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Van Kampen Diagrams and E-unitary Coextensions

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1 Introduction

Semidirect products and HNN extensions of groups resemble each other in the sense that both have a presentation described by a pseudo-commutative law, however, no approach to unify and generalize them has appeared so far. In [12], a unified treatment of semidirect products and HNN extensions of groups are presented. In this paer, a key lemma in [12] is presented.

First of all, we explain the generalization of semidirect products and HNN extensions given in [12]. Let G, H be groups. Each generator h of H corresponds to an isomorphism ϕ_h of a subgroup A_h of G onto a subgroup B_h of G. Let K be the group presented by

$$\mathrm{Gp}(G,\ H \mid ax = x(a\phi_x) \text{ for } \forall a \in A_x,\ \forall x \in X), \tag{1.1}$$

where H is generated by X. Is the natural homomorphism ξ_G of G into K an isomorphism? Is the natural homomorphism ξ_H of H into K an isomorphism?

It is shown in [12] that G is *strongly embedded into* K under a certain condition (we will not touch on the embedding problem in this paper). The embedding problem is inherently related to the algebraic system of partial automorphisms. The key idea is to apply inverse semigroup theory to group theory. In this paper, we present the key lemma concerning E-unitary coextensions of groups, which is crucial in [12]. Our approach is rather geometrical. We use *van Kampen diagrams*, which are very powerful method in group theory.

2 Inverse semigroups

Partial one-to-one mappings of a non-empty set constitute an algebraic system called the *symmetric inverse semigroup*. Conversely any abstract inverse semigroup is embedded in the symmetric inverse semigroup of a certain set (known as Wagner-Preston theorem). Thus, the theory of inverse semigroups is an ideal tool to formalize and study algebraic structure of a system of partial one-to-one morphisms and local properties of mathematical objects. As a matter of fact, inverse semigroup theory was initiated to study local properties of manifolds by Wagner [10]. Ehresmann [1] also considered inverse semigroups with some extra assumptions in the context of topology and differential geometry under the name of *pseudogroups*. After the introduction of the concept, numerous researches had been done mainly on its algebraic structure. Against its first motivation, only a little attention had been paid to the theory of partial automorphisms of a specific mathematical structure, such as groups, rings, vector spaces, topological spaces, manifolds, or graphs. The significance of partial automorphisms in other areas of mathematics has been rediscovered recently. For instance, the study on the

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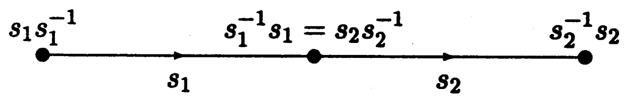
relationship between C^*-algebras, groupoids, and inverse semigroups can be found in [8]. The reader is referred to [3] for historical account and detailed information on the theory of inverse semigroups.

An *inverse semigroup* is a regular semigroup in which every element has the unique inverse (see [3]). The class of inverse semigroups forms a variety, and hence, every inverse semigroup has a presentation. We should remark that an inverse semigroup presentation is given by a set of generators and a set of *relations* which is a subset of $(X \cup X^{-1})^+ \times (X \cup X^{-1})^+$, whereas a group presentation is usually given by a set of generators and a set of *relators* which is a subset of $(X \cup X^{-1})^+$. We frequently consider a group presentation by a set of generators and a set of relations as well. A presentation for a group generated by X subject to R is denoted by $\mathrm{Gp}(X \mid R)$, and a presentation for an inverse semigroup generated by X subject to R is denoted by $\mathrm{Inv}(X \mid R)$ in this paper. The inverse semigroup S presented by $\mathrm{Inv}(X \mid R)$ is the freest inverse semigroup subject to the relation R in the following sense.

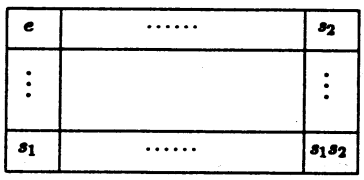
If a group H is a homomorphic image of an inverse semigroup S, then we say that S is an *inverse semigroup coextension* of H. We say that an inverse semigroup coextension S of H *has the same set of generators as* H if both S and H are generated by X and $\sigma \circ \iota_S = \iota_H$, where ι_S and ι_H are the natural mappings, and σ is a homomorphism of S onto H.

