数理解析研究所講究録 1220 巻 2001 年 63-77

Rigidity of infinitely renormalizable polynomials of higher degree

Hiroyuki Inou*
Department of Mathematics, Kyoto University

October 18, 2000

Abstract

Renormalization plays very important role in studying the dynamics of quadratic polynomials. We generalize renormalization to polynomials of higher degree so that many known properties still hold for the generalized renormalization. Furthermore, we show that the McMullen's result that any robust infinitely renormalizable quadratic polynomial carries no invariant line field is also extensible.

1 Preliminaries

1.1 Dynamics of rational maps

Let $f:\hat{\mathbb{C}}\to\hat{\mathbb{C}}$ be a rational map of degree $d\geq 2$. The *Fatou set* of f is the set of all $z\in\hat{\mathbb{C}}$ such that $\{f^n\}_{n>0}$ forms a normal family on some neighborhood of z. The *Julia set* $J(f)$ is the complement of the Fatou set. When f is a polynomial, the *filled Julia set* $K(f)$ is defined by:

$$K(f)=\left\{z\in\mathbb{C}\mid\{f^n(z)\}\text{ is bounded}\right\}.$$

The Julia set is equal to the boundary of $K(f)$.

Let $C(f)$ be the set of critical points of f. The *postcritical set* $P(f)$ is the closure of the union of the forward orbits of critical values of f, that is,

$$P(f)=\overline{\bigcup_{n>0}f^n(C(f))}.$$

Lemma 1.1. *For every point $x\in J(f)$ whose forward orbit does not intersect $P(f)$,*

$$\|(f^n)'(x)\|\to\infty$$

with respect to the hyperbolic metric on $\hat{\mathbb{C}}\setminus P(f)$.

See [Mc, Theorem 3.6].

*Partially supported by JSPS Research Fellowship for Young Scientists.

Lemma 1.2. *Any a rational map f of degree more than one satisfies one of the followings:*

1. $J(f) = \hat{\mathbb{C}}$ *and the action of f on $\hat{\mathbb{C}}$ is ergodic.*
2. *the spherical distance $d(f^n(x), P(f)) \to 0$ for almost every x in $J(f)$ as $n \to \infty$.*

See [Mc, Theorem 3.9].

1.2 Polynomial-like maps

A ***polynomial-like map*** is a triple (f, U, V) such that $f : U \to V$ is a holomorphic proper map between disks in $\mathbb{C}$ and U is a relatively compact subset of V. The ***filled Julia set*** $K(f, U, V)$ is defined by

$$K(f, U, V) = \bigcap_{n=1}^{\infty} f^{-n}(V)$$

and the ***Julia set*** is equal to the boundary of $K(f, U, V)$. The ***postcritical set*** $P(f, U, V)$ is defined similarly as in the case of rational maps.

Every polynomial-like map (f, U, V) of degree d is hybrid equivalent to some polynomial g of degree d. That is, there is a quasiconformal conjugacy ϕ from f to g defined near their respective filled Julia set and satisfies $\bar{\partial}\phi = 0$ on $K(f, U, V)$.

For $m > 0$, let $\mathrm{Poly}_d(m)$ be the set of all polynomials of degree d and all polynomial-like maps of degree d with $\mathrm{mod}(U, V) > m$. We consider the Carathéodory topology for the topology of $\mathrm{Poly}_d(m)$.

Then $\mathrm{Poly}_d(m)$ is compact up to affine conjugacy. Namely, any sequence $(f_n, U_n, V_n) \in \mathrm{Poly}_d(m)$ normalized so $U_n \supset \{\|z\| < r\}$ for some $r > 0$ and so the Euclidean diameter of $K(f_n, U_n, V_n)$ is equal to one, has a convergent subsequence [Mc, Theorem 5.8].

Moreover, if $(f, U, V) \in \mathrm{Poly}_d(m)$ has no attracting fixed point, then

$$\mathrm{diam}\, K(f, U, V) \leq C \,\mathrm{diam}\, P(f, U, V)$$

for some C depends only on d and m in the Euclidean metric [Mc, Corollary 5.10].

The following two lemmas are used repeatedly in the next section.

Lemma 1.3. *For $i = 1, 2$, let (f_i, U_i, V_i) be polynomial-like maps of degree d_i. Assume $f_1 = f_2 = f$ on $U = U_1 \cap U_2$. Let U' be a component of U with $U' \subset f(U') = V'$.*

Then $f : U' \to V'$ is a polynomial-like map of degree $d \leq \max(d_1, d_2)$ and

$$K(f, U', V') = K(f_1, U_1, V_1) \cap K(f_2, U_2, V_2) \cap U'.$$

Moreover, if $d = d_i$, then $K(f, U', V') = K(f_i, U_i, V_i)$.

See [Mc, Theorem 5.11].

Lemma 1.4. *Let f be a polynomial with connected filled Julia set. For any polynomial-like restriction (f^n, U, V) of degree more than one with connected filled Julia set K_n,* *1. The Julia set of (f^n, U, V) is contained in the Julia set of f.*

2. For any closed connected set $L \subset K(f)$, $L \cap K_n$ is also connected.

See [Mc, Theorem 6.13].

2 Renormalization

2.1 Definition of renormalization

Let f be a polynomial of degree d with connected Julia set. Fix a critical point $c_0 \in C(f)$.

Definition. f^n is called *renormalizable* about c_0 if there exist open disks $U, V \subset \mathbb{C}$ satisfying the followings:

1. c_0 lies in U.
2. (f^n, U, V) is a polynomial-like map with connected filled Julia set.
3. For each $c \in C(f)$, there is at most one i, $0 < i \leq n$, such that $c \in f^i(U)$.
4. $n > 1$ or $U \not\supset C(f)$.

