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Topics in dynamics of rational semigroups and fibered rational maps *

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Abstract

This article is a survey on dynamics of fibered rational maps, semigroups generated by rational or entire semigroups and random holomorphic dynamics.

1 Introduction

The modern theory of iteration of rational functions has been started in early 80's. Since then, so many articles concerning this field has been published. Some mathematicians pointed out that there are a large amount of similarities between the field of iteration of rational functions and that of Kleinian groups.

In early 90's, A.Hinkkanen and G.Martin discovered that the moduli space of discrete groups free on two generators (of given but fixed trace) is a one-complex-dimensional space which is modelled by the complement of the filled-in Julia set of some semigroup generated by (infinitely many) polynomials. After that, they started the research on dynamics of semigroups generatd by rational functions. ([HM1], [HM2], [HM3]). Of course, the field of 'semigroups generated by rational functions' contains iteration of rational functions, that of Kleinian groups and iterated function systems generated by some elements of Aut($\overline{\mathbb{C}}$). From the early 90's F.Ren's group in China has studied the same subject and has obtained the same results. ([GR]).

In recent several years some articles on the dynamics of semigroups generated by rational functions which had some results on completely invariant sets ([St1], [St2]), uniformly perfectness of Julia sets ([St3]) by R.Stankewitz, Teichmüller theory for semigroups ([Ha2]) by T.Harada, invariant measures and entropy ([Bo1], [S5]) by D.Boyd, H.Sumi, (semi)-hyperbolic dynamics and Hausdorff dimension of Julia sets ([S1], [S2], [S4], [S7]) by H.Sumi have been written. Sometimes the idea in iterated function systems (for example, [MU1],[MU2]) can be used. Difficult points are: that the Julia set of a rational semigroup may not be forward invariant and that the Julia set of sequence of words may not depend continuously on the sequence.

Since the middle of 90's S.Heinemann has studied the dynamics of skew product polynomials in $\mathbb{C}^2$. ([He1], [He2]). M.Jonsson followed the subject.([J1]). They discussed about decomposition of maximal entropy measures into fiberwise measures and hyperbolicity. They used the potential theory and current theory.

In 1997 O.Sester started the research on dynamics of skew product polynomials of which base spaces are arbitrary compact metric spaces.([Se1],[Se2]). He obtained many results especially on quadratic fibered polynomials. In [Se1] he constructed a compact connected configuration space which gives a combinatorial model of a subset of the parameter space. Then he explained how an abstract configuration can be realized by a quadratic fibered polynomial. In [S2] he discussed about hyperbolicity and generalized some results in [J1].

In [S4] that result on hyperbolicity by O.Sester was generalized to the case of semi-hyperbolic dynamics on fibered rational maps. This was a key to obtain uniformly perfectness of fiberwise Julia set, Johnness of the fiberwise attracting basins of semi-hyperbolic fibered rational maps and the upper estimate of Hausdorff dimension of Julia sets of semi-hyperbolic semigroups generated by rational functions.([S4], [S6]).

In [J2], [S5] and [S6], the entropy of fibered rational maps and the uniqueness of maximizing measures were discussed.

There is another context that is called random holomorphic dynamics. In 1991, J.E.Fornaess and N.Sibony showed that if f_c is a random polynomial map where c is taken over small polydisc, then for almost surely sequence, the Julia set has Lebesgue measure zero.([FS]). Developing the idea in this article, R.Brück, M.Büger and S.Reitz investigated the case of random quadratic polynomials in detail. They studied the Lebesgue measure and connectedness of random Julia sets and density of random orbits.([BBR], [Br], [Bu1], [Bu2]).

There are some works in which higher dimensional random complex dynamics or holomorphic semigroups are discussed. ([ZR],[FW],[Mae]).

2 Fibered rational maps

In this section we consider the fiber-preserving complex dynamics on fiber bundles. This setting sometimes gives us an integrated point of view among the field of skew product polynomials in higher dimensional comlex dynamics, dynamics of semigroups generated by rational functions and random complex dynamics.

Definition 2.1. ([J2]) A triplet (π, Y, X) is called a "$\overline{\mathbb{C}}$-bundle " if

1. Y and X are compact metric spaces,
2. $\pi : Y \to X$ is a continuous and surjective map,
3. There exists an open covering $\{U_i\}$ of X such that for each i there exists a homeomorphism $\Phi_i : U_i \times \overline{\mathbb{C}} \to \pi^{-1}(U_i)$ satisfying that $\Phi_i(\{x\} \times \overline{\mathbb{C}}) = \pi^{-1}(x)$ and $\Phi_j^{-1} \circ \Phi_i : (U_i \cap U_j) \times \overline{\mathbb{C}} \to (U_i \cap U_j) \times \overline{\mathbb{C}}$ is a Möbius map for each $x \in U_i \cap U_j$.

Remark 1. By the condition 3, each fiber $Y_x := \pi^{-1}(x)$ has a complex structure. We also have that given $x_0 \in X$ we may find a continuous family $i_x : \overline{\mathbb{C}} \to Y_x$ of homeomorphisms for x close to x_0. Such a family $\{i_x\}$ will be called a "lacal parameterization." Since X is compact, we may assume that there exists a compact subset M_0 of the set of Möbius transformations of $\overline{\mathbb{C}}$ such that $i_x \circ j_x^{-1} \in M_0$ for any two local parametrizatios $\{i_x\}$ and $\{j_x\}$. In this paper we always assume that.

Definition 2.2. ([J2]) We say that a $\overline{\mathbb{C}}$-bundle (π, Y, X) satisfies the "continuous forms condition" if for each $x \in X$ there exists a smooth $(1,1)$-form $\omega_x > 0$ inducing the metric on Y_x and $x \mapsto \omega_x$ is continuous. That is, if $\{i_x\}$ is a local parametrization, then the pull back $i_x^* \omega_x$ is a positive smooth forms on $\overline{\mathbb{C}}$ depending continuously on x.

Definition 2.3. Let (π, Y, X) be a $\overline{\mathbb{C}}$-bundle. Let $f : Y \to Y$ and $g : X \to X$ be continuous maps. We say that f is a rational map fibered over g if

1. $\pi \circ f = g \circ \pi$
2. $f|_{Y_x} : Y_x \to Y_{g(x)}$ is a rational map for any $x \in X$. That is, $(i_{g_x})^{-1} \circ f \circ i_x$ is a rational map from $\overline{\mathbb{C}}$ to itself for any local parametrization i_x at $x \in X$ and $i_{g(x)}$ at $g(x)$.

Notation: If $f : Y \to Y$ is a rational map fibered over $g : X \to X$, then we put $f_x^n = f^n|_{Y_x}$ for any $x \in X$ and $n \in \mathbb{N}$. Furthermore we put $d_n(x) = \deg(f_x^n)$ and $d(x) = d_1(x)$ for any $x \in X$ and $n \in \mathbb{N}$.

Definition 2.4. Let (π, Y, X) be a $\overline{\mathbb{C}}$-bundle. Let $f : Y \to Y$ is a rational map fibered over $g : X \to X$. Then for any $x \in X$ we denote by F_x the set of points $y \in Y_x$ which has a neighborhood U in Y_x satisfying that $\{f_x^n\}_{n \in \mathbb{N}}$ is a normal family in U, that is, $y \in F_x$ if and only if the family $Q_x^n = i_{x_n}^{-1} \circ f_x^n \circ i_x$ of rational maps on $\overline{\mathbb{C}}$ (x_n denotes $g^n(x)$) is normal near $i_x^{-1}(y)$: note that by Remark 1, this does not depend on the choices local parametrizations at x and x_n. Still equivalently, F_x is the open subset of Y_x where the family $\{f_x^n\}$

of mappings from Y_x into Y is local equicontinuous. We put $J_x = Y_x \setminus F_x$. Furthermore, we put

$$\tilde{J}(f) = \overline{\bigcup_{x \in X} J_x}, \ \tilde{F}(f) = Y \setminus \tilde{J}(f).$$

Remark 2. There exists a fibered rational map $f : Y \to Y$ satysfying that $\bigcup_{x \in X} J_x$ is NOT compact.

Remark 3. In [D] it was shown that if M is a ruled surface and $f : M \to M$ is a non-constant holomorphic map, then f is actually a fibered rational map on the $\overline{\mathbb{C}}$-bundle M.

We can construct a fibered rational map on trivial bundle from a generator system of a semigroup generated by rational functions. To investigate the dynamics of semigroups, we sometimes study the fibered rational maps.

2.1 Potential Theory and Measure Theory

We need some notations from [J2] and [S4], concerning potential theoritic aspects. Let (π, Y, X) be a $\overline{\mathbb{C}}$-bundle satisfying the continuous forms condition with a family $\{\omega_x\}_{x \in X}$ of positive $(1,1)$-forms. Let $f : Y \to Y$ be a rational map fibered over $g : X \to X$. Let $x \in X$ be a point. We set $x_n = g^n(x)$ for each $n \in \mathbb{N}$. The form ω_x on Y_x induces a measure which is also called ω_x on Y_x or even on Y. As measures on Y we have that $x \mapsto \omega_x$ is weakly continuous. For each continuous function φ on Y_x let $(f_x^n)_* \varphi$ be the continuous function on Y_{x_n} defined by $((f_x^n)_* \varphi)(z) = \sum_{f_x^n(w)=z} \varphi(w)$ for each $n \in \mathbb{N}$. We define pullbacks of measures by duality: $\langle (f_x^n)^* \nu, \ \varphi \rangle = \langle \nu, \ (f_x^n)_* \varphi \rangle$. Let $\mu_{x,n}$ be the probability measure on Y_x defined by $\mu_{x,n} = \frac{1}{d_n(x)} (f_x^n)^* \omega_{x_n}$.

We will lift $f_x : Y_x \to Y_{x_1}$ to self maps of $\overline{\mathbb{C}}$ and $\mathbb{C}^2_* := \mathbb{C}^2 \setminus \{0\}$. Let i_x and i_{x_1} be local parametrizations near x and x_1. Define $Q_x : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ to be a rational map and $R_x : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ to be a homogeneous polynomial map, both of degree $d(x)$, such that

$$\sup\{|R_x(z,w)| : |(z,w)| = 1\} = 1$$

and such that

$$f_x \circ i_x = i_{x_1} \circ Q_x, \ Q_x \circ \pi' = \pi' \circ R_x,$$

where we denote by π' the projection from $\mathbb{C}^2_*$ to $\overline{\mathbb{C}}$. Given the local parametrizations i_x and i_{x_1} these properties determine Q_x uniquely, and R_x uniquely up to multiplication by a complex number of unit modulus.

Now consider and orbit $(x_j)_{j \in \mathbb{N}}$ in X, select parametrizations at each point x_j and let R_{x_j} be the corresponding homogeneous selfmaps of $\mathbb{C}^2_*$. Let R_x^n be the composition $R_{x_n} \circ \cdots \circ R_x$. Then R_x^n is a homogeneous polynomial mapping of $\mathbb{C}^2_*$ of degree $d_n(x)$. Notice that R_x^n is determined, up to multiplication of by a complex number of unit modulus, by the local parametrizations at x and x_n.

Given a local parametrization $i_x : \overline{\mathbb{C}} \to Y_x$ there exists a smooth potential $G_{x,0}$ for ω_x in the sense that $\omega_x = dd^c(G_{x,0} \circ s \circ i_x^{-1})$, where s is any local section of π' and $d^c = \frac{i}{2\pi}(\overline{\partial} - \partial)$.

Define the plurisubharmonic function $G_{x,n}$ on $\mathbb{C}^2_*$ by

$$G_{x,n} = \frac{1}{d_n(x)} G_{x,0} \circ R_x^n.$$

If we change the local parametrizations at x_n and the potential $G_{x,0}$, then the modified plurisubharmonic function $\tilde{G}_{x,n}$ satisfies that there exists a constant $C > 0$ such that

$$|G_{x,n}(z,w) - \tilde{G}_x^n(z,w)| \leq \frac{C}{d_n(x)}, \tag{1}$$

for all $x \in X$, (z,w) and $n \in \mathbb{N}$. By (1) and the arguments in [J2] and [S4], we get the following.

Proposition 2.5. *Let (π, Y, X) be a $\overline{\mathbb{C}}$-bundle satisfying the continuous forms condition with a family $\{\omega_x\}_{x \in X}$ of positive $(1,1)$-forms. Let $f : Y \to Y$ be a rational map fibered over $g : X \to X$. Assume that $d(x) \geq 2$ for each $x \in X$. Then we have the following.*

1. *$\mu_{x,n}$ converges to a probability measure μ_x on Y_x weakly as $n \to \infty$ for each $x \in X$.*

2. *$G_{x,n}$ converges to a continuous plurisubharmonic function G_x locally uniformly on $\mathbb{C}^2_*$ as $n \to \infty$ for each $x \in X$. This function does not depend on the choice of local parametrizations at $x_j, j \geq 1$ and potentials $G_{x,0}$.*

3. *$\mu_x = (i_x^{-1})_*(dd^c(G_x \circ s))$ where s is a local section of $\pi' : \mathbb{C}^2_* \to \overline{\mathbb{C}}$. Further $G_x(z,w) \leq \log|(z,w)| + O(1)$ as $|(z,w)| \to \infty$ and $G_x(\lambda z, \lambda w) = G_x(z,w) + \log \lambda$ for each $\lambda \in \mathbb{C}$, for each $x \in X$.*

4. *$G_{x_1} \circ R_x = d(x) \cdot G_x$ for each $x \in X$.*

5. *if $x \to x'$ then $G_x \to G_{x'}$ uniformly on $\mathbb{C}^2_*$.*

6. *$(f_x)_* \mu_x = \mu_{x_1}$, $(f_x)^* \mu_{x_1} = d(x_1) \cdot \mu_x$ for each $x \in X$.*

7. *μ_x puts no mass on polar subsets of Y_x for each $x \in X$.*

8. *$x \mapsto \mu_x$ is continuous with respect to the weak topology of measures in Y.*

9. *$supp(\mu_x) = J_x$ for each $x \in X$.*

10. *J_x has no isolated points for each $x \in X$.*

11. *$x \mapsto J_x$ is lower semicontinuous with respect to the Hausdorff metric in the space of non-empty compact subsets of Y. That is, if $x, x^n \in X, x^n \to x$ as $n \to \infty$ and $y \in Y_x$, then there exists a sequence (y_n) of points in Y with $y_n \in Y_{x^n}$ for each $n \in \mathbb{N}$ such that $y_n \to y$ as $n \to \infty$.*

2.2 Entropy

Now we show some results on entropy of rational maps on $\overline{\mathbb{C}}$-bundles using the arguments in [J2].

