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Accumulation of stretching rays for real cubic polynomials

Shizuo Nakane Tokyo Institute of Polytechnics
(Joint work with Y. Komori)

In this note, we investigate the accumulation of stretching rays on the parabolic arc $Per_1(1)$ for the family of real cubic polynomials. We know that stretching rays with irrational Böttcher vectors do not land at any point on $Per_1(1)$. That is, their accumulation sets are non-trivial arcs. Here we characterize the accumulation set of each stretching ray in terms of the Böttcher vector map. As a corollary, it follows that any point on $Per_1(1)$ belongs to the accumulation set of some stretching ray.

1 Stretching rays : general settings

Let $\mathcal{P}_d$ be the family of monic centered polynomials of degree $d \geq 2$. For $P \in \mathcal{P}_d$, let φ_P be its Böttcher coordinate defined in a neighborhood of ∞. It satisfies $\varphi_P(P(z)) = \varphi_P(z)^d$ and tangent to identity at ∞. Let $h_P(z) = \log_+ |\varphi_P(z)|$ be the Green function for P, which is continued continuously to the whole plane by the functional equation $h_P(P(z)) = dh_P(z)$ and is harmonic in $\mathbf{C} - K(P)$, the complement of the filled-in Julia set. Put $G(P) = \max\{h_P(\omega); \omega \text{ is a critical point of } P\}$. Then φ_P can be continued analytically to $U_P = \{z; h_P(z) > G(P)\}$. For a complex number $u \in H_+ = \{u = s + it \in \mathbf{C}, s > 0\}$, put $f_u(z) = z|z|^{u-1}$ and we define a P-invariant almost complex structure σ_u by

$$\sigma_u = \begin{cases} (f_u \circ \varphi_P)^* \sigma_0 \text{ on } U_P, \\ \sigma_0 \text{ on } K(P). \end{cases}$$

Then, by the Measurable Riemann Mapping Theorem, σ_u is integrated by a qc-map F_u such that $P_u = F_u \circ P \circ F_u^{-1} \in \mathcal{P}_d$. Since the same theorem says F_u depends holomorphically on u, so does P_u. Thus we define a holomorphic map $W_P : H_+ \to \mathcal{P}_d$ by $W_P(u) = P_u$. The Böttcher coordinate φ_{P_u} of P_u is equal to $f_u \circ \varphi_P \circ F_u^{-1}$. This operation is called *wringing*. Since P_u is hybrid equivalent to P, it holds $P_u \equiv P$ for $P \in \mathcal{C}_d$, the *connectedness locus*. For $P \in \mathcal{E}_d$, the *escape locus*, we define the *stretching ray* through P by

$$R(P) = W_P(\mathbf{R}_+) = \{P_s; s \in \mathbf{R}_+\}.$$

For example, in case $d = 2$, stretching rays coincide with the external rays for the Mandelbrot set.

2 Stretching rays : special case of real cubic polynomials

We consider the family of real cubic polynomials in the first quadrant:

$$P(z) = P_{A,B}(z) = z^3 - 3Az + \sqrt{B}; A, B > 0.$$

We investigate the accumulation sets of stretching rays above the parabolic arc :

$$Per_1(1) : B = 4(A + 1/3)^3; 0 \leq A \leq 1/9.$$

For $Q \in Per_1(1)$, Q has a parabolic fixed point $\beta_Q = \sqrt{A + 1/3}$ with multiplier 1 and both critical points escape to ∞ above $Per_1(1)$.

We set $\zeta_P(z) = \dfrac{\log \log \varphi_P(z)}{\log 3}$ and define, for $P \in \mathcal{E}_3^2$ (the real *shift locus*, i.e. the locus where both critical points escape), the *Böttcher vector* $\eta(P)$ by

$$\eta(P) = \frac{\log h_P(-\sqrt{A}) - \log h_P(\sqrt{A})}{\log 3} = \zeta_P(P(-\sqrt{A})) - \zeta_P(P(\sqrt{A})).$$

Lemma 2.1 *On the stretching ray $R(P)$ through $P \in \mathcal{E}_3^2$, $\eta(P_s)$ is invariant.*

Thus each stretching ray in the shift locus $\mathcal{E}_3^2$ is a level curve $\eta(P) = \eta$ of the Böttcher vector map $P \mapsto \eta(P)$, which we denote by $R(\eta)$. Thus we have an explicit description of stretching rays in our family and we can draw their pictures.