A *renormalization* is a polynomial-like restriction (f^n, U, V) as above. We call n the period of a renormalization (f^n, U, V).

Note that the degree of a renormalization of f is not greater than 2^d.

Notation. Let $\rho = (f^n, U, V)$ be a renormalization. For $i = 1, \dots, n$ (or i may be regarded as an element of $\mathbb{Z}/n$),

- Let $n(\rho) = n$, $U(\rho) = U$ and $V(\rho) = V$.
- The filled Julia set of ρ is denoted by $K(\rho)$, the Julia set by $J(\rho)$, and the postcritical set by $P(\rho)$.
- The *i-th small filled Julia set* is denoted by $K(\rho, i) = f^i(K(\rho))$ and the *i-th small Julia set* $J(\rho, i) = f^i(J(\rho))$.
- The *ith small critical set* $C(\rho, i) = K(\rho, i) \cap C(f)$. Clearly, $C(\rho, i)$ may be empty for $0 < i < n$ (by the definition, $C(\rho, n)$ is nonempty).
- $\mathcal{K}(\rho) = \bigcup_{i=1}^n K(\rho, i)$ is the union of the small filled Julia sets. Similarly, $\mathcal{J}(\rho) = \bigcup_{i=1}^n J(\rho, i)$.
- $\mathcal{C}(\rho) = \bigcup_{i=1}^n C(\rho, i)$ is the set of critical points appeared in a renormalization ρ.
- $\mathcal{P}(\rho) = \overline{\bigcup_{k>0} f^k(\mathcal{C}(\rho))} \subset P(f) \cap \mathcal{K}(\rho)$.
- The *ith small postcritical set* is denoted by

$$P(\rho, i) = K(\rho, i) \cap \mathcal{P}(\rho).$$

- Let $V(\rho,i) = f^i(U)$ and $U(\rho,i)$ be the component of $f^{i-n}(U)$ contained in $V(\rho,i)$. Then $(f^n, U(\rho,i), V(\rho,i))$ is polynomial-like of same degree as (f^n, U, V). Moreover, it is also a renormalization of f if $C(\rho,i)$ is nonempty.

Let ρ and ρ' be renormalizations. Define an equivalence relation $\sim$ by

$$\rho \sim \rho' \Leftrightarrow n(\rho) = n(\rho') \text{ and } K(\rho) = K(\rho').$$

It implies that the dynamics of ρ and ρ' are equal. Let

$$\mathcal{R}(f, c_0) = \{\text{renormalizations of some iterates of } f \text{ about } c_0\} / \sim .$$

and for a subset $C_R \subset C(f)$ containing c_0,

$$\mathcal{R}(f, c_0, C_R) = \{\rho \in \mathcal{R}(f, c_0) \mid \mathcal{C}(\rho) = C_R\}.$$

We confuse the elements in $\mathcal{R}(f, c_0)$ with its representation and write like as $\rho = (f^n, U, V) \in \mathcal{R}(f, c_0)$.

The following three propositions are easily derived from Lemma 1.3 and Lemma 1.4.

Proposition 2.1. *Suppose two renormalizations $\rho = (f^n, U^1, V^1)$ and $\rho' = (f^n, U^2, V^2)$ of the same periods satisfy $C(\rho,i) = C(\rho',i)$ for any i $(0 \le i < n)$. Then their filled Julia sets are equal.*

Proposition 2.2. *Let $\rho_n = (f^n, U_n, V_n)$ and $\rho_m = (f^m, U_m, V_m) \in \mathcal{R}(f, c_0)$. Then there exists $\rho_l = (f^l, U_l, V_l) \in \mathcal{R}(f, c_0)$ with filled Julia set $K(\rho_l) = K(\rho_n) \cap K(\rho_m)$ where l is the least common multiple of n and m.*

Definition. For $\rho \in \mathcal{R}(f, c_0)$, the *intersecting set* $I(\rho)$ is defined by

$$I(\rho) = K(\rho) \cap \left(\bigcup_{i=1}^{n(\rho)-1} K(\rho, i) \right).$$

We say ρ is *intersecting* if $I(\rho)$ is nonempty.

Although we now define intersecting "set", it consists of at most one point.

Proposition 2.3. *If $\rho = (f^n, U, V) \in \mathcal{R}(f, c_0)$ is intersecting, then $I(\rho)$ consists of only one point which is a repelling fixed point of f^n.*

Although small Julia sets of a renormalization can meet at a repelling periodic point, the period of such point tends to infinity as the period of renormalization tends to infinity.

Theorem 2.4 (High periods). *For fixed $p > 0$, there are only finitely many $\rho \in \mathcal{R}(f, c_0)$ such that $K(\rho)$ contains a periodic point of period p.*

Therefore, infinitely renormalizable polynomial f satisfies that the filled Julia set of renormalization of period sufficiently large does not contain the fixed point of f. It implies that $\mathcal{K}(\rho)$ is disconnected when $n(\rho)$ is sufficiently large.

Since a repelling fixed point separates filled Julia set into finite number of components, components of $K(\rho) \setminus I(\rho)$ are finite. We say a renormalization is *simple* if $K(\rho)\backslash I(\rho)$ is connected, and *crossed* if it is disconnected. Let $\mathcal{SR}(f, c_0)$ be the set of all simple renormalizations in $\mathcal{R}(f, c_0)$. Similarly, $\mathcal{SR}(f, c_0, C_R)$ is the set of all $\rho \in \mathcal{SR}(f, c_0)$ with $C(\rho) = C_R$.

In the next section, we will show any infinitely renormalizable polynomial has infinitely many simple renormalizations. So simple renormalizations plays very important role in the case of infinitely renormalizable polynomials.

However, there even exist finitely renormalizable polynomials which is not simply renormalizable. See [Mc, §7.4].