Notation: Let (Y, d) be a metric space. Let $f : Y \to Y$ be a continuous mapping. For any compact subset Z of Y we denote by $h(f, Z)$ the entropy of f on Z. We set $h(f) = h(f, Y)$. For any f-invariant probability measure ν on Y we denote by $h_\nu(f)$ the metric entropy of f with respect to ν. If $g : X \to X$ is a continuous mapping on a compact metric space X and $\pi : Y \to X$ is a continuous mapping such that $g \circ \pi = \pi \circ f$, then we denote by $h_\nu(f|g)$ the metric entropy of f relative to g with respect to ν. See [J2] for these notations and definitions.

Theorem 2.6 ([J2],[S6]). *Let (π, Y, X) be a $\overline{\mathbb{C}}$-bundle. Let $f : Y \to Y$ be a rational map fibered over $g : X \to X$. Then the following holds.*

1. *$h(f, Y_x) \leq \limsup_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n-1} \log d(x_n)$ for any $x \in X$. If the function $d(x)$ is constant, then $h(f, Y_x) = \log d$.*

2. *If μ is an f-invariant probability measure on Y, then we have*
$$h_\mu(f|g) \leq \int_X \log d(x)\ d(\pi_* \mu)(x).$$

3. *$h(f) \leq \sup\{h_{\pi_*\mu}(g) + \int_X \log d(x)\ d(\pi_*\mu)(x))\}$, where the supremum is taken over all f-invariant probability measures μ on Y. If the function $d(x)$ is constant, then we have $h(f) = h(g) + \log d$.*

Theorem 2.7 ([J2],[S6]). *Let (π, Y, X) be a $\overline{\mathbb{C}}$-bundle satisfying the continuous forms condition with a family $(\omega_x)_{x\in X}$ of positive $(1,1)$-forms. Let $f : Y \to Y$ be a rational map fibered over $g : X \to X$. Assume that $d(x) \geq 2$ for any $x \in X$. Let μ' be a g-invariant Borel probability measure on X. Define the measure μ on Y by:*
$$\langle \mu, \varphi \rangle = \int_X (\int_{Y_x} \varphi(y)\ d\mu_x(y)) d\mu'(x)$$
for continuous funcitions φ on Y, where μ_x is the measure in Proposition 2.5. Then we have the following.

1. μ is f-invariant.

2. if μ' is ergodic, then so is μ.

3. if μ' is (strongly)mixing, then so is μ.

4. $h_\mu(f|g) = \sup_\nu h_\nu(f|g) = \int_X \log d(x)\ d\mu'(x)$, where the supremum is taken over all f-invariant probability measures ν satisfying $\pi_*\nu = \mu'$.

Problem. The interesting problems concerning the above result are:

1. the uniqueness of the measure μ with $\pi_*\mu = \mu'$ which gives us the equality in Therem 2.7.4.

2. the uniqueness of the maximal entropy measure of the fibered rational maps.

Here are some results concerning these problems.

Theorem 2.8 ([J2]). *Let (π, Y, X) be a $\overline{\mathbb{C}}$-bundle satisfying the continuous forms condition with a family $(\omega_x)_{x\in X}$ of positive $(1,1)$-forms. Let $f : Y \to Y$ be a rational map fibered over $g : X \to X$. Assume that there exist a constant integer $d \geq 2$ such that $d(x) = d$ for each $x \in X$. Let μ' be a g-invariant Borel probability measure on X. Define the measure μ on Y as in Theorem 2.7 Let ν be another invariant Borel probability measure for f such that $\pi_*\nu = \mu'$. Then the following holds.*

1. If $h_\nu(f|g) = \log d$, then $\nu = \mu$.

2. If $h_\nu(f) = h_\mu(f)$, then $\nu = \mu$.

Further, if g has a unique measure μ' of maximal entropy, then μ, defined as in Theorem 2.7, is the unique measure of maximal entropy for f.

Now we consider the case of skew product maps associated with finitely generated semigroups of rational functions. Let $\{f_1, \ldots, f_m\}$ be finitely many rational functions. We set $\Sigma_m = \{1, \ldots, m\}^{\mathbb{N}}$ and let $\sigma : \Sigma_m \to \Sigma_m$ be the shift map. Let $\tilde{f} : \Sigma_m \times \overline{\mathbb{C}} \to \Sigma_m \times \overline{\mathbb{C}}$ be the skew product map defined by: $\tilde{f}((x, y)) = (\sigma(x),\ f_{x_1}(y))$, where $x = (x_1, x_2 \ldots)$. We call this $\tilde{f}$ the skew product map associated with the generator system $\{f_1 \ldots, f_m\}$.

Let $\tilde{K}$ be a compact subset of $\Sigma_m \times \overline{\mathbb{C}}$ which is backward invariant under $\tilde{f}$. We define an operator $\tilde{B_a}$ on the space of complex valued continuous functions $C(\tilde{K})$ as follows. For each element $\tilde{\varphi} \in C(\tilde{K})$ we set

$$(\tilde{B_a}\tilde{\varphi})(z) = \Sigma_{\zeta\in\tilde{f}^{-1}(z)}\tilde{\varphi}(\zeta)\tilde{\psi_a}(\zeta)$$

where $\tilde{\psi_a}(\zeta) = \frac{a_{w_1}}{d_{w_1}}$ if $\pi_1(\zeta) = (w_1, w_2, \ldots)$.

$\tilde{B}_a$ is a bounded operator on $C(\tilde{K})$.

Notation: If G is a semigroup generated by non-constant rational functions on $\overline{\mathbb{C}}$ with the semigroup operation being the composition of maps, then G is called a **rational semigroup**. For any rational semigroup G, we denote by $F(G)$ the Fatou set for G:i.e. the set of points z which has a neighborhood where G is normal. We set $J(G) = \overline{\mathbb{C}} \setminus F(G)$ and this is called the Julia set for G. Further we set $E(G) = \{z \mid \sharp \bigcup_{g \in G} g^{-1}(z) < \infty\}$ and this is called the exceptional set for G. For more details, see the section of rational semigroups.

Theorem 2.9 ([S5]). *Let Gbe a rational semigroup generated by finitely many non-constant rational functions $\langle f_1, \dots, f_m \rangle$. Assume that there exists an element $g_0 \in G$ of degree at least two, the exceptional set $E(G)$ for G is included in $F(G)$ and $F(H) \supset J(G)$ where H is a rational semigroup defined by $H = \{h^{-1} \mid h \in Aut\ (\overline{\mathbb{C}}) \cap G\}$.(if H is empty, put $F(H) = \overline{\mathbb{C}}$.) Let $\tilde{f} : \Sigma_m \times \overline{\mathbb{C}} \to \Sigma_m \times \overline{\mathbb{C}}$ be the skew product map associated with the generator system $\{f_1, \dots, f_m\}$. Then all of the following hold.*

1. *For each weight $a = (a_1, \dots, a_m)$ with $\sum_{j=1}^m a_j = 1$ and $a_j > 0$, there exists a unique regular Borel probability measure $\tilde{\mu}_a$ on $\Sigma_m \times \overline{\mathbb{C}}$ for each compact set $\tilde{K}$ which is included in $\Sigma_m \times (\overline{\mathbb{C}} \setminus E(G))$ and backward invariant under $\tilde{f}$, we have*
$$\|\tilde{B}_a^n(\tilde{\varphi}) - \tilde{\mu}_a(\tilde{\varphi})\mathbf{1}\|_{\tilde{K}} \to 0, \quad \text{as } n \to \infty$$
for any continuous function $\tilde{\varphi}$ on $\tilde{K}$ where $\|\cdot\|_{\tilde{K}}$ denotes the supremum norm on $\tilde{K}$ and $\mathbf{1}$ denotes the constant function taking its value 1.

2. *$\tilde{B}_a^*(\tilde{\mu}_a) = \tilde{\mu}_a$ and $\tilde{\mu}_a$ is $\tilde{f}$-invariant. The projection of $\tilde{\mu}_a$ onto Σ_m is the Bernoulli measure with respect to the weight a.*

3. *The support of $\tilde{\mu}_a$ is equal to the "Julia set" $\tilde{J}$ of $\tilde{f}$.*

4. *$(\tilde{f}, \tilde{\mu}_a)$ is exact.*

5. *Let ν_a be the Bernoulli measure on Σ_m corresponding to the weight $a = (a_1, \dots, a_m)$. Then*
$$\sup_{\rho \in E(\tilde{f}, \nu_a)} h_\rho(\tilde{f}|\sigma) = \sum_{j=1}^m a_j \log \deg(f_j),$$
where $E(T, \nu_a)$ denotes the set of all ergodic $\tilde{f}$-invariant probability measures ρ on $\Sigma_m \times \overline{\mathbb{C}}$ satisfying $(\pi_1)_*(\rho) = \nu_a$ and $h_\rho(\tilde{f}|\sigma)$ denotes the "relative metric entropy" of $\tilde{f}$ with respect to ρ.

6. *The relative metric entropy of $\tilde{f}$ with respect to $\tilde{\mu}_a$ is:*

$$h_{\tilde{\mu}_a}(\tilde{f}|\sigma) = \sum_{j=1}^{m} a_j \log \deg(f_j)$$

and $\tilde{\mu}_a$ is the unique element of $E(\tilde{f}, \nu_a)$ satisfying the above.

7. *Let $\tilde{\mu}$ be the measure for the weight*

$$\tilde{a} = \Big(\frac{\deg(f_1)}{\sum_{j=1}^{m} \deg(f_j)}, \ldots, \frac{\deg(f_m)}{\sum_{j=1}^{m} \deg(f_j)}\Big).$$

Then $\tilde{\mu}$ is the unique maximal entropy measure and we have

$$h(\tilde{f}) = h_{\tilde{\mu}}(\tilde{f}) = \log\Big(\sum_{j=1}^{m} \deg(f_j)\Big).$$

In particular, the projection of maximal entropy measure of $\tilde{f}$ onto the base space Σ_m is equal to the Bernoulli measure corresponding to the above weight $\tilde{a}$.

Remark 4.
- David Boyd's invariant measure([Bo1]) is the projection of $\tilde{\mu}$ to $\overline{\mathbb{C}}$. To show the convergence of $\tilde{\mu}_a^n$ we developed the method in [Bo1]. Considering the projection of $\tilde{\mu}_a$ to $\overline{\mathbb{C}}$, the above result can be regarded as a generalization of the result on uniqueness of usual "self-similar measures" of iterated function systems generated by some similitudes.
- One of the motivations for the above result is to estimate the 'entropy of semigroup actions'. If G is a finitely generated semigroup acting on a compact metric space and $\mathcal{S} = \{f_1, \ldots, f_m\}$ is a fixed generator system of G, then we can define the entropy $h(G, \mathcal{S})$ of G with respect to $\mathcal{S}$ in the same way as that of the entropy of any group action with respect to a fixed generator system of the group. By definition, we have $h(G, \mathcal{S}) \leq h(\tilde{f})$, where $\tilde{f}$ is the skew product associated with the generator system $\mathcal{S}$.

2.3 Skew product polynomials on $\mathbb{C}^2$

In this subsection we introduce the works of S.Heinemann and M.Jonsson on skew product polynomials on $\mathbb{C}^2$. ([He1],[He2],[J1]). The first research on polynomial skew product on higher dimensional space was given by S.Heinemann. ([He1],[He2]).

Definition 2.10. A polynomial skew product on $\mathbb{C}^2$ of degree d is a map of the form $f(z,w) = (p(z), q(z,w))$, where p and q are polynomials of degree d and where $p(z) = z^d + O(z^{d-1})$ and $q(z,w) = w^d + O(w^{d-1})$.

Remark 5. If $f(z,w) = (p(z), q(z,w))$ is a polynomial skew product on $\mathbb{C}^2$, then f can be extended to a holomorphic map on $\mathbb{P}^2$. Also we can consider a fibered rational map $f : J(p) \times \overline{\mathbb{C}} \to J(p) \times \overline{\mathbb{C}}$ fibered over $p : J(p) \to J(p)$.

Theorem 2.11 ([J1]). *Let $f(z,w) = (p(z), q(z,w))$ be a polynomial skew product map on $\mathbb{C}^2$ of degree $d \geq 2$. Regarding f as a map on $\mathbb{C}^2$ (or $\mathbb{P}^2$), we associate a Green function G, measuring the rate of escape to infinity, a positive closed current $T = \frac{1}{2\pi} dd^c G$ and an invariant probability measure $\mu = T \wedge T$. Let μ' be the maximal entropy measure for $p : J(p) \to J(p)$. Let $\{\mu_x\}_{x \in J(p)}$ be the family of probability measures in Proposition 2.5 constructed by the fibered rational map $f : J(p) \times \overline{\mathbb{C}} \to J(p) \times \overline{\mathbb{C}}$ over $p : J(p) \to J(p)$. Then we have*

$$\mu = \int_{J(p)} \mu_x \, d\mu'(x).$$

In particular, the second Julia set J_2 for $f : \mathbb{P}^2 \to \mathbb{P}^2$, which is defined as the support of μ, satisfies the following:

$$J_2 = \tilde{J}(f),$$

where $\tilde{J}(f)$ is the set defined in Definition 2.4 for the fibered rational map $f : J(p) \times \overline{\mathbb{C}} \to J(p) \times \overline{\mathbb{C}}$ over $p : J(p) \to J(p)$. Moreover, J_2 is the closure of the repelling periodic points of $f : \mathbb{C}^2 \to \mathbb{C}^2$.

Remark 6. Concerning Theorem 2.11, see also Theorem 2.8. By Theorem 2.8, the map f on $\mathbb{C}^2$ (or $\mathbb{P}^2$) in Theorem 2.11 has the unique maximal entropy measure μ.

Let $f(z,w) = (p(z), q(z,w))$ be a polynomial skew product map on $\mathbb{C}^2$ of degree $d \geq 2$. Let μ be the maximal entropy measure for f in Theorem 2.11. We now investigate the Lyapunove exponent λ_1, λ_2 with $\lambda_1 \geq \lambda_2$ for f with respect to the measure μ. Let $\lambda(p)$ be the Lyapunov exponent for p. Then by Przytycki([P]), we know that

$$\lambda(p) = \log d + \int G_p \mu_{c,p},$$

where G_p is the Green function for p and $\mu_{c,p}$ is a *critical measure* defined by:

$$\mu_{c,p} = \sum_{p'(c)=0} \delta_c. \tag{2}$$

Define $H = \log|\partial q/\partial y|$. Define also a new critical measure $\mu_{c,q}$ by:

$$\mu_{c,q} = (\frac{1}{2\pi})^2 dd_y^c H \wedge dd_x^c G_p. \qquad (3)$$

Under these notations, we have the following result.

