数理解析研究所講究録 1220 巻 2001 年 123-127

BRAID INVARIANTS AND INSTABILITY OF PERIODIC SOLUTIONS OF TIME-PERIODIC 2-DIMENSIONAL ODE'S

Takashi MATSUOKA (松 岡 隆)
Department of Mathematics, Naruto University of Education
e-mail: matsuoka@naruto-u.ac.jp

ABSTRACT. We present a topological approach to the problem of the existence of unstable periodic solutions for 2-dimensional, time-periodic ordinary differential equations. This approach makes use of the braid invariant, which is one of the topological invariants for periodic solutions exploiting a concept in the low-dimensional topology. Using the braid invariant, an equivalence relation on the set of periodic solutions is defined. We prove that any equivalence class consisting of at least two solutions must contain an unstable one, except one particular equivalence class. Also, it is shown that more than half of the equivalence classes contain unstable solutions.

1. INTRODUCTION

Consider a 2-dimensional ordinary differential equation of the form:

$$\frac{dx}{dt} = f(x,t), \tag{1}$$

where $f : \mathbf{R}^2 \times \mathbf{R} \to \mathbf{R}^2$ is a Carathéodory map (i.e., f is continuous in x for almost all t and is measurable in t for each x) which is periodic with respect to t with period $\omega > 0$. Assume that there exists a unique solution $x(t)$ of the initial-value problem $x(0) = x_0$ for each point $x_0 \in \mathbf{R}^2$ and this solution is defined on an interval containing $[0, \omega]$. We shall study the problem of the existence of unstable periodic solutions of (1). The traditional approach to this problem is to make the linear analysis of the related variational equation, and it is known that in some sense, the linear analysis in the instability case is easier than that in the stability case (see e.g. [1], [2]). In this paper, we present a purely topological approach to the problem. This approach makes use of the braid invariant, which is one of the topological invariants for periodic solutions exploiting a concept in the low-dimensional topology (see [4], [9] for a survey). We shall only treat periodic solutions having period ω in order to make the argument simpler.

The detailed version of this paper will appear in Topological Methods in Nonlinear Analysis Vol. 14.

2. Braids of periodic solutions

Here we shall define a braid for a given set of ω-periodic solutions. For general references on braid theory, see, e.g., [3], [6]. Let n be a positive integer. We call a subset B of the product $\mathbf{R}^2 \times [0, \omega]$ an *n-braid* if the following conditions hold:

(i) B is a union of mutually disjoint n simple arcs.
(ii) Each arc joins a point $(x, 0) \in S \times \{0\}$ to $(\tau(x), \omega) \in S \times \{\omega\}$, where S is a set of n distinct points on the plane $\mathbf{R}^2$ and τ is a permutation defined on S.
(iii) Each arc intersects every plane $\mathbf{R}^2 \times \{t\}$, $0 \leq t \leq \omega$, exactly once.

These arcs are called the *strings* in B.

For an ω-periodic solution ξ of (1), let $\mathrm{str}(\xi)$ denote the simple arc in $\mathbf{R}^2 \times [0, \omega]$ defined by

$$\mathrm{str}(\xi) = \{(\xi(t), t) \mid 0 \leq t \leq \omega\}.$$

We call this arc the *string* corresponding to ξ.

In this paper, we shall always assume that the equation (1) has only finitely many ω-periodic solutions.

Definition 1. Let $\mathcal{P}$ be a set of ω-periodic solutions of (1), and n the cardinality of $\mathcal{P}$. Since the strings corresponding to the solutions in $\mathcal{P}$ are mutually disjoint, the union $\bigcup_{\xi \in \mathcal{P}} \mathrm{str}(\xi)$ of these strings forms an n-braid denoted by $b(\mathcal{P})$. We call it the *braid* of $\mathcal{P}$.

3. An equivalence relation on periodic solutions

Definition 2. Let B be a braid. A union B_0 of strings in B is called a *block* in B if there is a subset T of $\mathbf{R}^2 \times [0, \omega]$ such that

(i) T is the image of some embedding $\Lambda : D \times [0, \omega] \to \mathbf{R}^2 \times [0, \omega]$, where D is a closed disk, with $\Lambda(D \times \{t\}) \subset \mathbf{R}^2 \times \{t\}$ for each t.
(ii) If we denote by T_t the t-slice of T, i.e. the set $\{x \in \mathbf{R}^2 | (x, t) \in T\}$, then we have $T_0 = T_\omega$.
(iii) $B_0 = B \cap T$.