Theorem 2.5. *For $C_R \subset C(f)$, $\mathrm{sr}(f, C_R)$ is totally ordered with respect to division. Moreover, elements of $\mathcal{SR}(f, c_0, C_R)$ are uniquely determined by their period and their filled Julia sets form a decreasing sequence.*

2.2 Examples

In the last of this section, we present an example of finitely renormalizable polynomials. An example of infinitely renormalizable polynomials are given in §4.

Let

$$f(z) = z^3 - \frac{3}{4}z - \frac{\sqrt{7}}{4}i.$$

Then $C(f) = \{\pm 1/2\}$ and $\pm 1/2$ are periodic of period 2. Let $W_\pm$ be the Fatou component which contains $\pm 1/2$. Each of them is superattracting basin of period 2.

Every renormalization (f^n, U, V) must satisfy $U \supset W_-$ or W_+. So $n \leq 2$ and by symmetry, we will consider only the case $U \supset W_-$.

1. Let K be the connected component of the closure of $\bigcup_{n>0} f^{-n}(W_-)$ which contains W_- and let U_1 be a small neighborhood of K.

Then $\rho_1 = (f, U_1, f(U_1))$ is a renormalization with filled Julia set $K(\rho_1) = K_1$ which is hybrid equivalent to $z \mapsto z^2 - 1$.

2. Let U_2 be a small neighborhood of W_-. Then $\rho_{2,1} = (f^2, U_2, f^2(U_2))$ is a renormalization with filled Julia set $K(\rho_{2,1}) = \overline{W_-}$, which is hybrid equivalent to $z \mapsto z^2$.

3. Let K_2' be the connected component of $\overline{\bigcup_{n>0} f^{-2n}(W_- \cup W_+)}$ which contains W_- and let U_2' be a small neighborhood of K_2'.

Then $\rho_{2,2} = (f^2, U_2', f^2(U_2'))$ is a renormalization with filled Julia set K_2', which is hybrid equivalent to $z \mapsto z^3 - (3/\sqrt{2})z$.

4. Let K_2'' be the connected component of $\overline{\bigcup_{n>0} f^{-2n}(W_- \cup f(W_+))}$ which contains W_- and let U_2'' be a small neighborhood of K_2''.

Then $\rho_{2,3} = (f^2, U_2'', f^2(U_2''))$ is a renormalization with filled Julia set K_2'' and of degree 4.

Similarly, consider $\overline{\bigcup_{n>0} f^{-2n}(W_- \cup f(W_-) \cup W_+)}$ and then we can construct a polynomial-like map (f^2, U, V) of degree 6. But it is not a renormalization because $-\frac{1}{2}$ is contained in both U and $f(U)$.

Thus $\mathcal{R}(f, -1/2) = \{\rho_1, \rho_{2,1}, \rho_{2,2}, \rho_{2,3}\}$ and $\mathcal{SR}(f, -1/2) = \{\rho_1, \rho_{2,1}\}$.

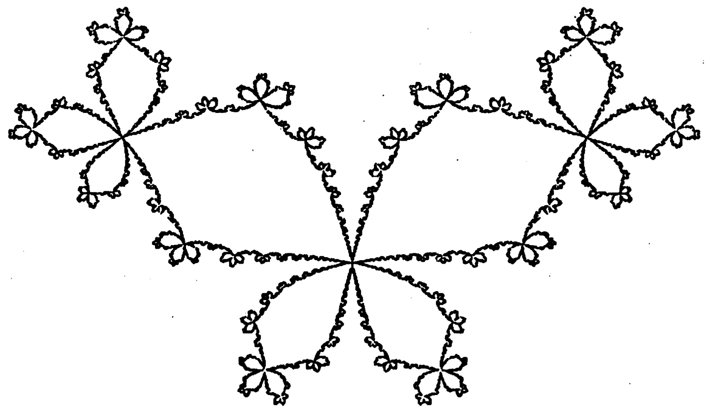

Figure 1: The Julia set of f.

3 Infinite renormalization

3.1 Infinite simple renormalization

For each $\rho \in \mathcal{R}(f, c_0, C_R)$, we construct a simple renormalization near the component of $\mathcal{K}(\rho)$ containing c_0 by using the Yoccoz puzzle. Then the period of new simple renormalization is equal to the number of components of $\mathcal{K}(\rho)$, which tends to infinity. So, we have the following:

Theorem 3.1. *If f is infinitely renormalizable, then f has infinitely many simple renormalizations.*

More precisely, if $\mathcal{R}(f, c_0, C_R)$ is infinite for some $C_R \subset C(f)$, then there exists some C'_R with $C_R \subset C'_R \subset C(f)$ such that $\mathcal{SR}(f, c_0, C_R)$ is also infinite.

Remark 3.2. We do not know whether C'_R coincide with C_R. However, when f is C'_R-robust infinitely renormalizable, then C'_R must be equal to C_R.

But if f is robust, C'_R must be equal to C_R. See Proposition 3.8.

3.2 Robust infinite renormalization

Consider the following assumption for a polynomial f with connected Julia set and a subset $C_R \subset C(f)$:

$$\mathrm{sr}(f, C_R) \text{ is infinite and } f(C_R) = f(C(f)). \tag{1}$$

Remark 3.3. This assumption corresponds to extracting an infinitely renormalizable "part" from an infinitely renormalizable polynomial.

More precisely, assume $\#\,\mathrm{sr}(f, C_R) = \infty$. For any $\rho = (f^n, U, V) \in \mathcal{SR}(f, c_0, C$
let g be a polynomial hybrid equivalent to ρ. There exist some c'_0 and C'_R

with $c'_0 \in C'_R \subset C(g)$ such that $\mathrm{sr}(g, C'_R)$ is infinite and each element ρ' of $\mathcal{SR}(g, c'_0, C'_R)$ corresponds to the renormalization in $\mathcal{SR}(f, c_0, C_R)$ with period $n(\rho') \cdot n$. Then we obtain $g(C'_R) = g(C(g))$ because the critical points of ρ is the union of $f^{-i}(C(\rho, i)) \cap U$ for $0 \le i < n$.