Theorem 2.12 ([J1]). *Under the above, we have the followings.*

1. $\lambda_1 = \log d + \int G_p \mu_{c,p}$,
2. $\lambda_2 = \log d + \int G \mu_{c,q}$.

Some types of polynomial skew products in $\mathbb{C}^2$ were investigated in [He1] and [He2] by S.Heinemann.

2.4 Quadratic fibered polynomials

In 1997 O.Sester investigated quadratic fibered polynomial maps in detail. ([Se1], [Se3]). Let

$$f_c : X \times \overline{\mathbb{C}} \to X \times \overline{\mathbb{C}}, \ f_c(x,y) = (g(x), y^2 + c(x))$$

be a fibered polynomial map, where X is a compact space, g is a continuous map on X and c is a continuous complex-valued function on X considered as a parameter. He constructed a compact connected configuration space which gives a combinatorial model of a subset of the parameter space. Then he explained how an abstract configuration can be realized by a quadratic fibered polynomial. He defined the fiberwise equipotential curves and external rays for fibered polynomial maps. Then he used the idea of 'Yoccoz puzzle' for quadratic fibered polynomial maps.

2.5 Semi-hyperbolicity

Notation :

- Let Z_1 and Z_2 be two topological spaces and $g : Z_1 \to Z_2$ be a map. For any subset A of Z_2, we denote by $c(g, A)$ the set of all connected components of $g^{-1}(A)$.
- for any $y \in \overline{\mathbb{C}}$ and $\delta > 0$, we put $B(y,\delta) = \{y' \in \overline{\mathbb{C}} \mid d(y,y') < \delta\}$, where d is the spherical metric. Similarly, for any $y \in \mathbb{C}$ and $\delta > 0$ we put $D(y,\delta) = \{y' \in \mathbb{C} \mid |y - y'| < \delta\}$.

Now we will define the (semi-)hyperbolicity of fibered rational maps.

Definition 2.13. Let f be a fibered rational map on a $\overline{\mathbb{C}}$-bundle. with continuous forms $\{\omega_x\}$. We say that f is expanding along fibers if there exists a positive constant C and a constant λ with $\lambda > 1$ such that for each $n \in \mathbb{N}$, we have

$$\inf_{z \in \tilde{J}(f)} \|(\tilde{f}^n)'(z)\| \geq C\lambda^n,$$

where we denote by $\| \cdot \|$ the norm of the derivative with respect to the metrics on fibers induced by $\{\omega_x\}$.

Definition 2.14. **(semi-hyperbolicity)** Let (π, Y, X) be a $\overline{\mathbb{C}}$-bundle. Let $f : Y \to Y$ be a rational map fibered over $g : X \to X$. Let $N \in \mathbb{N}$. We say that a point $z \in Y$ belongs to $SH_N(f)$ if there exists a positive number δ, a neighborhood U of $\pi(z)$ and a local parametrization $\{i_x\}$ in U such that for any $x \in U$, any $n \in \mathbb{N}$, any $x_n \in g^{-1}(x)$ and any $V \in c(i_x(B(i_{\pi(z)}^{-1}(z),\ \delta)),\ f_x^n)$, we have

$$\deg(f_x^n : V \to i_x(B(i_{\pi(z)}^{-1}(z),\ \delta))) \leq N.$$

We set

$$UH(f) = Y \setminus \bigcup_{N \in \mathbb{N}} SH_N(f).$$

We say that f is semi-hyperbolic (along fibers) if for any point $z \in Y$ there exists a positive integer $N \in \mathbb{N}$ satisfying that $z \in SH_N(f)$.

The result of the following lemma is a beauty deduced from semi-hyperbolicity.

Lemma 2.15 ([S4]). *Let V be a domain in $\overline{\mathbb{C}}$, K a continuum in $\overline{\mathbb{C}}$ with $diam_S K = a$. Assume $V \subset \overline{\mathbb{C}} \setminus K$. Let $f : V \to D(0,1)$ be a proper holomorphic map of degree N. Then there exists a constant $r(N,a)$ depending only on N and a such that for each r with $0 < r \leq r(N,a)$, there exists a constant $C = C(N,r)$ depending only on N and r satisfying that for each connected component U of $f^{-1}(D(0,r))$,*

$$diam_S\ U \leq C,$$

where we denote by $diam_S$ the spherical diameter. Also we have $C(N,r) \to 0$ as $r \to 0$.

We need some technical conditions.

Definition 2.16 (Condition(C1)). Let (π, Y, X) be a $\overline{\mathbb{C}}$-bundle. Let $f : Y \to Y$ be a rational fibered over $g : X \to X$. We say that f satisfies the condition (C1) if there exists a family $\{D_x\}_{x \in X}$ of topological discs with $D_x \subset Y_x$, $x \in X$ such that the following three conditions are satisfied:

1. $\overline{\bigcup_{n \geq 0} f_x^n(D_x)} \subset \tilde{F}(f)$ for each $x \in X$.

2. for any $x \in X$, we have that $\text{diam}_Y(f_x^{(n)}(D_x)) \to 0$, as $n \to \infty$.

3. $\inf_{x\in X}$ diam $_Y(D_x) > 0$.

Definition 2.17 (Condition(C2)). Let (π, Y, X) be a $\overline{\mathbb{C}}$-bundle. Let $f : Y \to Y$ be a rational map fibered over $g : X \to X$. We say that f satisfies the condition (C2) if for each $x_0 \in X$ there exists an open neighborhood O of x_0 and a family $\{D_x\}_{x\in O}$ of topological discs with $D_x \subset Y_x, x \in O$ such that the following three conditions are satisfied:

1. $\overline{\bigcup_{n\geq 0} f_x^n(D_x)} \subset \tilde{F}(f)$ for each $x \in O$.

2. for any $x \in O$, we have that $\text{diam}_Y(f_x^{(n)}(D_x)) \to 0$, as $n \to \infty$.

3. $x \mapsto D_x$ is continuous in O.

The following results(Theorem 2.18,2.19) are the key to investigate the dynamics of semi-hyperbolic fibered rational maps. The most important thing is the continuity of the map $x \mapsto J_x$ with respect to the Hausdorff topology. Note that there exists a fibered rational map such that $x \mapsto J_x$ is not continuous. The following results are also keys to get an upper estimate of Hausdorff dimension of semi-hyperbolic rational semigroups.

Theorem 2.18 ([S4]). *Let (π, Y, X) be a $\overline{\mathbb{C}}$-bundle. Let $f : Y \to Y$ be a rational map fibered over $g : X \to X$. Assume f satisfies the condition (C1). Let $z \in Y$ be a point with $z \in F_{\pi(z)}$. Let (i_x) be a local parametrization. Let U be a connected open neighborhood of $i_{\pi(z)}^{-1}(z)$ in $\overline{\mathbb{C}}$. Suppose that there exists a sequence (n_j) of $\mathbb{N}$ such that $R_j := i_{\pi f^{n_j}(z)}^{-1} \circ f_{\pi(z)}^{n_j} \circ i_{\pi(z)}$ converges to a non-constant map ϕ uniformly on U as $j \to \infty$. Further suppose $f_{\pi(z)}^{n_j}(z)$ converges to a point $z_0 \in Y$. Let $S_{i,j} = f_{g^{n_i}\pi(z)}^{n_j - n_i}$ for $1 \leq i \leq j$. We set*

$$V = \{a \in Y_{\pi(z_0)} \mid \exists \epsilon > 0, \limsup_{i\to\infty} \sup_{j>i} \sup_{d(\xi,y)\leq\epsilon, \xi\in Y_{\pi(z_0)}} d(S_{i,j} \circ \varphi(\xi), \xi) = 0\},$$

where φ is a map from $Y_{\pi(z_0)}$ onto $Y_{g^{n_i}\pi(z)}$ defined by the local triviality of Y around z_0. Then V is a non-empty open proper subset of $Y_{\pi(z_0)}$ and we have that

$$\partial V \subset \tilde{J}(f) \cap UH(f).$$

Remark 7. We call this domain V the *rotation domain.*

Theorem 2.19 ([S4],[S6]). *(**Key theorem**) Let (π, Y, X) be a $\overline{\mathbb{C}}$-bundle. Let $f : Y \to Y$ be a rational map fibered over $g : X \to X$. Assume f is semi-hyperbolic along fibers and satisfies the condition (C1). Then the following hold.*

1. *Let $z \in Y$ be any point with $z \in F_{\pi(z)}$. Then for any local parametrization (i_x) and any open connected neighborhood U of $i_{\pi(z)}^{-1}(z)$ in $\overline{\mathbb{C}}$, there exists no subsequence of $(i_{\pi f^n(z)}^{-1} \circ f_{\pi(z)}^n \circ i_{\pi(z)})_n$ converging to a non-constant map locally uniformly on U.*

2. $$\tilde{J}(f) = \bigcup_{x \in X} J_x.$$

3. *Suppose the condition (C2) is satisfied. Then there exist positive constants δ, L and $\lambda (0 < \lambda < 1)$ such that for any $n \in \mathbb{N}$,*

$$\sup\{diam_Y U \mid U \in c(\tilde{B}(z,\delta),\ f_{x_n}^n),\ z \in \tilde{J}(f),\ x_n \in g^{-n}(\pi(z))\} \leq L\lambda^n,$$

where we denote by $\tilde{B}(z,\delta)$ the ball in $Y_{\pi(z)}$ with the center z and the radius δ with respect to the metric in $Y_{\pi(z)}$ induced by the metric of Y.

4. *Assume that (π, Y, X) satisfies the continuous forms condition and that $d(x) \geq 2$ for each $x \in X$. Then we have that $x \mapsto J_x$ is continuous with respect to the Hausdorff metric in the space of compact subsets of Y.*

5. *Assume that (π, Y, X) satisfies the continuous forms condition with a family (ω_x) of positive $(1,1)$-forms and that $d(x) \geq 2$ for each $x \in X$. Then for any compact subset K of $\tilde{F}(f)$, we have that $\overline{\cup_{n\geq 0} f^n(K)} \subset \tilde{F}(f)$ and there exist constants $C > 0$ and $\tau < 1$ such that for each n, $\sup_{z\in K} \|(f^n)'(z)\| \leq C\tau^n$, where we denote by $\|(f^n)'(z)\|$ the norm of the derivative measured from $\omega_{\pi(z)}$ to $\omega_{g^n(\pi(z))}$. In particular, the condition (C2) is satisfied.*

Theorem 2.20 ([S6]). *(**measure zero**) Let (π, Y, X) be a $\overline{\mathbb{C}}$-bundle. Let $f : Y \to Y$ be a rational map fibered over $g : X \to X$. Suppose f is semi-hyperbolic along fibers and satisfies the condition (C2). Then for each $x \in X$, the 2-dimensional Lebesgue measure of J_x is equal to zero.*

Definition 2.21. Let C be a positive number. Let K be a closed subset of $\overline{\mathbb{C}}$. We say that K is C-uniformly perfect if for any doubly connected domain A in $\overline{\mathbb{C}}$ satisfying that both two connected components of $\overline{\mathbb{C}} \setminus A$ have non-empty intersection with K, the modulus of A is less than C.

Theorem 2.22 ([S6]). *(**uniform perfectness**) Let $(\pi, Y.X)$ be a $\overline{\mathbb{C}}$-bundle with continuous forms condition. Let $f : Y \to Y$ be a rational map fibered over $g : X \to X$ with $d(x) \geq 2$ for any $x \in X$. Suppose that f is semi-hyperbolic along fiberes and satisfies the condition (C1). Then there exists a positive constant C such that J_x is C-uniformly perfect for any $x \in X$.*

Notation: Let $y \in \mathbb{C}$ and $b \in \overline{\mathbb{C}}$ be two distinct points. Let E be a curve in $\overline{\mathbb{C}}$ joining y to b satisfying that $E \setminus \{b\} \subset \mathbb{C}$. For any $c \geq 1$ we set

$$\text{car } (E, c, y, b) = \bigcup_{z \in E \setminus \{y, b\}} D(z, \frac{|y-z|}{c}).$$

This is called the c-carrot with core E and vertex y joining y to b.

Definition 2.23. Let V be a subdomain of $\overline{\mathbb{C}}$. Let $c \geq 1$ be a number. We say that V is a c-John domain if there exists a point $y_0 \in \overline{V}$ satisfying that for any $y \in V \setminus \{y_0\}$ there exists a curve E joining y_0 to y such that $E \setminus \{y_0\} \subset \mathbb{C}$ and

$$\text{car } (E, c, y, y_0) \subset V.$$

In the above the point y_0 is called the center of John domain V.

Remark 8. Johnness implies many good properties ([NV], [Jone]).For example, if V is a John domain, then the following facts hold.

- If $\infty \in \overline{V}$, then the center of V is ∞.
- Let $a \in \partial V \setminus \{\infty\}$ and $b \in V$. Then there exists a curve E joining a to b and a constant c such that car $(E, c, a, b) \subset V$. In particular, a is accessible from b.
- V is finitely connected at any point in ∂V: that is, if $y \in \partial V$, then there exists an arbitrary small open neighborhood U of y in $\overline{\mathbb{C}}$ such that $U \cap V$ has only finitely many connected components.
- If V is simply connected and $\partial V \subset \mathbb{C}$, then we have that ∂V is locally connected.
- If $\partial V \subset \mathbb{C}$ then ∂V is holomorphic removable: that is, if $\varphi : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a homeomorphim and is holomorphic on $\overline{\mathbb{C}} \setminus \partial V$, then φ is holomorphic on $\overline{\mathbb{C}}$. From this fact, we can deduce that the 2-dimensional Lebesgue measure of ∂V is equal to zero.

Theorem 2.24 ([S6]). *(**Johnness**) Let $(\pi, Y = X \times \overline{\mathbb{C}}, X)$ be a trivial $\overline{\mathbb{C}}$-buundle. Let $f : Y \to Y$ be a rational map fibered over $g : X \to X$ satisfying that f_x is a polynomial with $d(x) \geq 2$ for any $x \in X$. Then there exists a positive constant c such that for any $x \in X$ the basin of infinity $A_x := \{y \in Y_x \mid f_x^n(y) \to \infty, \ n \to \infty\}$ in Y_x (here we identify f_x^n with a usual polynomial) satisfies that it is a c-John domain.*

Remark 9. In the Theorem 2.24 if X is a set consisting of one point, then f is semi-hyperbolic if and only if the basin of infinity is a John domain([CJY]).