We call T an isolating tube for B_0 with respect to B.

Example 1. It is clear that B is a block in itself, and any string in B is also a block in B. We give non-trivial examples in Figure 1 and Figure 2. Let B be the braid consisting of three strings s_1, s_2, s_3 as in Figure 1. Then the union $B_0 = s_1 \cup s_2$ is a block in B, and the set T drawn here gives an isolating tube for B_0. On the other hand, $s_1 \cup s_3$ is not a block, Indeed, if it were a block, then the string s_2 winds around s_1 and s_3 in the same number of times. However, these winding numbers are 1 and 0 respectively, and hence we get a contradiction. Consider next the braid B as in Figure 2. Then $s_1 \cup s_2$ and $s_4 \cup s_5$ are blocks in B with isolating tubes T, T' respectively. Also, $s_3 \cup s_4 \cup s_5$ is clearly a block. Furthermore, we can find an isolating block for the union $s_2 \cup s_3 \cup s_4 \cup s_5$, and so this union is a block.

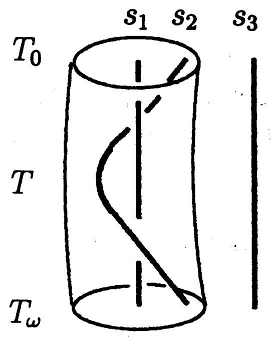

FIGURE 1

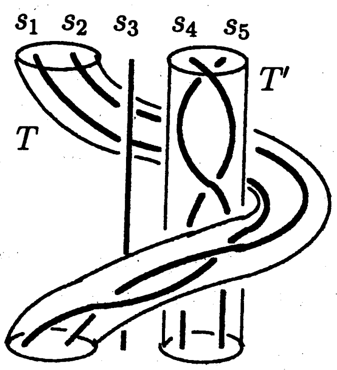

FIGURE 2

Let $\mathcal{P}_\omega$ denote the set of ω-periodic solutions.

Definition 3. Two ω-periodic solutions ξ_1 and ξ_2 are said to be *equivalent* if the braid $b(\{\xi_1, \xi_2\}) = \mathrm{str}(\xi_1) \cup \mathrm{str}(\xi_2)$ forms a block in $b(\mathcal{P}_\omega)$.

The choice of the term "equivalent" in this definition is reasonable as the following proposition shows:

Proposition 1. *The relation on $\mathcal{P}_\omega$ defined above is an equivalence relation.*

Example 2. (a) Suppose the equation (1) has three ω-periodic solutions ξ_i, $i = 1, 2, 3$ and the braid $b(\mathcal{P}_\omega)$ is as in Figure 1, where $s_i = \mathrm{str}(\xi_i)$. Then ξ_1 and ξ_2 are equivalent, since $b(\{\xi_1, \xi_2\}) = s_1 \cup s_2$ is a block in $B = b(\mathcal{P}_\omega)$. However, ξ_3 is not equivalent to ξ_1, since $s_1 \cup s_3$ is not a block. Thus, there are two equivalence classes $\{\xi_1, \xi_2\}$, $\{\xi_3\}$.

(b) Secondly, suppose (1) has five ω-periodic solutions ξ_i, $i = 1, \ldots, 5$, with the braid $b(\mathcal{P}_\omega) = s_1 \cup \cdots \cup s_5$ as in Figure 2, where $s_i = \mathrm{str}(\xi_i)$. Then, considering winding numbers also in this case, we see easily that there are three equivalence classes $\{\xi_1, \xi_2\}$, $\{\xi_3\}$, and $\{\xi_4, \xi_5\}$.

It should be noted that there is one exceptional equivalence class for which our main results, which will be stated in the next section, are not valid. This is the equivalence class consisting of the "peripheral" solutions defined below:

Definition 4. An ω-periodic solution ξ is said to be *peripheral* if one of the following conditions holds:

(i) $\mathcal{P}_\omega = \{\xi\}$, i.e., there are no other ω-periodic solutions.
(ii) There are at least two ω-periodic solutions and $b(\mathcal{P}_\omega - \{\xi\})$ is a block in $b(\mathcal{P}_\omega)$.