3 Preliminaries from parabolic implosion

In this section, we introduce some notions from parabolic implosion. For $Q \in Per_1(1)$, the immediate basin $\mathcal{B}_Q$ of the parabolic fixed point β_Q contains both critical points $\pm\sqrt{A}$ and $J(Q) = \partial\mathcal{B}_Q$ is a Jordan curve. Let $\phi_{Q,-}$ and $\phi_{Q,+}$ be the attracting and repelling Fatou coordinates respectively. They satisfy $\phi_{Q,\pm} \circ Q(z) = \phi_{Q,\pm}(z) + 1$ in the attracting and repelling petals respectively. Thus, there is an ambiguity of additive constant. Appropriately normalized, they are assumed to be symmetric with respect to the real axis. We define the *Fatou vector* $\tau(Q)$ of Q by

$$\tau(Q) = \phi_{Q,-}(-\sqrt{A}) - \phi_{Q,-}(\sqrt{A}).$$

Note that this definition does not depend on the choice of Fatou coordinates.

Lemma 3.1 *The Fatou vector gives a real analytic parametrization of* $Per_1(1), 0 <$

Let $\phi_{P,\pm}$ be the Fatou coordinates of $P \in \mathcal{E}_3^R$ above $Per_1(1)$ normalized in the same way. They are continuous up to $Per_1(1)$. After perturbation, the gate is open and the incoming Fatou coordinate $\phi_{P,-}$ can be regarded also as an outgoing Fatou coordinate and vice versa. Thus $\phi_{P,+}$ and $\phi_{P,-}$ differ only by an additive constant. We call this difference $\tilde{\sigma}(P) = \phi_{P,+}(z) - \phi_{P,-}(z)$ the *lifted phase* and its class $\sigma(P) = \{\tilde{\sigma}(P)\} \equiv \tilde{\sigma}(P) - [\tilde{\sigma}(P)]$ in $\mathbf{C}/\mathbf{Z}$ the *phase* of P. Since all the mappings are symmetric with respect to the real axis, the lifted phase is always real. Roughly speaking, minus the lifted phase is the time needed for the orbits of P to pass through the gate between two fixed points $\beta_P^{\pm}$.

Lemma 3.2 *The lifted phase $\tilde{\sigma}(P_s)$ tends to $-\infty$ as $s \to 0$ on a stretching ray.*

We also define, for $\tilde{\sigma} \in \mathbf{C}$, the *Lavaurs map* $g_{\tilde{\sigma}} : \mathcal{B}_Q \to \mathbf{C}$ of lifted phase $\tilde{\sigma}$ by $g_{\tilde{\sigma}} = \phi_{Q,+}^{-1} \circ T_{\tilde{\sigma}} \circ \phi_{Q,-}$, where $T_{\tilde{\sigma}}(w) = w + \tilde{\sigma}$.

Lemma 3.3 *Suppose $P_n \to Q \in Per_1(1)$ and $\sigma(P_n) \to \sigma \in \mathbf{C}/\mathbf{Z}$. Let $\tilde{\sigma}$ be any lift of σ. If we take $N_n \to \infty$ satisfying $N_n + \tilde{\sigma}(P_n) \to \tilde{\sigma}$, then $P_n^{N_n} \to g_{\tilde{\sigma}}$ locally uniformly on $\mathcal{B}_Q$.*

Since, in our case, $K(Q)$ is symmetric with respect to the real axis, connected and locally connected, its image in the repelling Fatou coordinate does not intersect the real axis. Then it follows $g_{\tilde{\sigma}}(\pm\sqrt{A}) \in \mathbf{C} - K(Q)$. Hence we can define the Böttcher vector $\eta(Q, \sigma)$ with phase σ also for $Q \in Per_1(1)$:

$$\eta(Q, \sigma) = \zeta_Q(g_{\tilde{\sigma}}(-\sqrt{A})) - \zeta_Q(g_{\tilde{\sigma}}(\sqrt{A})).$$

It depends only on the phase and not on the choice of lifted phase. Note that this definition depends on the choice of Fatou coordinates. In fact, if we add some constants to them, this changes the phase. This causes some difficulty in the next section. We define the *σ-impression* $I_\eta(\sigma)$ of $R(\eta)$ by the set of points $Q \in Per_1(1)$ such that there exists $P_n \in \overline{R(\eta)}$ satisfying $P_n \to Q$ and $\sigma(P_n) \to \sigma$. Apparently the accumulation set $I(\eta) = \overline{R(\eta)} - R(\eta)$ of $R(\eta)$ is the union of all $I_\eta(\sigma), \sigma \in \mathbf{R}/\mathbf{Z}$. By Lemma 3.3, it easily follows:

Lemma 3.4 *Under the same assumptions as in Lemma 3.3, $\eta(P_n) \to \eta(Q, \sigma)$. Consequently, we have $I_\eta(\sigma) \subset \{Q \in Per_1(1); \eta(Q, \sigma) = \eta\}$.*

We will show that the inclusion above actually is an identity.