3 Rational semigroups

For a Riemann surface S, let $\mathrm{End}(S)$ denote the set of all holomorphic endomorphisms of S. It is a semigroup with the semigroup operation being composition of maps. A *rational semigroup* is a subsemigroup of $\mathrm{End}(\overline{\mathbb{C}})$ without any constant elements. We say that a rational semigroup G is a *polynomial semigroup* if each element of G is a polynomial.

Definition 3.1. Let G be a rational semigroup. We set

$$F(G) = \{z \in \overline{\mathbb{C}} \mid G \text{ is normal in a neighborhood of } z\}, \ J(G) = \overline{\mathbb{C}} \setminus F(G).$$

$F(G)$ is called the *Fatou set* for G and $J(G)$ is called the *Julia set* for G. The backward orbit $G^-(z)$ of z and the set of exceptional points $E(G)$ are defined by: $G^-(z) = \cup_{g \in G} g^{-1}(z)$ and $E(G) = \{z \in \overline{\mathbb{C}} \mid \sharp G^-(z) \leq 2\}$.

Definition 3.2. A subsemigroup H of a semigroup G is said to be of finite index if there is a finite collection of elements $\{g_1, g_2, \dots , g_n\}$ of G such that $G = \cup_{i=1}^n g_i H$. Similarly we say that a subsemigroup H of G has cofinite index if there is a finite collection of elements $\{g_1, g_2, \dots , g_n\}$ of G such that for every $g \in G$ there exists a $j \in \{1, 2, \dots , n\}$ such that $g_j g \in H$.

Next results were shown in [HM1]. F.Ren's group in China has shown almost the same results(dealing with all meromorphic semigroups).

Theorem 3.3 ([HM1],[GR]). *Let G be a rational semigroup.*

1. *For each $f \in G$, we have $f(F(G)) \subset F(G)$ and $f^{-1}(J(G)) \subset J(G)$. Note that we do not have that the equality holds in general.*

2. *If a subsemigroup H of G is of finite or cofinite index, then $J(H) = J(G)$. In particular, when G is a rational semigroup generated by finite elements $\{f_1, f_2, \dots f_n\}$ and m is an integer, if we set*

$$H_m = \{g = f_{j_1} \cdots f_{j_k} \in G \mid m \text{ divides } k\},$$

$$I_m = \{g \in G \mid g \text{ is a product of some elements of word length } m\}$$

then $J(G) = J(H_m) = J(I_m)$. Here we say an element $f \in G$ is of word length m if m is the minimum integer such that $f = f_{j_1} \cdots f_{j_m}$.

3. *If $J(G)$ contains at least three points, then $J(G)$ is a perfect set.*

4. *If $J(G)$ contains at least three points, then $\sharp E(G) \leq 2$.*

5. *If a point z is not in $E(G)$, then for every $x \in J(G)$, x belongs to $\overline{G^-(z)}$. In particular if a point z belongs to $J(G) \setminus E(G)$, then $\overline{G^-(z)} = J(G)$.*

6. *If $J(G)$ contains at least three points, then $J(G)$ is the smallest closed backward invariant set containing at least three points. Here we say that a set A is backward invariant under G if for each $g \in G$, $g^{-1}(A) \subset A$.*

7. *If $J(G)$ contains at least three points, then*

$$J(G) = \overline{\{z \in \overline{\mathbb{C}} \mid z \text{ is a repelling fixed point of some } g \in G\}}$$

If G is generated by a compact subset of $\mathrm{End}(\overline{\mathbb{C}})$, then $J(G)$ has the *backward self-similarity.* That is,

Lemma 3.4 ([S4]). *Let G be a rational semigroup and assume G is generated by a precompact subset Λ of End$(\overline{\mathbb{C}})$. Then*

$$J(G) = \overline{\bigcup_{f \in \Lambda} f^{-1}(J(G))} = \bigcup_{h \in \overline{\Lambda}} h^{-1}(J(G)).$$

In particular if Λ is compact then we have $J(G) = \bigcup_{f \in \Lambda} f^{-1}(J(G))$.
We call this property the backward self-similarity of the Julia set.

3.1 Completely invariant sets

The Julia set of a rational semigroup may not be forward invariant. For example, $J(\langle z^2,\ 2z \rangle) = \{|z| \le 1.\}$. Hence a natural question is; what is the smallest compact subset of $\overline{\mathbb{C}}$ which is completely invariant under each element of the semigroup?

Definition 3.5 ([St1],[St2]). We say that a set Y is completely invariant under a map f if $f^{-1}(Y) \subset Y$ and $f(Y) \subset Y$. For any rational semigroup G, we denote by $CI(G)$ the smallest compact subset of $\overline{\mathbb{C}}$ which has at least three points and is completely invariant under each element of G. Actually this set exists. This set $CI(G)$ is called the completely invariant J-set for G. Furthermore we set $W(G) = \overline{\mathbb{C}} \setminus CI(G)$.

Theorem 3.6 ([St1]). *For polynomials f and g of degree at least two, $J(f) \neq J(g)$ implies $CI(\langle f, g \rangle) = \overline{\mathbb{C}}$.*

To show this result, the Green's function in the component of $W(G)$ which contains the infinity is used. Further for 'rational' semigroup, we have

Theorem 3.7 ([St2]). *Let G be a rational semigroup of which any element is of degree at least two. Then $W(G)$ can have only $0, 1, 2$ or infinitely many components.*

Here is a conjecture concerning the above problem.

Conjecture 3.8 (St2). *Let G be a rational semigroup of which any element is of degree at least two. Suppose there exist two maps f and g in G such that $J(f) \neq J(g)$. Then $CI(G) \neq \overline{\mathbb{C}}$ implies that $CI(G)$ is a simple closed curve in $\overline{\mathbb{C}}$.*

Remark 10. It is not known that if the Fatou set $F(G)$ must have only $0, 1, 2$, or infinitely many components when G is a finitely generated rational semigroup. However, for each positive n, an example of an infinitely generated polynomial semigroup G can be constructed with the property that $F(G)$ has exactly n components. These examples were constructed by David Boyd in [Bo2]

3.2 Uniformly perfectness

Uniformly perfectness is an important notion in the complex analysis, as we discussed in the section of semi-hyperbolicity of fibered rational maps. In [HM2] it was shown that the Julia set of finitely generated rational semigroup of which any element is of degree at least two is uniformly perfect. This result was generalized in [St3] as follows. The proof of the paper is more staight-forward than that in [HM2] and this result by Stankewitz is valid for Klenian groups, iteration of rational functions and iterated function systems. In fact, the result was given for a wide class of rational semigroups.

Theorem 3.9 ([St3]). *Let $G = \langle g_i : i \in I \rangle$ be a rational semigroup generated by the maps $\{g_i : i \in I\}$ such that the supremum of the Lipschitz constants of g_i with respect to the spherical metric in $\overline{\mathbb{C}}$ is bounded. Assume that $\sharp J(G) \geq 3$. Then the Julia set $J(G)$ is uniformly perfect.*

Now we consider the uniformly perfectness of attractors of semigroups.

Definition 3.10 ([St4]). Let U be a subdomain of $\mathbb{C}$ and K a compact subset of U. Let $\{g_i : i \in I\}$ be a family of non-constant maps from U to K such that there exists $0 < s < 1$ and a metric d on K which is compatible with the induced topology from $\mathbb{C}$ satisfying that $d(g_i(z), g_i(w)) \leq sd(z, w)$ for all $z, w \in K$ and all $i \in I$. Then We say that the semigroup G generated by $\{g_i : i \in I\}$ is a CIFS(Contracting Iterated Function System) on (U, K). For a CIFS on (U, K) we set $A'(G) = \overline{\{z \in K \mid \exists g \in G,\ g(z) = z\}}$ (the closure is taken in the topology of K.) and this set is called the attractor (in the sense of this subsection). If G is a CIFS on (U, K) and each element of G is analytic on U, then G is called an analytic CIFS.

Theorem 3.11 ([St4]). *Let $G = \langle g_i : i \in I \rangle$ be an analytic CIFS on (U, K). Let A' be the attractor. Suppose there exist $0 < \delta <$ $diam(A)$ and $C > 0$ such that we have the following:*

1. *if $a \in A'$ and $i \in I$, then g_i is one-to-one on $D(a, \delta)$*

2. *if $a \in A'$ and $g_i(a) = a'$ then the branch h_i of g_i^{-1} such that $h_i(a') = a$ is defined on $D(a', \delta)$,*

3. *if $a \in A'$ and $g_i(a) = a'$, then the branch h_i of g_i^{-1} such that $h_i(a') = a$ satisfies*
$$|h_i(z) - h_i(a')| \leq C|z - a'|$$
for all $z \in \overline{D(a', \delta/10)}$.

Then, if the attractor set A' has infinitely many points, then A' is uniformly perfect. Note that in the assumption we take the Euclidian metric.

Corollary 3.12 ([St4]). *Let $G = \langle g_i : i \in I \rangle$ be an analytic CIFS on (U, K). Let A' be the attractor. Suppose that there exists $\eta > 0$ where $|g_i'(a)| \geq \eta$ for all $a \in A'$ and all $i \in I$. If A' has infinitely many points, then A' is uniformly perfect.*

If the attractor set has a critical point of an element of G, then the attractor set may not be uniformly perfect. In fact,

Example 3.13 (St4). Let $G = \langle z^{23}, (z - 1/2)^2 + 1/2 \rangle$. Then G is actually an analytic CIFS on some (U, K) and the attractor set is NOT uniformly perfect.

3.3 Normality of inverse branches

We introduce a result by A.Hinkkanen and G.Martin.

Theorem 3.14 ([HM3]). *Let G be a rational semigroup whose every element has degree at least 2. Suppose that any seqence in G contains a subsequence, say f_j, such that each f_j can be factorized as $f_j = g_J \circ \varphi$ for rational functions g_j and φ that need not be elements of G, where φ is independent of j and has degree at least 2. Let $D \subset \overline{\mathbb{C}}$ be a domain. Let $\mathcal{F}$ be a family of single-valued mermorphic functions in D such that each element f of $\mathcal{F}$ is a branch of the inverse of some element of G in D. Then $\mathcal{F}$ is a normal family.*

Remark 11. Theorem 3.14 was used in [Bo1] and [S5] to show the convergence of the iteration of the operators $\tilde{B}_a$. (See Theorem 2.9). K.Maegawa investigated the normality of inverse branches of fibered rational maps in higher dimension.([Mae]).

3.4 Wandering or no wandering domains

Next we define stable basin, type of the basins and wandering domains.

Definition 3.15. Let G be a rational semigroup and U a connected component of $F(G)$.

- For each $g \in G$, we denote by U_g the connected component of $F(G)$ containing $g(U)$.
- U is called a wandering domain if there exist infinitely many distinct components U_j of $F(G)$ and elements g_j of G such that $g_j(U) \subset U_j$.
- We say that U is a stable basin if there is an element $g \in G \setminus \mathrm{Aut}\overline{\mathbb{C}}$ such that $g(U) \subset U$. And we set

$$G_U \overset{\text{def}}{=} \{g \in G \mid g(U) \subset U\}.$$

- Given a stable basin U for G we say that it is
  1. **attracting** if U is a subdomain of an attracting basin of each $g \in G_U$ of degree at least two;
  2. **superattracting** if U is a subdomain of a superattracting basin of each $g \in G_U$ of degree at least two;
  3. **parabolic** if U is a subdomain of a parabolic basin of each $g \in G_U$ of degree at least two;
  4. **Siegel** if U is a subdomain of a Siegel disk of each $g \in G_U$ of degree at least two;
  5. **Herman** if U is a subdomain of a Herman ring of each $g \in G_U$ of degree at least two.

Definition 3.16 ([HM1]). Let G be a rational semigroup containing an element g with $\deg(g) \geq 2$. We say that G is nearly abelian if there is a compact family of Möbius (or linear fractional) transformations $\Phi = \{\varphi\}$ with the following properties.

- $\varphi(F(G)) = F(G)$ for all $\varphi \in \Phi$
- for all $f,\ g \in G$ there is a $\varphi \in \Phi$ such that $fg = \varphi g f$

Theorem 3.17 ([HM1]). *Let G be a nearly abelian rational semigroup with an element in of degree at least two. Then for each $g \in G$ of degree at least two, we have $J(G) = J(g)$.*

Theorem 3.18 ([HM1]). *Let G be a nearly abelian rational semigroup with an element in of degree at least two. Then G has no wandering domains.*

There is an important example.

Theorem 3.19 ([HM1]). *There exists an infinitely generated polynomial semigroup which has a wandering domain.*

But here is a conjecture.

Conjecture 3.20 ([HM1]). *If G is a finitely generated rational semigroup, then there exists no wandering domain for the dynamics of G.*

Theorem 3.21 ([HM1]). *Let G be a rational semigroup with an element in of degree at least two. Suppose G has no wandering domains. Let U be any component of $F(G)$. Then the forward orbit of U under G, that is, $\{U_g\}_{g\in G}$, contains a stable basin W satisfying that G_W is a cofineit index subsemigroup of G.*

Theorem 3.22 ([HM1]). *Let G be a nearly abelian rational semigroup with an element in of degree at least two. Let U be a stable basin. Then U is either attracting, superattracting, parabolic, Siegel or Herman. In the Siegel case the basin U contains a single cycle fixed by each element of G_U. If U is of Siegel or Herman type, then G_U is abelian.*

One of conjectures in [HM1] was solved in [Ha1] by T.Harada.

Theorem 3.23 ([Ha1]). *If G is a nearly abelian polynomial semigroup and G contains some polynomials of degree at least two, then there exists a neighborhood of ∞ on which G is analytically conjugate into $\langle z \mapsto az^n : |a| = 1,\ n = 1, 2, 3, \dots \rangle$.*

3.5 Teichmüller theory for rational semigroups

In this subsection we introduce the Teichmüller theory for rational semigroups in [Ha2] by T.Harada. The following definitions are due to the paper [MS].

Definition 3.24. Let X be a Riemann surface and G be a subsemigroup of End(X). We denote by $D(X, G)$ the set of triplets $\{(\varphi, Y, H)\}$ where Y is a Riemann surface, H is a subsemigroup of End(Y), $\varphi : X \to Y$ is a quasi-conformal map, and they satisfy that $\varphi \circ G \circ \varphi^{-1} = H$. Two elements (φ_1, Y_1, H_1) and (φ_2, Y_2, H_2) of $D(X, G)$ are said to be equivalent if there exists a biholomorphic map $h : Y_1 \to Y_2$ such that $h \circ \varphi_1 = \varphi_2$. We denote by Def$(X, G)$ the equivalence classes and this is called the deformation space for (X, G). Further let $M(X, G)$ be a space of all measurable Beltrami differentials which is invariant under the action of G. It is a Banach space with the sup norm. We denote by $M_1(X, G)$ the open unit ball centered at zero in $M(X, G)$. For a measurable set E included in X, we denote by $M_1(E, G)$ the subspace of $M_1(X, G)$ that consists of all elements whose supports are included in E.