Proposition 2. *The set of peripheral solutions forms an equivalence class.*

We call this class consisting of all the peripheral solutions the *peripheral equivalence class*, and any other equivalence class a non-peripheral equivalence class. The equation (1) may not have any peripheral solution. In this case, the peripheral equivalence class is an empty set.

Example 3. If $\mathcal{P}_\omega$ is as in Example 2 (a), then ξ_3 is peripheral, since $b(\mathcal{P}_\omega - \{\xi_3\}) = s_1 \cup s_2$ is a block. Therefore, $\{\xi_3\}$ is the peripheral equivalence class. Also, if $\mathcal{P}_\omega$ is as in Example 2 (b), then $\{\xi_1, \xi_2\}$ is the peripheral equivalence class, since $b(\mathcal{P}_\omega - \{\xi_1\}) = s_2 \cup s_3 \cup s_4 \cup s_5$ is a block and this means that ξ_1 is peripheral.

4. Existence of unstable solutions

Definition 5. (cf. [7]) A solution x_0 of (1) defined for $0 \le t < \infty$ is *stable* (or Ljapunov stable) if for any $\epsilon > 0$, there is a $\delta > 0$ such that every solution $x(t)$ with $|x(0)-x_0(0)| < \delta$ is defined for all $0 \le t < \infty$ and satisfies $|x(t) - x_0(t)| < \epsilon$ for any t. Otherwise, x_0 is said to be unstable.

Theorem 1. *Any non-peripheral equivalence class consisting of at least two ω-periodic solutions contains an unstable one.*

In the case of an equivalence class with only one element, the following proposition provides a sufficient condition for its instability:

Proposition 3. *Suppose an ω-periodic solution ξ_0 is not peripheral and is a unique element in its equivalence class. Assume that there is a subset $\mathcal{P}$ of $\mathcal{P}_\omega$ containing ξ_0 such that $b(\mathcal{P})$ and $b(\mathcal{P} - \{\xi_0\})$ are blocks in $b(\mathcal{P}_\omega)$. Then ξ_0 is unstable.*

Theorem 1 and Poroposition 3 would suggest that not a few equivalence classes have an unstable solution. In fact, the following theorem holds:

Theorem 2. *More than half of the non-peripheral equivalence classes contain an unstable ω-periodic solution.*

Example 4. (a) Suppose $\mathcal{P}_\omega$ has the braid as in Figure 3. Then $\{\xi_4\}$ is the peripheral equivalence class, and the non-peripheral equivalence classes are $E_1 = \{\xi_1, \xi_2\}$ and $E_2 = \{\xi_3\}$. Since E_1 has two solutions, by Theorem 1, at least one of these solutions is unstable. Also, ξ_3 satisfies the assumption of Proposition 3 with $\mathcal{P} = \{\xi_1, \xi_2, \xi_3\}$. Hence ξ_3 is unstable. Thus, both E_1 and E_2 contain an unstable solution.

(b) We show that the estimate of the number of equivalence classes with unstable solutions given in Theorem 2 is the best possible one, by constructing an example. Consider the quotient space X obtained from the torus $T^2 = \mathbf{R}^2/\mathbf{Z}^2$ by identifying each point $x \in T^2$ with $-x$. A point of X represented by x will be denoted by the same symbol x. It is easy to see that X is homeomorphic to a sphere S^2. Let A be the matrix $\begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$. Then A induces a homeomorphism on X denoted by g_A. g_A has six fixed points, $s_0 = (0,0)$, $s_1 = (1/4, -1/4)$, $s_2 = (1/2, 0)$, $s_3 = (1/2, 1/2)$, $s_4 = (0, 1/2)$, and $s_5 = (1/4, 1/4)$. Since s_2, s_3, s_4 are degenerate fixed points, they are unstable. Since s_1 and s_5 are twisted saddles, one can alter these fixed points to stable ones by a local modification of g_A near these points without adding new fixed points. Identify $X - s_0$ with the plane $\mathbf{R}^2$. Then the restriction of g_A to $X - s_0$ gives an orientation-preserving homeomorphism $g : \mathbf{R}^2 \to \mathbf{R}^2$. We can choose an isotopy from id to g, and so we get a vector field on $\mathbf{R}^2 \times [0, \omega]$ which induces a time-periodic equation (1). This equation has five ω-periodic solutions $\xi_1, \dots, \xi_5$ which correspond to $s_1, \dots, s_5$ respectively. We see that the braid $b(\mathcal{P}_\omega)$ is as in Figure 4. Therefore each ω-solution is non-peripheral and is the unique element in its equivalence class. Thus, there are five non-peripheral equivalence classes. Since ξ_1, ξ_5 are stable and the other three are unstable, exactly three of them consist of unstable solutions.