We say a renormalization is ***robust infinitely renormalizable*** if it is hybrid equivalent to some robust infinitely renormalizable polynomial.

For each $n \in \mathrm{sr}(f, C_R)$, take a corresponding renormalization $\rho_n = (f^n, U_n, V_n) \in \mathcal{SR}(f, c_0, C_R)$. Then $\mathcal{P}(\rho_n) = P(f)$ for any $n \in \mathrm{sr}(f, C_R)$ and $\{K(\rho_n)\}_{n \in \mathrm{sr}(f, C_R)}$ forms a decreasing sequence by Theorem 2.5.

Proposition 3.4. *Let f, C_R and ρ_n as above. Then:*

1. *All periodic points of f are repelling.*
2. *The filled Julia set of f has no interior.*
3. *There exist no periodic points in* $\displaystyle\bigcap_{n \in \mathrm{sr}(f, C_R)} \mathcal{K}(\rho_n)$.
4. *there exist no periodic points in $P(f)$.*
5. *For any $n \in \mathrm{sr}(f, C_R)$, $P(f) \cap K(\rho_n, i)$ is disjoint from $K(\rho_n, j)$ if $i \ne j$.*

For each $n \in \mathrm{sr}(f, C_R)$, let $\delta_n(i)$ be a simple closed curve which separates $K(\rho_n, i)$ from $P(f) \setminus P(\rho_n, i)$. Since its homotopy class in $\mathbb{C} \setminus P(f)$ is uniquely determined, there exists a geodesic $\gamma_n(i)$ homotopic to $\delta_n(i)$. Let $\gamma_n = \gamma_n(n)$.

These geodesics are simple and mutually disjoint. Furthermore, the hyperbolic length $\ell(\gamma_n(i))$ of $\gamma_n(i)$ are comparable with $\ell(\gamma_n)$.

Definition. Suppose a polynomial f and $C_R \subset C(f)$ satisfies the condition (1). We say f is *robust* if there exists a subset $C_R \subset C(f)$ such that $\#\mathcal{SR}(f, C_R) = \infty$, $f(C_R) = f(C(f))$ and

$$\liminf_{n \in sr(f, C_R)} \ell(\gamma_n) < \infty$$

where $\ell(\cdot)$ is the hyperbolic length in $\mathbb{C} \setminus P(f)$.

Theorem 3.5. *Suppose f is robust. Then:*

1. *the postcritical set $P(f)$ is a Cantor set of measure zero.*
2. $\displaystyle\lim_{n \in \mathrm{sr}(f, C_R)} \left(\sup_{1 \le i \le n} \operatorname{diam} P(\rho_n, i) \right) = 0.$
3. *$f : P(f) \to P(f)$ is topologically conjugate to $\sigma : \Sigma \to \Sigma$ where*

$$\Sigma = \operatorname{proj\,lim}_{n \in \mathrm{sr}(f, C_R)} \mathbb{Z}/n$$

$$\sigma\left((i_n)_{n \in \mathrm{sr}(f, C_R)} \right) = (i_n + 1)_{n \in \mathrm{sr}(f, C_R)}.$$

Especially, $f|_{P(f)}$ is a homeomorphism.

Proof. By the collar theorem, there exists the standard collar $A_n(i)$ about the geodesic $\gamma_n(i)$ on the hyperbolic surface $\mathbb{C} \setminus P(f)$. Note that the collar theorem asserts that these collars are mutually disjoint. Each annulus $A_n(i)$ separates $P(\rho_n, i)$ from the rest of the postcritical set.

Consider a sequence of nested annuli $\{A_n(i_n)\}_{n \in \mathrm{sr}(f, C_R)}$, that is, for $m < n$, $A_n(i_n)$ lies in the bounded component of $A_m(i_m)$. Since $\mathrm{mod}(A_n(i_n))$ is a decreasing function of $\ell(\gamma_n)$ and $\liminf \ell(\gamma_n)$ is finite, the sum

$$\sum_{n \in \mathrm{sr}(f, C_R)} \mathrm{mod}\, A_n(i_n)$$

diverges to infinity. Note that $\ell(\gamma_n(i_n)) < C\,\ell(\gamma_n)$.

Thus the set $F = \bigcap F_n$ is totally disconnected and of measure zero, where F_n be the union of the bounded components of $\mathbb{C} \setminus (\bigcup_i A_n(i))$ (see [Mc, Theorem 2.16]).

Clearly, F contains $P(f)$. Furthermore, since each component of F_n intersects $P(f)$, we have $F = P(f)$. Therefore, the postcritical set has measure zero.

Each $P(\rho_n, i)$ lies in a single component of F_n. Since F is totally disconnected, the diameter of the largest component of F_n tends to zero as n tends to infinity, and so does $\sup_i \mathrm{diam}\, P(\rho_n, i)$.

For each $n \in \mathrm{sr}(f, C_R)$, let $\phi_n : P(f) \to \mathbb{Z}/n$ be the map which sends $P(\rho_n, i)$ to i mod n. These maps induces a continuous map $\phi : P(f) \to \Sigma$.

It is easy to confirm that ϕ has desired properties. Since Σ is a Cantor set, $P(f)$ is also a Cantor set. □

Corollary 3.6. *Suppose f is robust. Then if $n \in \mathrm{sr}(f, C_R)$ is sufficiently large, $\#C(\rho_n, i) \leq 1$ for any i.*

Proof. Otherwise, we may assume $C(\rho_n) = \{c_0, \dots c_r\}$ for all sufficiently large $n \in \mathrm{sr}(f, C_R)$ $(r \geq 1)$. Since $\mathrm{diam}(P(\rho_n, 1))$ tends to zero, all $f(c_j)$'s are equal.