Lemma 3.25. *The map*

$$\mathit{Def}(X, G) \ni (\varphi, Y, H) \mapsto \mu_\varphi \in M_1(X, G)$$

is bijective, where μ_φ is the Beltrami differential of φ.

Definition 3.26. We define the q.c. automorphism group $QC(X.G)$ as the set of all quasi-conformal maps ω from X to itself which satisfies $\omega \circ G \circ \omega^{-1} = G$. This group acts on Def$(X, G)$ as

$$w : (\varphi, Y, H) \mapsto (\varphi \circ \omega^{-1}, Y, H)$$

for $\omega \in QC(X, G)$. Its normal subgroup $QC_0(X, G)$ is defined as the group of all ω_0 admitting a uniformly quasiconformal isotopy ω_t *rel* the ideal boundary of X, such that $\omega_1 = id_X$ and

$$\omega_t \circ G \circ \omega_t^{-1} = G, \ (0 \le t \le 1).$$

The *Teichmüller space* Teich(X, G) for (X, G) is defined as: Def$(X, G)/QC_0(X, G)$. The *modular transformation group* Mod(X, G) is defined as a quotient group: $QC(X, G)/QC_0(X, G)$.

Definition 3.27. Let G be a countable rational semigroup. We denote by $C(G)$ the set of critical points of some element of G. We denote by $B(G)$ the set of fixed points of grand orbit relation of G. We denote by $\hat{J}_0(G)$ the grand orbit of $C(G) \cup B(G)$ under G. We denote by $\hat{J}(G)$ the closure of $\hat{J}_0(G)$. We set $\hat{\Omega}(G) = \overline{\mathbb{C}} \setminus \hat{J}(G)$. We resolve $\hat{\Omega}(G)$ to two parts. We define $\Omega^{dis}(G)$ as all points which have the discrete grand orbit and $\Omega^{fol}(G)$ as a complement of $\Omega^{dis}(G)$. Sometimes (G) is omitted.

Theorem 3.28 ([Ha2]). *Let G be a countable rational semigroup of which Julia set has at least three points. Then*

$$Teich(\overline{\mathbb{C}}, G) \cong M_1(\hat{J}, G) \times Teich(\Omega^{fol}, G) \times Teich(\Omega^{dis}/G, \emptyset),$$

where Ω^{dis}/G is a Riemann surface and the isomorphism is the one as complex Banace manifolds. And Teich$(\overline{\mathbb{C}}, G)$ has the unique complex structure which makes the canonical projection

$$Def(X, G) \to Teich(\overline{\mathbb{C}}, G)$$

is holomorphic.

Theorem 3.29 ([Ha2]). *Let G be a finitely generated rational semigroup of which Julia set has at least three points. Then Teich$(\overline{\mathbb{C}}, G)$ is a finite dimensional complex manifold.*

Theorem 3.30 ([Ha2]). *Let G be a finitely generated rational semigroup of which Julia set has at least three points. Suppose $G \cap Aut(\overline{\mathbb{C}}) = \emptyset$. Then the action of Mod$(\overline{\mathbb{C}}, G)$ to Teich$(\overline{\mathbb{C}}, G)$ is properly discontinuous.*

Remark 12. Most hope that there exists a nearly abelian finitely generated rational semigroup G (not generated by one map) such that $\Omega^{dis}(G)$ is not empty.

From the point of view of Theorem 3.6, some may think that the following conjecture is true.

Conjecture 3.31. *Let f and g be two rational maps of degree at least two. Suppose $J(f) \neq J(g)$ and $J(\langle f, g\rangle) = \overline{\mathbb{C}}$. Let μ be a Beltrami differential on $\overline{\mathbb{C}}$ with the norm less than one. Suppose μ is invariant under both f and g. Then $\mu = 0$.*

Remark 13. From Theorem 3.29, we at least know that the space of Beltrami differentials on $\overline{\mathbb{C}}$ with the norms less than one which are invariant under both f and g is a finite-dimensional ball.

3.6 Sub, Semi-hyperbolicity

Definition 3.32. Let G be a rational semigroup. We set

$$P(G) = \overline{\bigcup_{g \in G} \{ \text{ critical values of } g\}}.$$

We call $P(G)$ the post critical set of G. We say that G is *hyperbolic* if $P(G) \subset F(G)$. Also we say that G is *sub-hyperbolic* if $\sharp\{P(G) \cap J(G)\} < \infty$ and $P(G) \cap F(G)$ is a compact set.

We denote by $B(x, \epsilon)$ a ball of center x and radius ϵ in the spherical metric. We denote by $D(x, \epsilon)$ a ball of center $x \in \mathbb{C}$ and radius ϵ in the Euclidean metric. Also for any hyperbolic manifold M we denote by $H(x, \epsilon)$ a ball of center $x \in M$ and radius ϵ in the hyperbolic metric. For any rational map g, we denote by $B_g(x, \epsilon)$ a connected component of $g^{-1}(B(x, \epsilon))$. For each open set U in $\overline{\mathbb{C}}$ and each rational map g, we denote by $c(U, g)$ the set of all connected components of $g^{-1}(U)$. Note that if g is a polynomial and $U = D(x, r)$ then any element of $c(U, g)$ is simply connected by the maximal principle.

For each set A in $\overline{\mathbb{C}}$, we denote by A^i the set of all interior points of A.

Definition 3.33. Let G be a rational semigroup and N a positive integer. We set

$$\begin{aligned} & SH_N(G) \\ = \ & \{x \in \overline{\mathbb{C}} | \exists \delta(x) > 0, \ \forall g \in G, \ \forall B_g(x, \delta(x)), \ \deg(g : B_g(x, \delta) \to B(x, \delta)) \leq N\} \end{aligned}$$

and $UH(G) = \overline{\mathbb{C}} \setminus (\cup_{N \in \mathbb{N}} SH_N(G))$.

Definition 3.34. Let G be a rational semigroup. We say that G is *semi-hyperbolic* (resp. *weakly semi-hyperbolic*) if there exists a positive integer N such that $J(G) \subset SH_N(G)$(resp.$\partial J(G) \subset SH_N(G)$).

Theorem 3.35 ([S4]). *Let G be a rational semigroup. Assume that G is weakly semi-hyperbolic and there is a point $z \in F(G)$ such that the closure of the G-orbit $\overline{G(z)}$ is included in $F(G)$. Then for each $x \in F(G)$, $\overline{G(x)} \subset F(G)$ and there is no wandering domain.*

Definition 3.36. Let U be an open set in $\overline{\mathbb{C}}$. Let G be a semigroup generated by holomorphic maps from U to U. We say that a non-empty compact subset K of U is an *attractor* in U for G (in the sense of this subsection) if $g(K) \subset K$ for each $g \in G$ and for any open neighborhood V of K in U and each $z \in U$, $g(z) \in U$ for all but finitely many $g \in G$.

Definition 3.37. Let G be a rational semigroup. We set

$$A_0(G) = \overline{G(\{z \in \overline{\mathbb{C}} \mid \exists g \in G \text{ with } \deg(g) \geq 2,\ g(z) = z \text{ and } |g'(z)| < 1.\})},$$

$$\tilde{A}_0(G) = \overline{G(\{z \in F(G) \mid \exists g \in G \text{ with } \deg(g) \geq 2,\ g(z) = z \text{ and } |g'(z)| < 1.\})},$$

$$A(G) = \overline{G(\{z \in \overline{\mathbb{C}} \mid \exists g \in G,\ g(z) = z \text{ and } |g'(z)| < 1.\})},$$

$$\tilde{A}(G) = \overline{G(\{z \in F(G) \mid \exists g \in G,\ g(z) = z \text{ and } |g'(z)| < 1.\})},$$

where the closure in the definition of $\tilde{A}_0(G)$ and $\tilde{A}(G)$ is considered in $\overline{\mathbb{C}}$.

Theorem 3.38 ([S4]). *Let $G = \langle f_1, f_2, \dots, f_m \rangle$ be a finitely generated rational semigroup. Assume that $F(G) \neq \emptyset$, there is an element $g \in G$ such that* $\deg(g) \geq 2$ *and each element of Aut $\overline{\mathbb{C}} \cap G$(if this is not empty) is loxodromic. Also we assume all of the following conditions;*

1. *$\tilde{A}_0(G)$ is a compact subset of $F(G)$,*
2. *any element of G with the degree at least two has neither Siegel disks nor Hermann rings.*
3. *$\sharp(UH(G) \cap \partial J(G)) < \infty$ and all the fixed points of elements in G contained in $UH(G) \cap \partial J(G)$ are repelling.*

Then $\tilde{A}_0(G) = \tilde{A}(G) \neq \emptyset$ and for each compact subset L of $F(G)$,

$$\sup\{d(f_{i_n} \cdots f_{i_1}(z),\ \tilde{A}(G)) \mid z \in L,\ (i_n, \dots, i_1) \in \{1, \dots, m\}^n\} \to 0,$$

as $n \to \infty$, where we denote by d the spherical metric. Also $\tilde{A}(G)$ is the smallest attractor in $F(G)$ for G. Moreover we have that if (h_n) is a sequence in G consisting of mutually disjoint elements and converges to a map ϕ in a subdomain V of $F(G)$, then ϕ is constant taking its value in $\tilde{A}(G)$.

3.7 Conditions to be semi-hyperbolic

In this section we will show some conditions to be semi-hyperbolic.

Theorem 3.39 ([S4]). *Let $G = \langle f_1, f_2, \dots, f_m \rangle$ be a finitely generated rational semigroup. Let $z_0 \in J(G)$ be a point. Assume all of the following conditions:*

1. *there exists a neighborhood U_1 of z_0 in $\overline{\mathbb{C}}$ such that for any sequence $(g_n) \subset G$, any domain V in $\overline{\mathbb{C}}$ and any point $\zeta \in U_1$, we have that the sequence (g_n) does NOT converge to ζ locally uniformly on V.*
2. *there exists a neighborhood U_2 of z_0 in $\overline{\mathbb{C}}$ and a positive real number $\tilde{\epsilon}$ such that if we set*
$$T = \{c \in \overline{\mathbb{C}} \mid \exists j,\ f_j'(c) = 0,\ (G \cup \{id\})(f_j(c)) \cap U_2 \neq \emptyset\}$$
then for each $c \in T \cap C(f_j)$, we have $d(c,\ (G \cup \{id\})(f_j(c))) > \tilde{\epsilon}$.
3. *$F(G) \neq \emptyset$.*

Then $z_0 \in SH_N(G)$ for some $N \in \mathbb{N}$.

Now we get the sufficient and necessary condition to be semi-hyperbolic for a finitely generated rational semigroup.

Theorem 3.40 ([S4]). *Let $G = \langle f_1, f_2, \dots f_n \rangle$ be a finitely generated rational semigroup. Assume that there exists an element of G with the degree at least two, that each element of $Aut\ \overline{\mathbb{C}} \cap G$(if this is not empty) is loxodromic and that $F(G) \neq \emptyset$. Then G is semi-hyperbolic if and only if all of the following conditions are satisfied.*

1. *for each $z \in J(G)$ there exists a neighborhood U of z in $\overline{\mathbb{C}}$ such that for any sequence $(g_n) \subset G$, any domain V in $\overline{\mathbb{C}}$ and any point $\zeta \in U$, we have that the sequence (g_n) does NOT converge to ζ locally uniformly on V*
2. *for each $j = 1, \dots, m$ each $c \in C(f_j) \cap J(G)$ satisfies*
$$d(c,\ (G \cup \{id\})(f_j(c))) > 0$$

Theorem 3.41 ([S4]). *Let $G = \langle f_1, f_2, \dots, f_m \rangle$ be a finitely generated subhyperbolic rational semigroup. Assume that there exists an element of G with the degree at least two, that each element of $Aut\ \overline{\mathbb{C}} \cap G$(if this is not empty) is loxodromic, that there is no super attracting fixed point of any element of G in $J(G)$ and $F(G) \neq \emptyset$. Then G is semi-hyperbolic.*

3.8 Interior points of Julia sets

The Julia set of a rational semigroup may have non-empty interior points. In this subsection, we discuss about when Julia set of a rational semigroup has non-empty or empty interior points. Further we give a sufficient condition for Julia sets of semi-hyperbolic rational semigroups to be of 2-dimensional Lebesgue measure zero.

Firstable we give a sufficient condition for interior of Julia sets to be non-empty.

Theorem 3.42 **([HM2]).** *Let G be a rational semigroup. Suppose $J(G)$ is uniformly perfect. Further suppose that there exists an element $g \in G$ such that g has a superattracting fixed point z_0 in $J(G)$. Then z_0 is an interior point of $J(G)$.*

Hence by Theorem 3.9, we get the following Corollary. From this, we can easily get many examples of rational semigroups of which Julia sets have non-empty interior points.

Corollary 3.43 **([St3]).** *Let $G = \langle g_i : i \in I \rangle$ be a rational semigroup generated by the maps $\{g_i : i \in I\}$ such that the supremum of the Lipschitz constants of g_i with respect to the spherical metric in $\overline{\mathbb{C}}$ is bounded. Assume that $\sharp J(G) \geq 3$. If a point $z_0 \in J(G)$ is a superattracting point of some element of G, then z_0 is an interior point of $J(G)$.*

Now we give some sufficient conditions for interior of Julia sets to be empty.

Theorem 3.44 **([S2]).** *Let $G = \langle f_1, f_2, \dots f_n \rangle$ be a finitely generated rational semigroup. We assume that the set $\cup_{(i,j):i \neq j} f_i^{-1}(J(G)) \cap f_j^{-1}(J(G))$ does not contain any continuum. Then the Julia set $J(G)$ has no interior points.*

Definition 3.45. Let $G = \langle f_1, f_2, \dots, f_m \rangle$ be a finitely generated rational semigroup. We say that G satisfies the *open set condition* with respect to the generators $f_1, f_2, \dots, f_m$ if there exists an open set O such that for each $j = 1, \dots, m$, $f_j^{-1}(O) \subset O$ and $\{f_j^{-1}(O)\}_{j=1,\dots,m}$ are mutually disjoint.

