k)|p|^{-k(s+it)} \right) \\
&= \frac{1}{\rho_j(1)(1-|p|^{-2it})} \\
&\quad \prod_{(p)} \left(\frac{1}{1-\lambda_j(p)|p|^{-(s-it)}+|p|^{-2(s-it)}} - \frac{|p|^{-2it}}{1-\lambda_j(p)|p|^{-(s+it)}+|p|^{-2(s+it)}} \right) \\
&= \frac{1}{\rho_j(1)} \\
&\quad \prod_{(p)} \frac{1-|p|^{-2s}}{(1-\lambda_j(p)|p|^{-(s-it)}+|p|^{-2(s-it)})(1-\lambda_j(p)|p|^{-(s+it)}+|p|^{-2(s+it)})} \\
&= \frac{1}{\rho_j(1)} \frac{L(\phi_j, \frac{s-it}{2})L(\phi_j, \frac{s+it}{2})}{\zeta_K(s)}. \qquad (3.9)
\end{aligned}
$$

Therefore

$$
\begin{aligned}
J_j(t) &= I_j(1-it) \\
&= \frac{2\pi^{-1+it}}{\xi_K(1+it)} \frac{\Gamma(\frac{1+ir_j}{2})\Gamma(\frac{1+ir_j-2it}{2})\Gamma(\frac{1-ir_j}{2})\Gamma(\frac{1-ir_j-2it}{2})}{\Gamma(1-it)} R(1-it). \qquad (3.10)
\end{aligned}
$$

By Stirling's formula $|\Gamma(\sigma+it)| \sim e^{-\pi t/2}|t|^{\sigma-\frac{1}{2}}$, we see

$$\text{the gamma factors in (3.10)} \ll |t|^{-1} \qquad (3.11)$$

as $t \to \infty$. It is known that the Dedekind zeta function in (3.10) is estimated as

$$t^{-\epsilon} \ll |\zeta_K(1+it)| \ll t^{\epsilon}. \qquad (3.12)$$

Estimating the automorphic L-functions in (3.10) was recently done successfully by Sarnak and Petridis [SP]. They proved there exists $\delta > 0$ such that for any $\epsilon > 0$,

$$L(\phi_j, \frac{1}{2} + it) \ll_{j,\epsilon} |t|^{1-\delta+\epsilon} \tag{3.13}$$

as $|t| \to \infty$. The estimates (3.11)-(3.13) yield

$$J_j(t) \ll |t|^{-\delta+\epsilon}. \tag{3.14}$$

This implies Proposition 3.1. $\square$

We now turn to inner products of μ_t with incomplete Eisenstein series. Let $h(y)$ be a rapidly decreasing function at 0 and ∞, that is $h(y) = O_N(y^N)$ as $y \to \infty$ or 0 and $N \in \mathbf{Z}$. Let $H(s)$ be its Mellin transform

$$H(s) = \int_0^\infty h(y) y^{-s} \frac{dy}{y}.$$

Clearly $H(s)$ is entire in s and is of Schwartz class in t for each vertical line $\sigma + it$. The inversion formula gives

$$h(y) = \frac{1}{2\pi i} \int_{(\sigma)} H(s) y^s ds$$

for any $\sigma \in \mathbf{R}$. For such an h we form the convergent series

$$F_h(v) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} h(y(\gamma v)) = \frac{1}{2\pi i} \int_{(3)} H(s) E(v, s) ds,$$

which we call incomplete Eisenstein series.

Proposition 3.2. *For incomplete Eisenstein series $F(v)$, we have*

$$\int_X F(v) d\mu_t(v) \sim \frac{2}{\zeta_K(2)} \left(\int_X F(v) dV(v) \right) \log t$$

as $t \to \infty$.

Proof. Incomplete Eisenstein series decrease rapidly as $y \to \infty$ and belong to $C^\infty(X)$. Hence

$$\begin{aligned}\int_X F_h(v)d\mu_t(v) &= \int_X F_h(v)|E(v,1+it)|^2\frac{dzdy}{y^3}\\ &= \frac{1}{2\pi i}\int_X\int_{(3)} H(s)E(v,s)ds|E(v,1+it)|^2\frac{dzdy}{y^3}\\ &= \frac{1}{2\pi i}\int_0^\infty\int_{(3)} H(s)y^s ds\int_{F_L}|E(v,1+it)|^2\frac{dzdy}{y^3}\\ &= \frac{1}{2\pi i}\int_0^\infty\int_{(3)} H(s)y^s ds\left(\left|y^{1+it}+y^{1-it}\frac{\xi_K(it)}{\xi_K(1+it)}\right|^2\right.\\ &\qquad\left.+\left|\frac{2y}{\xi_K(1+it)}\right|^2\sum_{n\in O^*/\sim}|\sigma_{-2it}(n)K_{it}(4\pi|n|\omega y)|^2\right)\frac{dy}{y^3}\\ &= F_1(t)+F_2(t),\end{aligned}$$

where we put

$$F_1(t) = \frac{1}{2\pi i}\int_0^\infty\int_{(3)} H(s)y^s ds\left|y^{1+it}+y^{1-it}\frac{\xi_K(it)}{\xi_K(1+it)}\right|^2\frac{dy}{y^3}.$$