Let m_j be the multiplicity of the critical point c_j. Then the degree of the proper map $f : U \to U(\rho_n, 1)$ is equal to $(\sum m_j) + 1$, but the cardinality of $f^{-1}(f(c_0))$ (counted with multiplicity) is not less than $\sum(m_j + 1)$, that is a contradiction. □

Corollary 3.7. *If f is robust, then $C_R \subset P(f)$.*

Proof. Let $n \in \mathrm{sr}(f, C_R)$ sufficiently large so that Corollary 3.6 holds. Then for $c \in C_R$, we have $C(\rho_n, i) = \{c\}$ for some i. Therefore, the inverse image of $f(c)$ by the proper map $f : U(\rho_n, i) \to f(U(\rho_n, i))$ consist only of c. Since $f : P(\rho_n, i) \to P(\rho_n, i+1)$ is a homeomorphism by Theorem 3.5 and $f(c) \in P(f)$, we have $c \in P(f)$.

□

The next corollary gives an answer of Remark 3.2 in the case of robust infinitely renormalizable case.

Corollary 3.8. *Suppose a renormalization $\rho \in \mathcal{SR}(f, c_0, C_R)$ is robust. Then every renormalization ρ about c_0 satisfies $C(\rho) \supset C_R$.*

Especially, if $\mathcal{R}(f, c_0, C'_R)$ is infinite for some $C'_R \subset C_R$, then $C'_R = C_R$.

Proof. By Theorem 3.5, $\mathcal{P}(\rho)$ is a Cantor set of measure zero and the forward orbit of c_0 is dense in P.

Therefore, for any $\rho \in \mathcal{R}(f, c_0)$, $\mathcal{K}(\rho)$ must contain $\mathcal{P}(\rho)$. By Corollary 3.7, $\mathcal{K}(\rho) \supset C_R$. □

Figure 2: The Julia set of $h(z) = z^2 - 1.401155189\ldots$ h is infinitely renormalizable with $\mathrm{sr}(h, \{0\}) = \{3^n\}$.

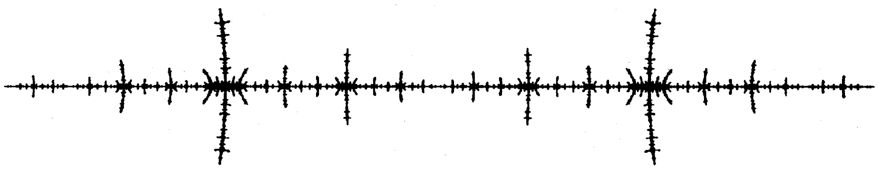

Figure 3: The Julia set of $g(z) = -z^3 - 3c^2z$ and $f = -g$, where $c = 0.87602957776\ldots$ g is constructed by the intertwining surgery (see [EY]) from two h's. f is infinitely renormalizable and each renormalization has degree 4.

4 Robust rigidity

In the rest of the paper, we will prove the main theorem, which is the following.

Theorem 4.1 (Robust rigidity). *A robust infinitely renormalizable polynomial carries no invariant line field on its Julia set.*

Since hybrid equivalence preserves invariant line field on the Julia set, we can easily apply the result to polynomials whose dynamics on its Julia set is essentially robust infinitely renormalizable.

For example,

Corollary 4.2. *Let f be a polynomial of degree $d \geq 2$. Suppose every critical point $c \in C(f)$ satisfies one of the followings:*

1. *c is preperiodic.*
2. *the forward orbit of c tends to an attracting cycle.*
3. *there exists some robust infinitely renormalizable renormalization ρ (see Remark 3.3) and $n > 0$ such that $f^n(c)$ lies in $J(\rho)$ and the forward orbit of c does not accumulate to $I(\rho)$.*

Then f carries no invariant line field on its Julia set.

The corollary is an easy consequence of Theorem 4.1 and the following lemma:

Lemma 4.3. *Let f be a polynomial of degree $d \geq 2$. Suppose there exist simple renormalizations $\rho_1, \dots, \rho_J$ such that $\mathcal{P}(\rho_j)$'s are pairwise disjoint and every critical point $c \in C(f)$ satisfies one of the followings:*

1. *c is preperiodic.*
2. *the forward orbit of c tends to an attracting cycle.*
3. *there exist $n > 0$ and j such that $f^n(c)$ lies in $K(\rho_j)$ and the forward orbit of c does not accumulate to $I(\rho)$.*

Then almost every x in $J(f)$ eventually mapped onto $\bigcup K(\rho_j)$ by f.

Proof. By Lemma 1.2, the Euclidean distance $d(f^n(x), P(f))$ tends to 0 for almost every $x \in J(f)$. Now we consider such $x \in J(f)$. Since there exist only countably many eventually periodic points, we may assume x is not eventually periodic.

For any $\epsilon > 0$, there exists $N > 0$ and j such that $d(f^N(x), P(\rho_j)) < \epsilon$ and $d(f^n(x), P(f)) < \epsilon$ for any $n \geq N$.

When ϵ is sufficiently small, it implies that $d(f^n(x), P(\rho_j, n - N)) < \epsilon$ for any $n > N$. Thus $f^N(x)$ lies in $K(\rho_j)$. □

Proof of Theorem 4.2. By the assumption, there exists robust infinitely renormalizable renormalizations $\rho_1, \dots, \rho_J$ satisfies the assumption of Lemma 4.3. Note that by Corollary 3.8, $\mathcal{C}(\rho_j)$'s are pairwise disjoint and if $\mathcal{P}(\rho_i) \cap \mathcal{P}(\rho_j)$ is nonempty, then $\mathcal{P}(\rho_i) = \mathcal{P}(\rho_j)$ by Theorem 3.5. So the postcritical sets are also pairwise disjoint.