Proposition 3.46 **([S4]).** *Let $G = \langle f_1, f_2, \dots, f_m \rangle$ be a finitely generated rational semigroup. Assume that G satisfies the open set condition with respect to the generators $f_1, f_2, \dots, f_m$ and $O \setminus J(G) \neq \emptyset$ where O is an open set in the definition of the open set condition. Then $J(G)^i = \emptyset$ where we denote by $J(G)^i$ the interior of $J(G)$.*

Now we give a sufficient condition for Julia sets of semi-hyperbolic rational semigroups to be of 2-dimensional Lebesgue measure zero.

Theorem 3.47 **([S4]).** *Let $G = \langle f_1, f_2, \dots, f_m \rangle$ be a finitely generated rational semigroup which is semi-hyperbolic, contains an element with the degree at least two and satisfies the open set condition with respect to the* *generators $f_1, f_2, \cdots, f_m$. Let O be an open set in Definition 3.45. Assume that $\sharp(\partial O \cap J(G)) < \infty$. Then the 2-dimensional Lebesgue measure of $J(G)$ is equal to 0.*

3.9 Hausdorff dimension of Julia sets

We introduce a result on an upper estimate of Haussdorff dimension of semi-hyperbolic rational semigroups. Actually the key is the Theorem 2.19.

Definition 3.48. Let G be a rational semigroup and δ a non-negative number. We say that a Borel probability measure μ on $\overline{\mathbb{C}}$ is δ-subconformal if for each $g \in G$ and for each Borel measurable set A

$$\mu(g(A)) \leq \int_A \|g'(z)\|^\delta d\mu,$$

where we denote by $\|\cdot\|$ the norm of the derivative with respect to the spherical metric. For each $x \in \overline{\mathbb{C}}$ and each real number s we set

$$S(s,\ x) = \sum_{g \in G} \sum_{g(y)=x} \|g'(y)\|^{-s}$$

counting multiplicities and

$$S(x) = \inf\{s \mid S(s,\ x) < \infty\}.$$

If there is not s such that $S(s,\ x) < \infty$, then we set $S(x) = \infty$.Also we set

$$s_0(G) = \inf\{S(x)\},\ s(G) = \inf\{\delta \mid \exists\mu : \delta\text{-subconformal measure}\}$$

It is not difficult for us to prove the next result using the same method as that in [Sul].

Theorem 3.49 ([S2]). *Let G be a rational semigroup which has at most countably many elements. If there exists a point $x \in \overline{\mathbb{C}}$ such that $S(x) < \infty$ then there is a $S(x)$-subconformal measure. In particular, we have $s(G) \leq s_0(G)$.*

Proposition 3.50 ([S4]). *Let $G = \langle f_1, f_2, \cdots, f_m \rangle$ be a finitely generated rational semigroup. Assume that G satisfies the open set condition with respect to the generators $f_1, f_2, \cdots, f_m$ and $O \setminus J(G) \neq \emptyset$ where O is an open set in the definition of the open set condition. If there exists an attractor in $F(G)$ for G, then*

$$s_0(G) \leq 2.$$

Theorem 3.51 ([S4]). *Let G be a rational semigroup generated by a generator system $\{f_\lambda\}_{\lambda\in\Lambda}$ such that $\cup_{\lambda\in\Lambda}\{f_\lambda\}$ is a compact subset of End($\overline{\mathbb{C}}$). Let $\tilde{f}$ be a rational skew product constructed by the generator system. Assume $\tilde{f}$ is semi-hyperbolic along fibers and satisfies the condition C2. Then we have*

$$\dim_H(J(G)) \leq s(G).$$

Theorem 3.52 ([S4]). *Let $G = \langle f_1, f_2, \dots, f_m\rangle$ be a finitely generated rational semigroup which is semi-hyperbolic. Assume that G contains an element with the degree at least two, each element of Aut $\overline{\mathbb{C}} \cap G$(if this is not empty) is loxodromic and $F(G) \neq \emptyset$. Then we have*

$$\dim_H(J(G)) \leq s(G) \leq s_0(G).$$

Proof. By Theorem 3.51 and Theorem 3.49. □

Remark 14 (S2,S4). Let $G = \langle f_1, f_2, \dots, f_m\rangle$ be a finitely generated hyperbolic rational semigroup which satisfies the strong open set condition (i.e. G satisfies the open set condition with an open set O satisfying $O \supset J(G)$.). We assume that when $n = 1$ the degree of f_1 is at least two. By the results in [S4](Theorem 3.2 and the proof, Theorem 3.4 and Corollary 3.5), we have

$$0 < \dim_H J(G) = s(G) = s_0(G) < 2.$$

Example 3.53 (S4). Let $G = \langle f_1, f_2\rangle$ where $f_1(z) = z^2 + 2,\ f_2(z) = z^2 - 2$. Since $P(G) \cap J(G) = \{2, -2\}$ and $P(G) \cap F(G)$ is compact, we have G is sub-hyperbolic. By Theorem 3.41, G is also semi-hyperbolic. Since $f_j^{-1}(D(0,2)) \subset D(0,2)$ for $j = 1,2$ and$f_1^{-1}(D(0,2)) \cap f_2^{-1}(D(0,2)) = \emptyset$, G satisfies the open set condition. Also $J(G)$ is included in $B = \cup_{j=1}^2 f_j^{-1}(\overline{D(0,2)})$. Since $B \cap \partial D(0,2) = \{2, -2, 2i, -2i\}$, we get $\sharp(J(G) \cap \partial D(0,2)) < \infty$. By Corollary 3.47, we have $m_2(J(G)) = 0$, where we denote by m_2 the 2-dimensional Lebesgue measure. By Theorem 3.52 and Proposition 3.50, we have also

$$dim_H(J(G)) \leq s(G) \leq s_0(G) \leq 2.$$

3.10 Using thermodynamical formalisms

Let G be a rational semigroup generated by $\{f_1, \dots, f_m\}$. Under the same notation as those in subsection of Entropy, let $\tilde{f} : \Sigma_m \times \overline{\mathbb{C}} \to \Sigma_m \times \overline{\mathbb{C}}$ be the skew product map associated with the generator system $\{f_1, \dots, f_m\}$. That is, $\tilde{f}((w, x)) = (\sigma(w),\ f_{w_1}(x))$ where σ is the shift map on Σ_m and $w = (w_1, w_2, \dots)$.

In this section, we assume that $\tilde{f}$ is expanding along fibers (see the definition in subsection of semi-hyperbolicity in fibered rational maps.)

We recall the following sufficient condition to be expanding.

Theorem 3.54 ([S2]). *Let $G = \langle f_1, f_2, \dots f_m \rangle$ be a finitely generated hyperbolic rational semigroup. Assume that G contains an element with the degree at least two and each Möbius transformation in G is neither the identity nor an elliptic element. Then the skew product map $\tilde{f} : \Sigma_m \times \overline{\mathbb{C}} \to \Sigma_m \times \overline{\mathbb{C}}$ associated with the generator system $\{f_1, \dots, f_m\}$ is expanding along fibers.*

For each $j = 1, \dots, m$, let φ_j be a Hölder continuous function on $f_j^{-1}(J(G))$. We set for each $(w, x) \in \tilde{J}$, $\varphi((w, x)) = \varphi_{w_1}(x)$. Then φ is a Hölder continuous function on $\tilde{J}$. We define an operater L on $C(\tilde{J}) = \{\psi : \tilde{J} \to \mathbb{C} \mid$ continuous $\}$ by

$$L\psi((w,x)) = \sum_{\tilde{f}((w',y))=(w,x)} \frac{\exp(\varphi((w',y)))}{\exp(P)} \psi((w',y)),$$

counting multiplicities, where we denote by $P = P(\tilde{f}|_{\tilde{J}}, \varphi)$ the pressure of $(\tilde{f}|_{\tilde{J}}, \varphi)$.

Lemma 3.55. *With the same notations as the above, let $G = \langle f_1, f_2, \dots f_m \rangle$ be a finitely generated expanding rational semigroup. Then for each set of Hölder continuous functions $\{\varphi_j\}_{j=1,\dots,m}$, there exists a unique probability measure τ on $\tilde{J}$ such that*

- $L^*\tau = \tau$,
- *for each $\psi \in C(\tilde{J})$, $\|L^n\psi - \tau(\psi)\alpha\|_{\tilde{J}} \to 0, n \to \infty$, where we set $\alpha = \lim_{l\to\infty} L^l(1) \in C(\tilde{J})$ and we denote by $\|\cdot\|_{\tilde{J}}$ the supremum norm on $\tilde{J}$,*
- *$\alpha\tau$ is an equilibrium state for $(\tilde{f}|_{\tilde{J}}, \varphi)$.*

Lemma 3.56. *Let $G = \langle f_1, f_2, \dots f_m \rangle$ be a finitely generated expanding rational semigroup. Then there exists a unique number $\delta > 0$ such that if we set $\varphi_j(x) = -\delta \log(\|f_j'(x)\|), j = 1, \dots, m,$ then $P = 0$.*

From Lemma 3.55, for this δ there exists a unique probability measure τ on $\tilde{J}$ such that $L_\delta^*\tau = \tau$ where L_δ is an operator on $C(\tilde{J})$ defined by

$$L_\delta\psi((w,x)) = \sum_{\tilde{f}((w',y))=(w,x)} \frac{\psi((w',y))}{\|(f_{w_1'})'(y)\|^\delta}.$$

Also δ satisfies that

$$\delta = \frac{h_{\alpha\tau}(\tilde{f})}{\int_{\tilde{J}} \tilde{\varphi}\alpha d\tau} \le \frac{\log(\sum_{j=1}^m \deg(f_j))}{\int_{\tilde{J}} \tilde{\varphi}\alpha d\tau},$$

where $\alpha = \lim_{l\to\infty} L_\delta^l(1)$, we denote by $h_{\alpha\tau}(\tilde{f})$ the metric entropy of $(\tilde{f}, \alpha\tau)$ and $\tilde{\varphi}$ is a function on $\tilde{J}$ defined by $\tilde{\varphi}((w,x)) = \log(\|f_{w_1}'(x)\|)$.

By these argument, we get the following result.

Theorem 3.57 ([S7]). *Let $G = \langle f_1, f_2, \dots f_m \rangle$ be a finitely generated expanding rational semigroup and δ the number in the above argument. Then*

$$\dim_H(J(G)) \le s(G) \le \delta.$$

Moreover, if the sets $\{f_j^{-1}(J(G))\}$ are mutually disjoint, then $\dim_H(J(G)) = \delta < 2$ and $0 < H_\delta(J(G)) < \infty$, where we denote by H_δ the δ-Hausdorff measure.

Corollary 3.58 ([S7]). *Let $G = \langle f_1, f_2, \dots f_m \rangle$ be a finitely generated expanding rational semigroup. Then*

$$\dim_H(J(G)) \le \frac{\log(\sum_{j=1}^m \deg(f_j))}{\log \lambda},$$

where λ denotes the number in Definition 2.13.(See Theorem 3.54).

Example 3.59. Let $G = \langle f_1, f_2 \rangle$ where $f_1(z) = z^2$ and $f_2(z) = 2.3(z-3) + 3$. Then we can see easily that $\{|z| < 0.9\} \subset F(G)$ and G is expanding. By the corollary 3.58, we get

$$\dim_H(J(G)) \le \frac{\log 3}{\log 1.8} < 2.$$

In particular, $J(G)$ has no interior points.

3.11 Lower estimate of Hausdorff dimension of Julia sets

Now we consider a generalization of Mañé's result([Ma3]).

Lemma 3.60 ([S5]). *Let $G = \langle f_1, f_2, \dots f_m \rangle$ be a finitely generated rational semigroup. Assume that the sets $\{f_i^{-1}(J(G))\}_{j=1,\dots,m}$ are mutually disjoint. We define a map $f : J(G) \to J(G)$ by $f(x) = f_i(x)$ if $x \in f_i^{-1}(J(G))$ (Note that $J(G) = \cup_{i=1}^m f_i^{-1}(J(G))$). If μ is an ergodic invariant probability measure for $f : J(G) \to J(G)$ with $h_\mu(f) > 0$, then $\int_{J(G)} \log(\|f'\|)\, d\mu > 0$ and $HD(\mu) = \frac{h_\mu(f)}{\int_{J(G)} \log(\|f'\|)\, d\mu}$, where we set*

$$HD(\mu) = \inf\{\dim_H(Y) \mid Y \subset J(G),\ \mu(Y) = 1\}.$$

The following result is shown from Lemma 3.60 and Theorem 2.9.

Theorem 3.61 ([S5]). *Let $G = \langle f_1, f_2, \dots f_m \rangle$ be a finitely generated rational semigroup. Assume that $F(H) \supset J(G)$ where $H = \{h^{-1} \mid h \in Aut(\overline{\mathbb{C}}) \cap G\}$ (if $H = \emptyset$, put $F(H) = \overline{\mathbb{C}}$.) Also assume that the sets $\{f_i^{-1}(J(G))\}_{j=1,\dots,m}$ are mutually disjoint. Then*

$$\dim_H(J(G)) \ge \frac{\log(\sum_{j=1}^m \deg(f_j))}{\int_{J(G)} \log(\|f'\|)\, d\mu},$$

where $\mu = (\pi_2)_* \tilde{\mu}_a$, $a = (\frac{d_1}{d}, \dots, \frac{d_m}{d})$ and $f(x) = f_i(x)$ if $x \in f_i^{-1}(J(G))$.

4 Semi-hyperbolic transcendental semigroups

An *entire semigroup* is a semigroup generated by non-constant entire functions in $\mathbb{C}$. In [KS] H.kriete and H.Sumi investigate a sufficient condition to be semi-hyperbolic for (not necessarily finitely generated) entire semigroups. The result is similar to that in the subsection of conditions to be semi-hyperbolic for rational semigroups. In fact, originally the idea to get the sufficient conditions to be semi-hyperbolic for finitely generated rational semigroups has come from the idea in [KS].

5 Random holomorphic dynamics

There are so many pubulished articles concerning the random dynamical systems. (for example, [A],[K]). Very recently, some have been investigating the random holomorphic dynamics. ([Br],[Bu1], [Bu2], [BBR],[FS],[FW],[Ro], [ZR]). We introduce some results of them.

5.1 Classification of sequences of polynomials

First we introduce the works of Büger's in which the classification of sequences of polynomials was given.