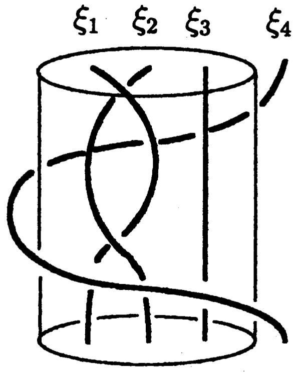

FIGURE 3

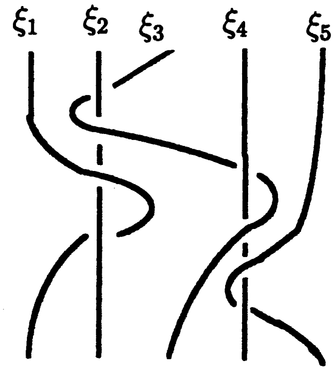

FIGURE 4

The results of this paper are proved by using a combination of the Nielsen fixed point theory and the Nielsen-Thurston classification theory of surface maps up to isotopy.

Remark . The content of this paper is closely related to that of a previous paper [8] of the author. It considers an orientation-preserving embedding of the 2-dimensional closed disk into itself, and includes some results on the existence of unstable fixed points for such embeddings. Consider the case where the initial-values of the ω-periodic solutions of (1) are contained in a disk D which is mapped into itself under the Poincaré operator $U : \mathbf{R}^2 \to \mathbf{R}^2$ associated with (1). Then we can apply the results in [8] to the embedding $U : D \to D$, and we obtain several results on the existence of unstable ω-periodic solutions of (1). These results are slightly stronger than those given here, since they are valid for all equivalence classes including the peripheral one. In this sense, the present paper can be regarded as a generalization of [8] to the general case where U may not have an invariant disk.

REFERENCES

[1] J. Andres, *Existence, uniqueness, and instability of large-period harmonics to the third-order nonlinear ordinary differential equations*, J. Math. Anal. Appl. **199** (1996), 445–457.

[2] J. Andres, *Concluding remarks to problem of Moser and conjecture of Mawhin*, Annal. Math. Silesianae **10** (1996), 57–65.

[3] J. S. Birman, *Braids, Links, and Mapping Class Groups*, Ann. Math. Studies, vol. 82, Princeton Univ. Press, Princeton, 1974.

[4] P. Boyland, *Topological methods in surface dynamics*, Topology and its Appl. **58** (1994), 223–298.

[5] A. Fathi, F. Laudenbach, and V. Poénaru, *Travaux de Thurston sur les surfaces*, Astérisque **66-67** (1979).

[6] V. L. Hansen, *Braids and Coverings: Selected Topics*, London Math. Soc. Student Texts **18**, Cambridge Univ. Press, Cambridge, 1989.

[7] M. A. Krasnosel'skiĭ, *The Operator of Translation Along the Trajectories of Differential Equations*, Translations of Math. Monographs, vol. 19, Amer. Math. Soc. 1968.

[8] T. Matsuoka, *Fixed point index and braid invariant for fixed points of embeddings on the disk*, to appear in Top. Appl.

[9] F. A. McRobie and J. M. T. Thompson, *Braids and knots in driven oscillators*, International J. of Bifurcation and Chaos **3** (1993), 1343–1361.

[10] W. P. Thurston, *On the geometry and dynamics of diffeomorphisms of surfaces*, Bull. Amer. Math. Soc. **19** (1988), 417-431.