Thus

$$E = \bigcup_{j=1}^{J} \bigcup_{k>0} f^{-k}(K(\rho_j))$$

has full measure in $J(f)$.

By Theorem 4.1, f carries no invariant line field on E and so does on $J(f)$. □

In the rest of this section, we state the outline of the proof of Theorem 4.1. The proof is based on the McMullen's proof in the quadratic case [Mc].

The proof is divided into two cases, whether $L = \liminf \ell(\gamma_n)$ is zero or positive. However, both proofs goes very similarly. We pass to a subsequence in $\mathrm{sr}(f, C_R)$ so that after properly rescaling, f^n converge to some proper map $f_\infty : U \to V$ if we restricted f^n on some neighborhood of the small postcritical set $P(\rho_n)$. (We use some proper map $f^n : X_n \to Y_n$ constructed from $\rho_n \in \mathcal{SR}(f, c_0, C_R)$ to obtain good estimates. Only when the case L is sufficiently small, they are polynomial-like.)

Now suppose f carries an invariant line field μ on its Julia set. We will construct a g-invariant univalent line field ν on V. Then it is a contradiction because $U \cap V$ contains a critical point of g.

To construct ν, we will use the following two lemmas:

Lemma 4.4. *Suppose holomorphic maps $f_n : (U_n, u_n) \to (V_n, v_n)$ between disks converge to some non-constant map $f : (U, u) \to (V, v)$ in the Carathéodory topology.*

If f_n-invariant line field μ_n converges in measure to some line field μ, then μ is f-invariant.

See [Mc, Theorem 5.14].

Lemma 4.5. *Suppose a measurable line field μ on $\mathbb{C}$ is almost continuous at a point x and $|\mu(x)| = 1$. Let $(V_n, v_n) \to (V, v)$ be a convergence sequence of pointed disks, and let $h_n : V_n \to \mathbb{C}$ be a sequence of univalent maps. Suppose $h_n'(v_n) \to 0$ and*

$$\sup \frac{|x - h_n(v_n)|}{|h_n'(v_n)|} < \infty.$$

Then there exists a subsequence such that $h_n^*(\mu)$ converges in measure to a univalent line field on V.

See [Mc, Theorem 5.16].

We take a point $x \in J(f)$ having good properties and for infinitely many $n \in \mathrm{sr}(f, C_R)$, and take an inverse branch h_n of f^k for some k which sends some neighborhood of the small postcritical set univalently near x.

Then $h_n^*(\mu) = \mu$ by f-invariance, so $h_n^*(\mu)$ is f^n-invariant line field on Y_n. We apply Lemma 4.5 and obtain a univalent line field μ on V. By Lemma 4.4, μ is f_∞-invariant.

However, since f_∞ has a critical point c in $U \cap V$, $\mu(c)$ must be equal to zero, and it is a contradiction.

Many estimates in McMullen's proof can be applied similarly to our case. However, the main difficulty is to avoid critical points of f. For example, we will construct univalent maps h_n by choosing an inverse branch of iterates of f, so we must check that the forward orbit x does not pass near critical points outside C_R.

5 Thin rigidity

Now we will give the proof of Theorem 4.1. For simplicity, we consider only the case $\liminf \ell(\gamma_n)$ is sufficiently small. We use the same notations as in Section 3.

We say a renormalization (f^n, U_n, V_n) is *unbranched* if $V_n \cap P(f) = P_n$.

Lemma 5.1. *There exists some $L > 0$ (depends only on d) which satisfies the following:*

If $\ell(\gamma_n) < L$, then we can take an unbranched representation (f^n, U_n, V_n) of ρ_n with $\mathrm{mod}(U_n, V_n) > m(\ell(\gamma_n))$ where $m(\ell)$ is a positive function which tends to infinity as $\ell \to 0$.

Proof. Let A_n be the standard collar of γ_n in $\mathbb{C} \setminus P(f)$ and let B_n be the component of $f^{-n}(A_n)$ which has the same homotopy class in $\mathbb{C} \setminus P(f)$. Let D_n (*resp.* E_n) be the union of B_n (*resp.* A_n) and the bounded component of $\mathbb{C} \setminus B_n$ (*resp.* $\mathbb{C} \setminus A_n$). Then $f^n : D_n \to E_n$ is a proper critically compact map.

Then there exists some $M > 0$ such that if $\mathrm{mod}(P(\rho_n), E_n) > M$ then we can take $U_n' \subset D_n$ and $V_n' \subset E_n$ as follows: (f^n, U_n', V_n') is a renormalization and $\mathrm{mod}(U_n', V_n') > m(\mathrm{mod}(P(\rho_n), E_n))$ (see [Mc, Theorem 5.12]). Since $E_n \cap P(f) = P(\rho_n)$, (f^n, U_n', V_n') is unbranched.

Since $\mathrm{mod}(P(\rho_n), E_n) \geq \mathrm{mod}\, A_n$, there exists some $L > 0$ such that if $\ell(\gamma_n) < L$ then we can take an unbranched renormalization (f^n, U_n', V_n') with $\mathrm{mod}(U_n', V_n') > m(\ell(\gamma_n))$. □

Therefore, we will prove the following:

Theorem 5.2 (Polynomial-like rigidity). *Let f as above. Suppose there exists some $m > 0$ such that (f^n, U_n, V_n) is unbranched with $\mathrm{mod}(U_n, V_n) > m$ for infinitely many $n \in \mathrm{sr}(f, C_R)$.*

Then f carries no invariant line field on its Julia set.