Definition 5.1 ([Bu1]). Given a sequence (f_n) of polynomials of one complex variable with the degrees at least two, the *Fatou set* for the sequence is the set of points in $\overline{\mathbb{C}}$ each of which has a neighborhood where the sequence is normal. The *Julia set* is the complement in $\overline{\mathbb{C}}$. For any connected component V of the Fatou set of a sequence, we denote by $\mathcal{L}(V)$ the set of all limit functions. If all elements of $\mathcal{L}(V)$ are constant functions, we call V a contracting domain, otherwise an expanding domain.

Definition 5.2 ([Bu1]). Let (f_n) be a sequence of polynomials of degree at least two. We say that A hyperbolic domain $M \subset \overline{\mathbb{C}}$ is called *invariant*, if $f_n(M) \subset M$ for all $n \in \mathbb{N}$. We say that (f_n) belongs to

1. - the class $\mathcal{P}_1$ if there is an invariant domain M, $\infty \in M$, such that $f_n \circ \cdots \circ f_1 \to \infty (n \to \infty)$, locally uniformly in M,
2. - the class $\mathcal{P}_2$ if $f_n \circ \cdots \circ f_1 \to \infty (n \to \infty)$, locally uniformly in some neighborhood of ∞, although there is no invariant domain M such that $\infty \in M$,
3. - the class $\mathcal{P}_3$, if ∞ belongs to the Julia set of (f_n).

We say (f_n) belongs to class $\mathcal{Q}$ if for each $n \in \mathbb{N}$, there exists a complex number c_n such that $f_n(z) = z^2 + c_n$.

Theorem 5.3 ([Bu1]). *If $(f_n) \in \mathcal{P}_1$, then the Julia set of the sequence is equal to the boundary of the attracting basin of infinity. Moreover, the attracting basin of infinity is equal to $\bigcup_{n=0}^{\infty}(f_n \circ \cdots \circ f_1)^{-1}(M)$.*

Definition 5.4 ([Bu1]). Let (f_n) be a sequence of polynomials. If for every domain D which intersects the Julia set J of the sequence there is an integer n such that $(f_n \circ \cdots \circ f_1)^{-1}(f_n \circ \cdots \circ f_1(D \cap J)) = J$, we call J *self-similar.*

Theorem 5.5 ([Bu1]). *If $(f_n) \in \mathcal{P}_1$, then the Julia set of the sequence is self-similar. Moreover, the Julia set is perfect or finite. In the finite case we can find $n \in \mathbb{N}$ such that $f_n \circ \cdots \circ f_1(J)$ consists of a single point.*

Theorem 5.6 ([Bu1]). *Let (f_n) be a sequence of polynomials*

$$f_n(z) = \sum_{k=0}^{d_n} a_{k_n} z^k, \;\; a_{d_n,n} \neq 0, \;\; d_n \geq 2,$$

such that:

1. $\inf\{|a_{d_n,n}| : n \in \mathbb{N}\} > 0$,
2. $\max\{|a_{k_n}| : 0 \leq k < d_n\} = O(|a_{d_n,n}|)$
   Then (f_n) is contained in $\mathcal{P}_1$. If, in addition, (f_n) satisfies
3. $\log^+ |a_{d_n,n}| = O(d_n)$, *then the Julia set is perfect.*

Theorem 5.7 ([Bu2]). *Let $(f_n) \in \mathcal{Q}$, and (c_n) be a sequence of complex numbers such that $f_n(z) = z^2 + c_n$. Then (f_n) belongs to*

1. *class $\mathcal{P}_1$, if and only if (c_n) is bounded,*
2. *class $\mathcal{P}_2$, if and only if (c_n) is not bounded, but* $\log^+ |c_n| = O(2^n)$,
3. *class $\mathcal{P}_3$, if and only if* $\limsup_{n\to\infty}(\log^+ |c_n|)/2^n = +\infty$.

In particular, $\mathcal{Q}$ is the disjoint union of $\mathcal{P}_1 \cap \mathcal{Q}$, $\mathcal{P}_2 \cap \mathcal{Q}$ and $\mathcal{P}_3 \cap \mathcal{Q}$.

We investigate the class $\mathcal{P}_1$.

Theorem 5.8 ([Bu2]).
1. *Let $(f_n) \in \mathcal{P}_1 \cap \mathcal{Q}$ and V be a contracting domain. Suppose $V \subset \mathbb{C}$. Then $\mathcal{L}(V)$ is compact and $\mathcal{L}(V) \cap \{z \mid |z| \leq 1/2\} \neq \emptyset$.*
2. *For every compact set $L \subset \mathbb{C}$ which satisfies $L \cap \{z \mid |z| \leq 1/2\} \neq \emptyset$ we can find a sequence $(f_n) \in \mathcal{P}_1 \cap \mathcal{Q}$ whose Fatou set contains a contracting domain V such that $\mathcal{L}(V) = L$.*

Theorem 5.9 ([Bu2]). *Let $(f_n) \in \mathcal{P}_1 \cap \mathcal{Q}$, and V be an expanding domain. Then the set $\mathcal{L}(V)$ contains infinitely many functions.*

Next we investigate the class $\mathcal{P}_2$.

Theorem 5.10 ([Bu2]). *Let $D \subset \mathbb{C}$ be a bounded domain. Then we can find a sequence $(f_n) \in \mathcal{P}_2 \cap \mathcal{Q}$ such that $\overline{D}$ is included in the Julia set of (f_n).*

Theorem 5.11 ([Bu2]).
1. *Let $(f_n) \in \mathcal{P}_2 \cap \mathcal{Q}$, and V be a contracting domain, $\mathcal{L}(V) \neq \{\infty\}$. Then $\mathcal{L}(V)$ is closed and $\infty \in \mathcal{L}(V)$.*
2. *For every closed set $L \subset \overline{\mathbb{C}}$, $\infty \in L$, which satisfies $L \cap \{|z| \leq 1/2\} \neq \emptyset$, there exists a sequence $(f_n) \in \mathcal{P}_2 \cap \mathcal{Q}$ whose Fatou set contains a contracting domain V such that $\mathcal{L}(V) = L$.*

Theorem 5.12 ([Bu2]). *There is a sequence $(f_n) \in \mathcal{P}_2 \cap \mathcal{Q}$ such that the Fatou set contains an expanding domain.*

Next we investigate the class $\mathcal{P}_3$.

Theorem 5.13 ([Bu2]). *Let $(f_n) \in \mathcal{P}_3 \cap \mathcal{Q}$. Then $(f_n \circ \cdots \circ f_1)$converges to ∞ locally uniformly in the Fatou set.*

5.2 Results for generic sequences

In 1991 J.Fornaess and N.Sibony started to investigate the behavior of the generic sequences for the random iteration of rational functions.

Notation. Let W be a connected open set in $\mathbb{C}$. We consider a holomorphic function $R : W \times \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ such that for each $c \in W$, $R_c(z) := R(c, z)$ is a rational function of degree d. In this subsection we will always assume that $R(c, z)$ is *generic*, i.e. that for every $z \in \overline{\mathbb{C}}$, the function $c \to R(c, z)$ is not constant. For any point $c_0 \in W$ and any number $\delta > 0$ with $\overline{D(c_0, \delta)} \subset W$, we set $X(c_0, \delta) = \overline{D(c_0, \delta)}^{\mathbb{N}}$. Let $g : X(c_0, \delta) \to X(c_0, \delta)$ be the shift map. We construct a fibered rational map $f : X(c_0, \delta) \times \overline{\mathbb{C}} \to X(c_0, \delta) \times \overline{\mathbb{C}}$ fibered over $g : X(c_0, \delta) \to X(c_0, \delta)$ defined as:

$$f(x, y) = (g(x), R_{x_1}(y))$$

for $(x, y) \in X(c_0, \delta) \times \overline{\mathbb{C}}$, $x = (x_1, x_2, \dots)$.

For any probability measure μ on $\overline{D(c_0, \delta)}$ we set $\tilde{\mu} = \otimes_{k=1}^{\infty} \mu$.

Theorem 5.14 ([FS]). *Under the above, let $c_0 \in W$ and suppose R_{c_0} has k attractive cycles, $\gamma_1, \dots, \gamma_k$, $k \geq 1$. For each $1 \leq j \leq k$ let V_j be a neighborhood of γ_j. We will assume that V_j is contained in the basin of attraction of γ_j. Then, there is a $\delta_0 > 0$ such that for each $0 < \delta < \delta_0$ there exist continuous functions $h_1, \dots, h_k$ defined on $\overline{\mathbb{C}}$ with the following properties:*

1. $0 \leq h_j(y) \leq 1$, $\sum_{j=1}^{k} h_j(y) = 1$,

2. for each $y \in \overline{\mathbb{C}}$ there exist disjoint open sets $U_{j,y}$ in $K(c_0, \delta)$ with $\tilde{\lambda}(U_{j,y}) = h_j(y)$ where λ is the normalized Lebesgue measure on $\overline{D(c_0, \delta)}$, and if $x \in U_{j,y}$ then for all large enough $n \in \mathbb{N}$, $f_x^n(y)(z) \in V_j$.

Corollary 5.15 ([FS]). *Under the same condition of Theorem 5.14, let V be a neighborhood of $\{\gamma_1, \dots, \gamma_k\}$. Then there exists $\delta_0 > 0$ such that for each $\leq \delta \leq \delta_0$ there exists a set $\mathcal{E} \subset K(c_0, \delta)$ of full measure with respect to the measure $\tilde{\lambda}$, where λ denotes the normalized Lebesgue measure in $\overline{D(c_0, \delta)}$, such that if $x \in \mathcal{E}$, a.e. in $\overline{\mathbb{C}}$, $f_x^n(y) \in V$ for all large enough n. In particular, J_x is of Lebesgue measure 0 in $\overline{\mathbb{C}}$ for $x \in \mathcal{E}$.*

We next show an ergodic property of random iteration of rational functions. Let $c_0 \in W$. Let $M(\overline{\mathbb{C}})$ denote the set of Borel probability measures on $\overline{\mathbb{C}}$. Fix a small $\delta > 0$ such that $\overline{D(c_0, \delta)} \subset W$. We define the operator $T : M(\overline{\mathbb{C}}) \to M(\overline{\mathbb{C}})$ by:

$$(T\nu)(B) = \int_{c \in D(c_0, \delta)} \nu(R_c^{-1}(B))\, d\lambda(c),$$

where B is a Borel set in $\overline{\mathbb{C}}$ and λ is the normalized Lebesgue measure on $\overline{D(c_0 . \delta)}$. Then the following holds.

Theorem 5.16 ([FS]). *Under the above, Suppose R_{c_0} has no superattracting cycles. Then there is a lower semicontinuous function $h_\delta : \overline{\mathbb{C}} \to (0, \infty]$ such that for every $M(\overline{\mathbb{C}})$ the sequence $T^n \nu$ converges to $h_\delta \sigma$ where σ denotes the normalized Lebesgue measure on $\overline{\mathbb{C}}$.*

Moreover, for $y \in \overline{\mathbb{C}}$, $B \subset \overline{\mathbb{C}}$ Borel set, there exists $\mathcal{E}_{y,B} \subset X(c_0, \delta)$ of full measure with respect to $\tilde{\lambda}$ such that for every $x \in \mathcal{E}_{y,B}$ we have

$$\lim_{k \to \infty} \frac{1}{k} \sum_{n \leq k} \chi_B(f_x^n(y)) = \int_B h_\delta \, d\sigma.$$

If $R(c, z)$ is regular, then $h_\delta : \overline{\mathbb{C}} \to (0, \infty)$ is continuous.

5.3 Random iteration of quadratic polynomials

Developing some ideas of random iteration of rational functions in [FS], R.Brück, M.Büger and S.Reitz investigated the case of random quadratic polynomials in detail.

In this subsection we discuss about the following fibered quadratic polynomials. Let K be a compact subset of $\mathbb{C}$. Let $X(K) = K^{\mathbf{N}}$ and $g : X \to X$ be the shift map. We consider the following map f defined as:

$$f(x, y) = (g(x), y^2 + x_1),$$

where $(x, y) \in X(K) \times \overline{\mathbb{C}}$ and $x = (x_1, x_2, x_3, \dots)$. For a probability measure μ on K, we set $\tilde{\mu} = \otimes_{k=1}^{\infty} \mu$.

Theorem 5.17 ([BBR]). *Under the above, let $K = \overline{D(0,1/4)}$ and λ be a Borel probability measure on K with the support equal to K. We denote by $\mathcal{B}$ the set of points $x \in X(K)$ satifying that the orbit $\{\pi_{\overline{\mathbb{C}}}(f_x^n(y))\}_{n\in\mathbb{N}}$ is dense in $\overline{D(0,1/2)}$ for all $y \in \overline{D(0,1/2)}$. Then we have*

$$\tilde{\lambda}(\mathcal{B}) = 1.$$

Theorem 5.18 ([BBR],[Br]). *Let $K \subset \mathbb{C}$ be a compact set. Suppose $int(K) \cap (\mathbb{C} \setminus \mathcal{M}) \neq \emptyset$, where $\mathcal{M}$ denotes the Mandelbrot set. Let μ be a Borel probability measure on K which is absolutely continuous with respect to the Lebesgue measure on K and which satisfies $int(supp(\mu)) \cap (\mathbb{C} \setminus \mathcal{M}) \neq \emptyset$. Then we have the following.*

1. *Let $R > 0$. Then for every $y \in \overline{\mathbb{C}}$ there exists an open set $U_y \subset X(K)$ with the following properties:*

   (a) $\tilde{\mu}(U_y) = 1$,

   (b) for every $x \in U_y$, there holds $|\pi_{\overline{\mathbb{C}}}(f_x^k(y))| > R$ for all sufficiently large k.

   In particular, for almost all $x \in X(K)$ with respect to $\tilde{\mu}$ we have that the 2-dimensional Lebesgue measure of J_x is equal to zero.

2. *We denote by $\mathcal{D}_\infty$ the set of points $x \in X(K)$ satifying that J_x has infinitely many components. Then we have*

$$\tilde{\mu}(\mathcal{D}_\infty) = 1.$$

Theorem 5.19 ([Br]). *Let K be a bounded set such that $K \cap (\mathbb{C} \setminus \mathcal{M}) \neq \emptyset$. We set:*

$$\mathcal{D}_N = \{x \in X(K) \mid J_x \text{ has more than } N \text{ components}\}, \tag{4}$$

$$\mathcal{D}_\infty = \{x \in X(K) \mid J_x \text{ has infinitely many components }\}, \tag{5}$$

$$\mathcal{T} = \{x \in X(K) \mid J_x \text{ is totally disconnected }\}. \tag{6}$$

Then

1. *$\mathcal{T}$ is dense in $X(K)$.*
2. *$\mathcal{D}_N$ is an open and dense subset of $X(K)$ for each $N \in \mathbb{N}$.*
3. *$\mathcal{D}_\infty$ is a countable intersection of dense open subsets of $X(K)$ and has empty interior.*

Question.