Since $\left|\frac{\xi_K(it)}{\xi_K(1+it)}\right| = 1$, we have

$$F_1(t) = 2\int_0^\infty h(y)\frac{dy}{y} + (\text{a rapidly decreasing function of } t). \tag{3.15}$$

Whereas

$$F_2(t) = \frac{2}{\pi i|\xi_K(1+it)|^2}\int_{(3)} H(s)\sum_{n\in O^*/\sim}\frac{|\sigma_{-2it}(n)|^2}{|n|^s}\int_0^\infty |K_{it}(4\pi\omega y)|^2 y^s\frac{dy}{y}ds.$$

The series is computed as follows:

$$\begin{aligned}
\sum_{n\in O^*/\sim} \frac{|\sigma_a(n)|^2}{|n|^s} &= \prod_{(p):\ \text{prime ideal}} \sum_{k=0}^{\infty} \frac{\sigma_a(p^k)\sigma_{-a}(p^k)}{|p|^{ks}} \\
&= \prod_{(p)} \sum_{k=0}^{\infty} \frac{1}{|p|^{ks}} \left(\frac{1-|p|^{a(k+1)}}{1-|p|^a}\right)\left(\frac{1-|p|^{-a(k+1)}}{1-|p|^{-a}}\right)^2 \\
&= \prod_{(p)} \frac{1}{(1-|p|^a)(1-|p|^{-a})} \\
&\qquad \sum_{k=0}^{\infty}\left(2|p|^{-ks} - |p|^{(a-s)k+a} + |p|^{(-a-s)k-a}\right) \\
&= \prod_{(p)} \frac{1}{(1-|p|^a)(1-|p|^{-a})} \\
&\qquad \left(\frac{2}{1-|p|^{-s}} - \frac{|p|^a}{1-|p|^{a-s}} - \frac{|p|^{-a}}{1-|p|^{-a-s}}\right) \\
&= \prod_{(p)} \frac{1+p^{-s}}{(1-p^{-s})(1-p^{-(s-a)})(1-p^{-(s+a)})} \\
&= \frac{\zeta_K(\frac{s}{2})^2\zeta_K(\frac{s-a}{2})\zeta_K(\frac{s+a}{2})}{\zeta_K(s)}. \qquad (3.17)
\end{aligned}$$

The y-integral in (3.16) is evaluated in terms of the Γ function as before. We obtain

$$\begin{aligned}
F_2(t) &= \frac{2}{\pi i|\xi_K(1+it)|^2}\int_{(3)} H(s) \sum_{n\in O^*/\sim} \frac{|\sigma_{-2it}(n)|^2}{|n|^s} \int_0^{\infty} |K_{it}(4\pi\omega y)|^2 y^s \frac{dy}{y} ds \\
&= \frac{2}{\pi i|\xi_K(1+it)|^2}\int_{(3)} \frac{H(s)\zeta_K(\frac{s}{2})^2|\zeta_K(\frac{s}{2}+it)\Gamma(\frac{s}{2}+it)|^2\Gamma(\frac{s}{2})^2}{(4\pi\omega)^s\zeta_K(s)\Gamma(s)} ds \\
&= \frac{2}{\pi i|\xi_K(1+it)|^2}\int_{(3)} B(s)ds, \qquad (3.18)
\end{aligned}$$

where we put

$$B(s) = \frac{H(s)\zeta_K(\frac{s}{2})^2|\zeta_K(\frac{s}{2}+it)\Gamma(\frac{s}{2}+it)|^2\Gamma(\frac{s}{2})^2}{(4\pi\omega)^s\zeta_K(s)\Gamma(s)}. \qquad (3.19)$$

By Stirling's formula to estimate the gamma factors and from the fact that $H(\sigma+it)$ is rapidly decreasing in t, we can shift the integral in (3.18) to $\mathrm{Re}(s)=1$:

$$F_2(t) = \frac{4\mathrm{Res}_{s=2}B(s)}{|\xi_K(1+it)|^2} + \frac{2}{\pi i|\xi_K(1+it)|^2}\int_{(1)} B(s)ds. \qquad (3.20)$$

The second term in (3.20) is evaluated by Heath-Brown [H] as

$$\zeta_K(\frac{1}{2}+it) \ll t^{\frac{1}{3}+\epsilon}$$

for any fixed $\epsilon > 0$. We find that

$$\frac{2}{\pi i|\xi_K(1+it)|^2}\int_{(1)} B(s)ds \ll_\epsilon t^{-\frac{1}{3}+\epsilon}.$$

This corresponds to the bound (3.14).