Corollary 5.3 (Thin rigidity). *Let f as above. There exists some $L > 0$ such that*

$$\liminf_{n \in \mathrm{sr}(f, C_R)} \ell(\gamma_n) < L$$

Then f carries no invariant line field on its Julia set.

Lemma 5.4. *Assume an unbranched renormalization (f^n, U_n, V_n) satisfies* $\mathrm{mod}(U_n,$ *$m > 0$. Let E be a component of $f^{-1}(J(\rho_n, i))$ that is not $J(\rho_n, i-1)$.*

Then in the hyperbolic metric on $\mathbb{C} \setminus P(f)$, the diameter of E is bounded in terms of m.

Note that E does not intersects $P(f)$ because $P(f) \subset \mathcal{J}_n$.

Proof. Since $\mathrm{mod}(J(\rho_n), V_n) > \mathrm{mod}(U_n, V_n) > m$, $\mathrm{mod}(J(\rho_n, i), V_n(\rho_n, i))$ is greater than $m/2^d$.

Let W be the component of $f^{-1}(V(\rho_n, i))$ which contains E. Then $f : W \to V(\rho_n, i)$ is a branched cover of degree less than d. Note that all critical points of this map lie in E. Hence $\mathrm{mod}(E, W)$ is greater than $m/(2^d d)$.

Therefore, the diameter of E with respect to the hyperbolic metric on W is bounded in terms of m. By the Schwarz-Pick lemma, the hyperbolic diameter of E on $\mathbb{C} \setminus P(f)$ is also bounded. □

Lemma 5.5. *Suppose there exists some $m > 0$ such that for infinitely many $n \in \mathrm{sr}(f, C_R)$, (f^n, U_n, V_n) is unbranched with $\mathrm{mod}(U_n, V_n) > m$.*
Then f is robust and

$$P(f) = \bigcap_{n \in \mathrm{sr}(f, C_R)} \mathcal{J}(\rho_n).$$

Proof. For $n \in \mathrm{sr}(f, C_R)$ with $\mathrm{mod}(U_n, V_n) > m$, let A_n be an annulus in $V_n \setminus U_n$ enclosing $K(\rho_n)$ with $\mathrm{mod}\, A_n > m$.

Then the hyperbolic length in A_n of the core curve of A_n is less than π/m. Since the core curve of A_n is homotopic to γ_n in $\mathbb{C} \setminus P(f)$, $\ell(\gamma_n)$ is also less than π/m by the Schwarz-pick lemma. Therefore, f is robust.

By Theorem 3.5, the postcritical set is a Cantor set of measure zero and $\sup_i \mathrm{diam}\, P(\rho_n, i) \to 0$. Since $\mathrm{mod}(U_n, V_n) > m$, we have $\mathrm{diam}\, J(\rho_n, i) < C\, \mathrm{diam}\, P(\rho_n, i)$ for some C which depends only on m and d. Thus $\sup \mathrm{diam}\, J(\rho_n, i) \to 0$ as well. Since $J(\rho_n, i)$ intersects $P(f)$, the theorem follows. □

Lemma 5.6. *Under the same assumption as Lemma 5.5, almost every $x \in J(f)$ satisfies the following:*

1. *The forward orbit of x does not intersects $P(f)$.*
2. *$\|(f^n)'(x)\| \to \infty$ with respect to the hyperbolic metric on $\mathbb{C} \setminus P(f)$.*
3. *For each $n \in \mathrm{sr}(f, C_R)$, there exists some $k > 0$ such that $f^k(x) \in \mathcal{J}(\rho_n)$.*
4. *For each $k > 0$, there exists some $n \in \mathrm{sr}(f, C_R)$ such that $f^k(x) \notin \mathcal{J}(\rho_n)$.*

Proof. Since $P(f)$ is measure zero, so is $\bigcup f^{-k}(P(f))$, which implies *1.*

2. follows from Lemma 1.1.

By Lemma 1.2, $d(f^k(x), P(f)) \to 0$ for almost every $x \in J(f)$. Since $P(f) = \bigcup P(\rho_n, i)$, if $f^k(x)$ is sufficiently close to $P_n(i)$, then $f^{k+1}(x)$ must be close to $P(\rho_n, i+1)$ (note that $P(f) \cap I(\rho_n) = \emptyset$).

Therefore, $f^{k+nj-i}(x)$ lies in U_n for all $j > 0$. However, this means that $f^{k+n-i}(x) \in J(\rho_n)$, so we proved *3.*

By Lemma 5.5, $\mathrm{area}(\mathcal{J}(\rho_n))$ tends to zero. Therefore, $\bigcap_n f^{-k}(\mathcal{J}(\rho_n))$ is measure zero for any k and we have now proved *4.* □

We now prove Theorem 5.2.

Proof of Theorem 5.2. Let

$$\mathrm{usr}(f, C_R, m) = \{n \in \mathrm{sr}(f, C_R) \mid \rho_n \text{ is unbranched and } \mathrm{mod}(U_n, V_n) > m\}.$$

Suppose $\#\mathrm{usr}(f, C_R, m) = \infty$ and there exists an f-invariant line field supported on $E \subset J(f)$ of positive Lebesgue measure.

Fix a point $x \in E$ which satisfies the conditions in Lemma 5.6 and where μ is almost continuous. For each $n \in \mathrm{usr}(f, C_R, m)$, let $k(n) \geq 0$ be the smallest number which satisfies $f^{k(n)+1}(x) \in \mathcal{J}(\rho_n)$ and assume $f^{k(n)+1}(x) \in J(\rho_n, i(n)+1)$. Note that $k(n) \to \infty$.

Let $n_0 = \min \mathrm{usr}(f, C_R, m)$. Consider sufficiently large $n \in \mathrm{usr}(f, C_R, m)$ so that $k(n) > k(n_0)$. Especially, $k(n)$ is positive so $f^{k(n)}(x)$ does not lie in $\mathcal{J}(\rho_n)$.