4 Accumulation sets of stretching rays

In the last conference, we have shown the following.

Theorem 4.1 *Suppose η is irrational. Then the stretching ray $R(\eta)$ does not land on $Per_1(1)$. Hence its accumulation set $I(\eta)$ is a non-trivial arc on $Per_1(1)$.*

Figure 1: Non-landing stretching rays

In this section, we will characterize the sets $I_\eta(\sigma)$ and $I(\eta)$ in terms of the Böttcher vector map $\eta(Q,\sigma)$. Note that the map $\eta(Q,\sigma)$ and the set $I_\eta(\sigma)$ depend on the choice of Fatou coordinates.

Proposition 4.1 $I_\eta(\sigma) = \{Q \in Per_1(1); \eta(Q,\sigma) = \eta\}$.

As a corollary, we have

Corollary 4.1 *$R(\eta)$ accumulates at Q if and only if $\eta \in \eta(Q, \mathbf{R}/\mathbf{Z})$. Consequently, for any $Q \in Per_1(1)$, there exist at least one stretching rays accumulating at Q.*

John Milnor drew pictures of non-landing stretching rays, found that their oscillation is extremely regular and suggested the problems whether the set $I_\eta(\sigma)$ consists of a single point and whether it depends continuously on η and σ. See Figure 1. Considering Proposition 4.1, these are true if and only if the map $Q \mapsto \eta(Q,\sigma)$ is monotone increasing for any σ.

The proof of Proposition 4.1 relies on the following.

Lemma 4.1 *Fix σ and suppose $Q_0 \in I$, a connected component of $Per_1(1) - \tau^{-1}(\mathbf{Z})$, satisfies $\eta(Q_0,\sigma) = \eta_0$. Then there exists a normalization of Fatou coordinates such that the map $Q \mapsto \tilde{\eta}(Q,\sigma)$ in these new Fatou coordinates is monotone increasing on I and $\tilde{\eta}(Q_0,\sigma) = \eta_0$.*

The proof of Lemma 4.1 is done by qc-deformation of the Böttcher vectors just as for the proof of Lemma 3.1. But the conclusion is somewhat different: we have to lose the freedom of additive constant for Fatou coordinates, since Böttcher vectors depend on the choice of Fatou coordinates. In Lemma 4.1, the normalization of Fatou coordinates depends on σ. So we have not solved above problems yet.

proof. Suppose $\eta_0 \in (k, k+1)$ for some $k \in \mathbf{Z}$. Let $R = \varphi_{Q_0}(g_{Q_0,\sigma}(\sqrt{A_{Q_0}}))$ and, on the annulus $R \leq |z| \leq R^3$ in the Böttcher coordinate, we take a qc-map

$\ell = \ell_\eta$ such that it changes only radial coordinates, maps η_0 to η *mod* $\mathbf{Z}$ in the ζ-coordinate for any $\eta \in (k, k+1)$ and is identity on the boundary. We can extend it to the complement of the closed unit disk so that it commutes with $z \mapsto z^3$. Then $\sigma_\eta = \varphi_{Q_0}^* \ell^* \sigma_0$ is Q_0-invariant in the complement of $K(Q_0)$. Pulling it back by $g_{Q_0,\sigma}$, we extend it also inside $K(Q_0)$. Actually it extends to the complement of $K(Q_0, \sigma) = J(Q_0, \sigma)$, the set of points which do not escape by Q_0 and $G_{Q_0,\sigma}$ and $J(Q_0, \sigma)$ has measure 0. Let $\chi = \chi_\eta$ be the qc-map integrating σ_η so that $Q_\eta = \chi \circ Q_0 \circ \chi^{-1} \in Per_1(1)$. Its Böttcher coordinate is expressed by $\varphi_{Q_\eta} = \ell \circ \varphi_{Q_0} \circ \chi^{-1}$.