1. ([Br]) Is $\mathcal{T}$ of the second Baire category in $X(K)$?
2. ([BBR]) By zero-one law, we know that $\tilde{\mu}(\mathcal{T})$ is equal to 0 or 1. Which is true?

5.4 Rational semigroups and random iteration

We introduce a work of S.Rohde's in which some relationship between rational semigroups and random iteration composed in the opposite way.

Let K be a closed disk centered at the origin. Let $R(c,z) : K \times \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a holomorphic family such that the degree of $R_c(z) := R(c,z)$ is at least two. In this subsection we always assume that there exists a point $c_0 \in K$ such that $z \mapsto \frac{\partial R}{\partial c}(c_0, z)$ is not identically zero.

For $x \in X(K) = K^{\mathbb{N}}$, we denote by $\hat{F}_x$ the set of points $z \in \overline{\mathbb{C}}$ satisfying that z has a neighborhood where the sequence $(R_{x_1} \circ R_{x_2} \circ \cdots \circ R_{x_n})_{n \in \mathbb{N}}$ is normal, where $x = (x_1, x_2, \dots)$. We set $\hat{J}_x = \overline{\mathbb{C}} \setminus \hat{F}_x$. We call $\hat{J}_x$ the *opposite Julia set* for the sequence x. Let λ be the normalized Lebesgue measure. We set $\tilde{\lambda} = \otimes_{k=1}^{\infty} \lambda$.

Theorem 5.20 ([Ro]). *Under the above, Let G be the rational semigroup generated by $\{R_c \mid c \in K\}$. Then $J(G)$ contains interior points and for almost $x \in X(K)$ with respect to $\tilde{\lambda}$, the opposite Julia set $\hat{J}_x$ is equal to $J(G)$.*

Similarly, let $K = \{1, \dots n\}$. Let $\{R_1, \dots, R_n\}$ be some rational functions of degree at least two. We define the opposite Julia set $\hat{J}_x$ for $x \in X(K) = K^{\mathbb{N}}$ as the above. Let λ be any probability measure on K such that $\lambda(j) > 0$ for any $j = 1, \dots, n$. We set $\tilde{\lambda} = \otimes_{k=1}^{\infty} \lambda$. Then we have the following.

Theorem 5.21 ([Ro]). *Under the above, let G be the rational semigroup generated by $\{R_1, \dots, R_n\}$. Then for almost $x \in X(K)$ with respect to $\tilde{\lambda}$ the opposite Julia set $\hat{J}_x$ is equal to $J(G)$.*

6 Higher dimensional cases

6.1 Attracting currents and measures

We introduce the J.Fornaess and B.Weickert's work([FW]) in which they showed that for the random iteration which is generated by a holomorphic family in $\mathbb{P}^k$, there exists a positive closed $(1,1)$ current and a measure on $\mathbb{P}^k$ which are invariant and which attract all positive closed $(1,1)$ currents and all measures, respectively, under normalized pull-back and averaging by the maps.

Let $K = \overline{D(0,\delta)} \subset \mathbb{C}^k$. Let $R : K \times \mathbb{P}^k \to \mathbb{P}^k$ be a holomorphic family such that $R_c(z) := R(c,z)$ is of degree $d > 1$ for each $c \in K$. We also assume that $c \to R(c,z)$ is finite-to-one and hence open. Let $X(K) = K^{\mathbb{N}}$. Let λ be the normalized Lebesgue measure on K. Let $\tilde{\lambda} = \otimes_{j=1}^{\infty} \lambda$.

Just as in the subsection 'Potential theory and measure theory' in the section of fibered rational maps, we construct a family of Green functions $\{G_x\}_{x \in X(K)}$ on $\mathbb{C}^{k+1}$. We denote by T_x the unique positive closed $(1,1)$

current in $\mathbb{P}^k$ such that $\pi^* T_x = dd^c(G_x)$ where $\pi : \mathbb{C}^{k+1} \setminus \{0\} \to \mathbb{P}^k$ is the natural projection.

Further, we set $EG(y) = \int_{X(K)} G_x(y)\ d\tilde{\lambda}(x)$. Then we can see EG is plurisubharmonic and continuous on $\mathbb{C}^{K+1} \setminus \{0\}$. We also see that there exists a unique positive closed $(1,1)$ current T on $\mathbb{P}^k$ satysfying that $\pi^* T = dd^c(EG)$.

Let $\mathcal{S}$ be the space of positive closed $(1,1)$ current S on $\mathbb{P}^k$ such that $\|S\| = 1$. We define an operator Θ on $\mathcal{S}$ as:

$$\Theta(S) = \frac{1}{d} \int_K R_c^* S.$$

Similarly, define the measure μ_x, considered as a (k,k) current, on $\mathbb{P}^k$, by the equation $\pi^* \mu_x = (dd^c G_x)^k$. We set $\mu = \int_{X(K)} \mu_x$. Let $\mathcal{P}$ be the space of Borel probability measure η on $\mathbb{P}^k$. We define an operator Ω on $\mathcal{P}$ as:

$$\Omega(\eta) = \frac{1}{d^k} \int_K R_c^* \eta,$$

where R_c^* is the operator defined just as in the subsection 'Potential theory and measure theory' in the section of fibered rational maps. Then we have the following result.

Theorem 6.1 ([FW]). *Under the above, we have the following.*

1. $T = \int_{X(K)} T_x$.
2. $\Theta(T) = T$.
3. *For any* $s \in \mathcal{S}$, *we have* $\Theta^n(S) \to T$ *as* $n \to \infty$ *in the weak topology of currents.*
4. $\Omega(\mu) = \mu$.
5. *For any* $\eta \in \mathcal{P}$, *we have* $\Omega^n(\eta) \to \mu$ *as* $n \to \infty$ *in the weak topology of currents.*
6. *The support of* T_x *is equal to the Julia set of* $(R_{x_n} \circ \cdots \circ R_{x_1})_{n \in \mathbb{N}}$.
7. *The support of* T *is equal to the Julia set of semigroup generated by* $\{R_c \mid c \in K\}$.

6.2 Fibered holomorphic maps and semigroups

We introduce some works of K.Maegawa's. ([Mae]).

Let X be a compact metric space.

Just as in the section of fibered rational maps in $\overline{\mathbb{C}}$-bundles, let $f : X \times \mathbb{P}^k \to X \times \mathbb{P}^k$ be a fibered holomorphic map fibered over a continuous map $g : X \to X$. We define fiberwise Fatou sets $\{F_x\}$ and Julia sets $\{J_x\}$. We use the same notations as those in the section of fibered rational maps. We assume that the degree of f_x is at least two for any $x \in X$. (Here we do not assume that the degree of f_x is constant with respect to $x \in X$.)

Theorem 6.2 (Mae). *Under the above,*

1. *Let $x \in X$. We have that $y \in \pi_{\mathbb{P}^k}(F_x)$ if and only if there exists a neighborhood U of y in $\mathbb{P}^k$ such that there exists a saubsequence of $\{f_x^n\}_{n\in\mathbb{N}}$ which converges to a map locally uniformly in U.*

2. *Suppose that $X = K^{\mathbb{N}}$ for some compact subset K of the space of all holomorphic maps on $\mathbb{P}^k$, that $g : X \to X$ is the shift map and that $f(x, y) = (g(x), R(x_1, y)$ where $R(c, z)$ is a continuous family of holomorphic maps on $\mathbb{P}^k$. Then we have that a point $y \in \mathbb{P}^k$ belongs to the Fatou set of semigroup G generated by $\{R_c \mid c \in K\}$ if and only if there exists a neighborhood U such that for each $x \in X(K)$, there exists a subsequence of $\{f_x^n\}_{n\in\mathbb{N}}$ converging to a map locally uniformly on U. In particular, we have*

$$\pi_{\mathbb{P}^k}(\overline{\bigcup_{x\in X} J_x}) = J(G).$$

Remark 15. He generalized a result concerning the normality of the family of inverse branches of maps in semigroups in [HM3] to higher dimensional case also.([Mae])

6.3 Other works

In [ZR] W.Zhang and F.Ren discussed about the random iteration of holomorphic self-maps over bounded domains in $\mathbb{C}^n$. In [Hi] A.Hirachi discussed about the skew product maps associated with finitely many Hénon maps on $\mathbb{C}^2$. He constructed a family of Green function $\{G_x\}$.

7 Problems

- Consider Collet-Eckmann and expansive fibered rational maps.

- Similarly, consider Collet-Eckmann and expansive rational semigroups. Consider the dynamical behavior and get some estimate of Hausdorff dimension of Julia sets of such rational semigroups, using Poincaré series or ergodic theory. Use some ideas in Iterated function systems, for example those in [MU1] and [MU2].

- Get some estimate of entropy of finitely generated semigroups with respect to the some generator systems. Note that by Theorem 2.9, we have an upper estimate of that. What happens when the entropy of a finitely generated rational semigroup with respect to a generator system is maximal i.e. attains the log of the sum of degrees of generators.

- Develop the ergodic theory for fibered rational maps, like the works of M.Denker and Urbański's.
- Investigate the non-constant limit functions of fibered rational maps and investigate the 'rotation domains.' (See Theorem 2.18.)
- Construct the Teichmüller theory for fibered rational maps.
- Develop the theory of random holomorphic dynamics to a more general one. For example, take other measures than Bernoulli measures or Lebesgue measures. Consider stochastic process with holomorphic dynamics. What happens for pathwise dynamics and Julia sets? What can we say about almost sure paths?

8 Note

In this note, we use the same notations as those in 'Skew product maps ralated to finitely generated rational semigroup' by Hiroki Sumi. ([S5]) We will give a precise proof of a statement in it.

Proposition 8.1. *Let $G = \langle f_1, \dots, f_m \rangle$ be a finitely generated rational semigroup. Let $\tilde{f} : \Sigma_m \times \overline{\mathbb{C}} \to \Sigma_m \times \overline{\mathbb{C}}$ be the skew product map associated with the generator system $\{f_1, \dots, f_m\}$. Then we have the following.*

$$\tilde{J} = \cap_{n \geq 0} \tilde{f}^{-n}(\Sigma_m \times J(G)) \tag{7}$$

$$\pi_{\overline{\mathbb{C}}}(\tilde{J}) = J(G). \tag{8}$$

Proof. First we will show (8). Since $J_w \subset J(G)$ for each $w \in \Sigma_m$, we have $\pi_{\overline{\mathbb{C}}}(\tilde{J}) \subset J(G)$. We set

$$R(G) = \{z \in \overline{\mathbb{C}} \mid \exists g \in G,\ g(z) = z, |g'(z)| > 1\}.$$

We consider several cases.

Case 1. $\sharp J(G) \geq 3$. Then by Hinkkanen and Martin, we have $\overline{R(G)} = J(G)$. Since we have $R(G) \subset \pi_{\overline{\mathbb{C}}}(\tilde{J})$ and $\pi_{\overline{\mathbb{C}}}(\tilde{J})$ is a compact set, we get $\overline{R(G)} \subset \pi_{\overline{\mathbb{C}}}(\tilde{J})$. Hence we get (8).

Case 2. $\sharp J(G) = 0$. Then it is trivial to see (8).

Case 3. $J(G) = \{a\}$. Then each element of G belongs to $\mathrm{Aut}(\overline{\mathbb{C}})$. We have for each $g \in G, g(a) = a$. If there exists an element $g \in G$ with $|g'(a)| < 1$, then the repelling fixed point b of g is different from a and $b \in J(G)$. It is a contradiction. Hence we have for each $g \in G$, $|\, g'(a)| \geq 1$. If each element of G is elliptic, then each disc $D \subset \overline{\mathbb{C}}$ satisfies diam $g(D) =$ diam D and so G is equicontinuous in $\overline{\mathbb{C}}$. This is a contradiction. Hence there exists an element $g \in G$ such that a is a repelling or parabolic fixed point, then $a \in \pi_{\overline{\mathbb{C}}}(\tilde{J})$ and (8) holds.

Case 4. $J(G) = \{a_1, a_2\}$ and $a_1 \neq a_2$. Then each element of G belongs to Aut$(\overline{\mathbb{C}})$. Also we have $g(J(G)) = J(G)$ for each $g \in G$. From this, there is no parabolic element in G. Since $J(G) \neq \emptyset$, We must have non-elliptic element in G. Hence we have a loxodromic element $g \in G$. We can assume that a_1 is a repelling fixed point of g. Then we have

$$a_1 \in \pi_{\overline{\mathbb{C}}}(\tilde{J}). \tag{9}$$

If there exists a number j such that $f_j(a_2) = a_1$, then we have $a_2 \in \pi_{\overline{\mathbb{C}}}(\tilde{J})$ and so (8) holds. Now let us assume $f_j(a_2) = a_2$ for each $j = 1, \dots m$. If a_2 is a repelling fixed point of some element in $\{f_j\}$, then $a_2 \in \pi_{\overline{\mathbb{C}}}(\tilde{J})$ and (8) holds. If $|f_j'(a_2)| \leq 1$ for each j then we have f_j is elliptic or loxodromic for each j. Then there exists a disc D around a_2 such that $f_j(D) \subset D$ for each j. Then $a_2 \in F(G)$ and this is a contradiction.

Now we will show (7). Since $J_w \subset J(G)$ for each $w \in \Sigma_m$, we have $\tilde{J} \subset \cap_{n \geq 0} \tilde{f}^{-n}(\Sigma_m \times J(G))$. Let $(w, x) \in \Sigma_m \times \overline{\mathbb{C}}$ be a point satisfying that $\tilde{f}^n((w, x)) \in \Sigma_m \times J(G)$ for each $n \in \mathbb{N}$. Suppose $(w, x) \in \tilde{F}$. We will show it causes a contradiction. There exists a cylinder set $U = \{w' \in \Sigma_m \mid w'_j = w_j,\ j = 1, \dots n\}$ and an open neighborhood V of x such that $U \times V \subset \tilde{F}$. Then we have

$$\tilde{f}^n(U \times V) = \Sigma_m \times f_{w_n} \circ \cdots \circ f_{w_1}(V) \subset \tilde{F}. \tag{10}$$

In particular, we have $\Sigma_m \times \{f_{w_n} \circ \cdots \circ f_{w_1}(x)\} \in \tilde{F}$. By (8), we get $f_{w_n} \circ \cdots \circ f_{w_1}(x) \in F(G)$. It is a contradiction. $\square$

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