Therefore, $f^{k(n)}(x)$ lies in a component E of $f^{-1}(J(\rho_n, i(n)+1))$, which is not equal to $J(\rho_n, i(n))$. But since $k(n) > k(n_0)$, we have $f^{k(n)}(x) \in \mathcal{J}(\rho_{n_0})$. Hence E lies in $\mathcal{J}(\rho_{n_0})$ and does not contain any critical points.

By Lemma 5.4, the hyperbolic diameter of E in $\mathbb{C}\setminus P(f)$ is bounded in terms of m. Moreover, there exists a univalent branch $\tilde{h}_n$ of $f^{-k(n)-1}$ on $V(\rho_n, i(n)+1)$ which sends $f^{k(n)+1}(x)$ to x.

Take $j(n)$ so that $i(n) < j(n) \leq n$, $C(\rho_n, j(n))$ is nonempty, and there exists a univalent inverse branch

$$V(\rho_n, j(n)) \xrightarrow{f^{-1}} V(\rho_n, j(n)-1) \xrightarrow{f^{-1}} \cdots \xrightarrow{f^{-1}} V(\rho_n, i(n)+1).$$

Let h_n be the composition of the map above and $\tilde{h}_n$. Namely, h_n is a univalent branch of $f^{-j(n)+i(n)-k(n)}$ on $V(\rho_n, j(n))$ which sends $f^{j(n)-i(n)+k(n)}(x)$ to x.

Let $J_n^* = h_n(J(\rho_n, j(n)))$. Then $f^{k(n)}(J_n^*) = E$. Since $\|(f^k)'(x)\|$ tends to infinity,

$$\operatorname{diam} J_n^* \to 0$$

with respect to the hyperbolic metric on $\mathbb{C}\setminus P(f)$ by Koebe distortion theorem.

There exists some $c \in C_R$, such that for infinitely many $n \in \operatorname{usr}(f, C_R, m)$, c lies in $J(\rho_n, j(n))$. Furthermore,

$$(f^n, U_n(j(n)), V_n(j(n)))$$

is also unbranched and satisfies

$$\operatorname{mod}(U_n(j(n)), V_n(j(n))) > m/2^d.$$

Hence by replacing c_0, m, U_n and V_n with c, $m/2^d$, $U_n(j(n))$ and $V_n(j(n))$ respectively, we may suppose $j(n) = n$ for infinitely many $n \in \operatorname{usr}(f, C_R, m)$.

For such n, let

$$\begin{aligned} A_n(z) &= \frac{z - c_0}{\operatorname{diam}(J(\rho_n))} \\ g_n &= A_n \circ f^n \circ A_n^{-1} \\ y_n &= A_n(h_n^{-1}(x)). \end{aligned}$$

Then

$$(g_n, A_n(U_n), A_n(V_n)))$$

is a polynomial-like map with $\operatorname{diam}(J(g_n)) = 1$ and $\operatorname{mod}(A_n(U_n), A_n(V_n)) > m$. After passing to a subsequence, g_n converges to some polynomial-like map (or polynomial) (g, U, V) with $\operatorname{mod}(U, V) \geq m$ in the Carathéodory topology.

Let $k_n = h_n \circ A_n^{-1}$ defined on $A_n(V_n)$. Then $k_n(y_n) = x$ and $\nu_n = k_n^*(\mu)$ is g_n-invariant line field on $A_n(V_n)$. Since $\operatorname{diam}(J(g_n)) = 1$, while $\operatorname{diam}(k_n(J(g_n))) = \operatorname{diam}(J_n^*) \to 0$, $k_n'(y_n) \to 0$ by Koebe distortion theorem.

y_n lies in $J(g_n)$ and $J(g_n)$ is surrounded by an annulus of definite modulus. Hence after passing to a further subsequence, y_n converges to some $y \in V$.

By Lemma 4.5, there exists a further subsequence such that μ_n converges to a univalent line field μ on V.

The critical point 0 lies in $J(g) \subset U \cap V$. However, it contradicts the fact that the univalent line field μ is g-invariant by Lemma 4.4.

Therefore, f carries no invariant line field on its Julia set. $\square$

References

[Bu] P. Buser, *Geometry and Spectra of Compact Riemann Surfaces*, Birkhauser Boston, 1992.

[DH] A. Douady and J. Hubbard, *On the dynamics of polynomial-like mappings*, Ann. sci. Éc. Norm. Sup., **18** (1985) 287-343.

[EY] A. Epstein and M. Yampolsky, *Geography of the cubic connectedness locus* I: *Intertwining surgery*, SUNY at Stony Brook IMS preprint 1996/10.

[Hu] J. H. Hubbard, Local connectivity of Julia sets and bifurcation loci: three theorems of J.-C. Yoccoz, *Topological Methods in Modern Mathematics* (*Stony Brook, NY, 1991*), 467-511, Publish or Perish, Houston, TX, 1993.

[Mc] C. McMullen, *Complex Dynamics and Renormalization*, Annals of Math Studies, vol. 135, 1994.

[MS] C. McMullen and D. Sullivan. *Quasiconformal homeomorphisms and dynamics* III: *The Teichmüller space of a holomorphic dynamical system*, Adv. Math. **135** (1998) 351-395.

[Mi1] J. Milnor, *Remarks on Iterated Cubic Maps*, SUNY at Stony Brook IMS preprint 1990/6.

[Mi2] J. Milnor, *Local Connectivity of Julia Sets: Expository Lectures*, SUNY at Stony Brook IMS preprint 1992/11.

[Mi3] J. Milnor, *Dynamics in One Complex Variable*, Friedr. Vieweg & Sohn, Braunschweig, 1999.

[St] N. Steinmetz, *Rational Iteration*, de Gruyter, 1993.