We normalize the attracting Fatou coordinate by $\tilde{\phi}_{Q,-}(\sqrt{A_Q}) = 0$, which is preserved under qc-deformation and repelling Fatou coordinates by $\tilde{\phi}_{Q,+}(\varphi_Q^{-1}(R)) = \sigma$. Then $\tilde{\phi}_{Q,+}^{-1}(\sigma) = \varphi_Q^{-1}(R)$ is preserved under the above qc-deformation since R is fixed by ℓ. If $\eta = \eta_0$, $Q_{\eta_0} = Q_0$ and we have

$$\tilde{\phi}_{Q_0,+}(\varphi_{Q_0}^{-1}(R)) = \tilde{\phi}_{Q_0,+}(g_{Q_0,\sigma}(\sqrt{A_{Q_0}})) = T_\sigma(\tilde{\phi}_{Q_0,-}(\sqrt{A_{Q_0}})) = T_\sigma(0) = \sigma.$$

Hence the original Fatou coordinate $\phi_{Q_0,+}$ for Q_0 coincides with $\tilde{\phi}_{Q_0,+}$.

We show $\tilde{\eta}(Q_\eta, \sigma) = \eta$. In order that, we have only to show $g = \chi \circ g_{Q_0,\sigma} \circ \chi^{-1}$ is equal to the Lavaurs map $g_{Q_\eta,\sigma}$ of Q_η.

Since σ_η is, by definition, $g_{Q_0,\sigma}$-invariant, g is holomorphic. Then $T = \tilde{\phi}_{Q_\eta,+} \circ g \circ \tilde{\phi}_{Q_\eta,-}^{-1}$ is a holomorphic map from attracting petal to the repelling petal. It easily follows that T commutes with T_1. Then T induces a cylinder isomorphism from the attracting cylinder onto the repelling one, hence is a translation. Next we show $T = T_\sigma$, which completes the proof. The critical point $\sqrt{A}$ is preserved by qc-map χ and takes 0 in the attracting Fatou coordinate. On the other hand, the value $\tilde{\phi}_{Q,+}^{-1}(\sigma)$ is preserved by χ. When $\eta = \eta_0$, $T = T_\sigma$ maps $0 = \tilde{\phi}_{Q_0,-}(\sqrt{A_{Q_0}})$ to $\sigma = \tilde{\phi}_{Q_0,+}(\varphi_{Q_0}^{-1}(R))$. Hence, this is true also for any η and we have $T = T_\sigma$ for any η. □

The proof of Proposition 4.1 is now easy. We have only to show $\{Q \in Per_1(1);\ \eta(Q, \sigma) = \eta_0\} \subset I_{\eta_0}(\sigma)$. Suppose $\eta(Q_0, \sigma) = \eta_0$. We apply Lemma 4.1. Then the map $Q \mapsto \tilde{\eta}(Q, \sigma)$ for the new Fatou coordinates is monotone increasing. Let P_n be a sequence on $R(\eta_0)$ satisfying $P_n \to Q_1$ and $\sigma(P_n) \to \sigma$ with respect to the new Fatou coordinates. Then $\tilde{\eta}(Q_1, \sigma) = \eta_0$. Since $Q \mapsto \tilde{\eta}(Q, \sigma)$ is monotone, this implies $Q_1 = Q_0$ and we have $\tilde{I}_{\eta_0}(\sigma) = \{Q_0\}$. Since the repelling Fatou coordinates coincide at Q_0 in both normalization, it follows, by the following Lemma 4.2, $Q_0 \in I_{\eta_0}(\sigma)$ in the original normalization. This completes the proof of Proposition 4.1. □

Take two Fatou coordinates $\phi_{P,\pm}^1$ and $\phi_{P,\pm}^2$ for P. They differs only by constants : $\phi_{P,\pm}^2(z) - \phi_{P,\pm}^1(z) = \delta_{P,\pm}$ depending on P. Hence their lifted phases satisfy

$$\tilde{\sigma}_2(P) = \phi_{P,+}^2(z) - \phi_{P,-}^2(z) = \tilde{\sigma}_1(P) + \delta_{P,+} - \delta_{P,-}.$$

And, if $P_n \to Q$ and $\sigma_1(P_n) \to \sigma_1$, we have

$$\sigma_2(P_n) = \sigma_1(P_n) + \delta_{P_n,+} - \delta_{P_n,-} \to \sigma_2 \equiv \sigma_1 + \delta_{Q,+} - \delta_{Q,-} \quad (mod\, \mathbf{Z}).$$

Lemma 4.2 *Denote the corresponding impressions by $I_\eta^1(\sigma_1)$ and $I_\eta^2(\sigma_2)$. If $Q \in I_\eta^1(\sigma_1)$, then $Q \in I_\eta^2(\sigma_2)$. Especially, if $\delta_{Q,\pm} = 0$, then $\sigma_2 = \sigma_1$ and $Q \in I_\eta^2(\sigma_2)$ if and only if $Q \in I_\eta^1(\sigma_1)$.*

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