Next we deal with the residue term in (3.20), which is more complicated. Write $B(s)$ as $\zeta_K(\frac{s}{2})^2G(s)$ where $G(s)$ is holomorphic at $s=2$. Put

$$\zeta_K(s/2) = \frac{A_{-1}}{s-2} + A_0 + O(s-2) \quad (s \to 2).$$

In the expansion of

$$B(s) = \left(\frac{A_{-1}}{s-2} + A_0 + O(s-2)\right)^2 \left(G(2) + G'(2)(s-2) + O(s-2)^3\right),$$

the coefficient of $(s-2)^{-1}$ gives the residue

$$\mathrm{Res}_{s=2}B(s) = G(2)A_{-1}\left(2A_0 + A_{-1}\frac{G'}{G}(2)\right).$$

A simple calculation gives

$$G(2) = \frac{H(2)|\zeta_K(1+it)\Gamma(1+it)|^2\Gamma(\frac{1}{2})^2}{(4\pi\omega)^2\zeta_K(2)} = \frac{H(2)|\xi_K(1+it)|^2}{4\zeta_K(2)}$$

and

$$\frac{G'}{G}(2) = \frac{H'}{H}(2) + \frac{\zeta_K'(1+it)}{2\zeta_K(1+it)} + \frac{\zeta_K'(1-it)}{2\zeta_K(1-it)} + \frac{\Gamma'(1+it)}{2\Gamma(1+it)} + \frac{\Gamma'(1-it)}{2\Gamma(1-it)} + C$$

with C being independent of t. For the Weyl-Hadamard-De La Vallée Poussin bound [T, (6.15.3)] and its generalization to Dirichlet L-functions by Landau, we have

$$\frac{\zeta_K'(1+it)}{\zeta_K(1+it)} \ll \frac{\log t}{\log\log t}.$$

This together with $\frac{\Gamma'}{\Gamma}(1+it) \sim \log t$ gives

$$\mathrm{Res}_{s=2}B(s) = \frac{H(2)|\xi_K(1+it)|^2}{2\zeta_K(2)}\log t + O\left(\frac{\log t}{\log\log t}\right).$$

Finally the first term of (3.20) is evaluated as

$$\frac{4\mathrm{Res}_{s=2}B(s)}{|\xi_K(1+it)|^2} = \frac{2H(2)}{\zeta_K(2)}\log t + O(1).$$

Taking into account that

$$H(2) = \int_0^\infty h(y)\frac{dy}{y^3} = \int_X F_h(z)\frac{dzdy}{y^3}$$

we reach the conclusion. □

Proposition 3.3. *Let F be a continuous function of compact support in X. Then*

$$\int_X F(v)d\mu_t(v) \sim \frac{2}{\zeta_K(2)}\left(\int_X F(v)dV(v)\right)\log t$$

as $t \to \infty$.

Proof. The space of all incomplete Eisenstein series and cusp forms is dense in the space of continuous functions vanishing in the cusp. For any $\epsilon > 0$, we can find $G = G_1 + G_2$ with G_1 the finite sum of cusp forms and G_2 in the space of incomplete Eisenstein series, such that $\|G - F\|_\infty < \epsilon$. The difference $H = G - F$ is sufficiently small and rapidly decreasing in the cusp. Namely, it is majorized in terms of another incomplete Eisenstein series

$$H_1(v) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} h_1(y(\gamma v))$$

as

$$H_1(v) \geq |H(v)|$$

satisfying

$$\int_X H_1(v)dV(v) < C(K)\epsilon$$

with some constant $C(K)$ depending only on the field K. Hence the conclusion. □

Propositions 2.3 implies Theorem 1.1 by standard approximation arguments.

REFERENCES

[A] T. Asai, *On a certain function analogous to* $\log|\eta(z)|$., Nagoya Math. J. **40** (1970), 193-211.

[E] J. Elstrodt, F. Grunewald and J. Mennicke, *Eisenstein series for imaginary quadratic number fields*, Contemporary Math. **53** (1986), 97-117.

[GR] I.S. Gradshteyn, I.M. Ryzhik, *Table of integrals, series, and products*, Academic Press, 1994.

[H] D.R. Heath-Brown, *The growth rate of the Dedekind zeta-function on the critical line*, Acta Arith. **49** (1988).

[HR] D. Hejhal and B. Rackner, *On the topography of Maass waveforms for* $PSL(2,\mathbf{Z})$, Experimental Math. **1** (1992), 275-305.

[LS] W. Luo and P. Sarnak, *Quantum ergodicity of eigenfunction on* $PSL_2(Z)\backslash H^2$ (to appear).

[M] T. Meurman, *On the order of the Maass L-function on the critical line*, Colloquia Math. Soc. János Bolyai **51** (1990), North-Holland, Budapest.

[S] P. Sarnak, *The arithmetic and geometry of some hyperbolic three manifolds*, Acta math. **151** (1983), 253-295.

[SP] P. Sarnak and Y. Petridis, Quantum ergodicity for $SL(2,O)\backslash H^3$ and estimates for L-functions (preprint) (2000).

[T] E.C. Titchmarsh, *The theory of the Riemann zeta function* (1951), Oxford.