An inverse semigroup coextension S of H is called *E-unitary* if $\sigma^{-1}(1) = E(S)$. In such a case, the group H is the *maximal group homomorphic image of* S. We remark that the maximal group homomorphic image H of an inverse semigroup S has the universal mapping property: for any homomorphism ρ of S into any group Q, there exists a homomorphism ν of H into Q such that $\nu \circ \sigma = \rho$. An inverse semigroup S is called *E-unitary* if any element above an idempotent is an idempotent (see [3]). It is easy to see that an inverse semigroup is E-unitary if and only if it is an E-unitary coextension of its maximal group homomorphic image.

A *groupoid* is a small category in which every morphism is invertible. Any inverse semigroup S can be endowed with a groupoid structure ([6, 9]) and we denote it by $\mathrm{Gpd}(S)$ in this paper. The set of objects of $\mathrm{Gpd}(S)$ is $E(S)$. For $e, f \in E(S)$, a morphism of e to f is any element $s \in S$ such that $e = ss^{-1}$ and $f = s^{-1}s$. The inverse of the morphism s is given by s^{-1}. If $e \in E(S)$, then e is the *identity morphism* of the object e. The *trace product* of an inverse semigroup S is the groupoid product in $\mathrm{Gpd}(S)$, that is, the trace product for s_1, s_2 in S is defined to be s_1s_2 if and only if $s_1^{-1}s_1 = s_2s_2^{-1}$ in S. Then s_1s_2 is a morphism from $s_1s_1^{-1}$ to $s_2^{-1}s_2$ since $(s_1s_2)(s_1s_2)^{-1} = s_1s_2s_2^{-1}s_1^{-1} = s_1s_1^{-1}s_1s_1^{-1} = s_1s_1^{-1}$ and $(s_1s_2)^{-1}(s_1s_2) = s_2^{-1}s_1^{-1}s_1s_2 = s_2^{-1}s_2s_2^{-1}s_2 = s_2^{-1}s_2$. Using the terminology in semigroup theory, we define the trace product s_1s_2 if and only if there exists $e \in E(S)$ such that $s_1 \mathcal{L} e \mathcal{R} s_2$, where $\mathcal{L}$ and $\mathcal{R}$ are Green's relations (see [3]). We illustrate the morphism s_1s_2 in $\mathrm{Gpd}(S)$ and the location of e, s_1, s_2, s_1s_2 in $\mathcal{D}$ class of S in Figure 1. Furthermore, $\mathrm{Gpd}(S)$ is endowed with an order structure, however, we do not touch on it.



Morphism s_1s_2 in $\mathrm{Gpd}(S)$



$\mathcal{D}$ class of S

Figure 1: Trace product

Let τ be a mapping of a set X into an inverse semigroup S. Let $w = h_1h_2 \cdots h_n$ be a word in $(X \cup X^{-1})^+$. Then w is called a *trace product* in S with respect to τ if $\tau(h_1)\tau(h_2)\cdots\tau(h_n)$ is a trace product in $\mathrm{Gpd}(S)$, that is, every consecutive pair of $\tau(h_i)$ and $\tau(h_{i+1})$ forms a trace product in S. The word w is called a *cyclic trace product* in S with respect to τ if every cyclic conjugate of w is a trace product in S with respect to τ. The language of all trace products and all cyclic trace products with respect to τ are denoted by $\mathrm{TRACE}(S, X, \tau)$ and $\mathrm{LOOP}(S, X, \tau)$, respectively. Note that $\mathrm{LOOP}(S, X, \tau) \subset \mathrm{TRACE}(S, X, \tau) \subset (X \cup X^{-1})^+$ in general, whereas $\mathrm{TRACE}(G, X, \tau) = \mathrm{LOOP}(G, X, \tau) = (X \cup X^{-1})^+$ for any group G and any mapping τ of X into G.

Lemma 2.1 *Suppose $w = h_1h_2\cdots h_n \in (X \cup X^{-1})^+$ and τ is a mapping of X into an inverse semigroup S.*
(1) If w lies in $\mathrm{TRACE}(S, X, \tau)$ and $w = w_1w_2$, where $w_1, w_2 \in (X \cup X^{-1})^+$, then $w_1^{-1}w_1 = w_2w_2^{-1}$ in S under τ.
(2) If w lies in $\mathrm{LOOP}(S, X, \tau)$, then so does any cyclic conjugate of w.
(3) If w lies in $\mathrm{LOOP}(S, X, \tau)$ and $E(S)$, then so does any cyclic conjugate of w.
(4) Suppose $w \in \mathrm{TRACE}(S, X, \tau)$ and $w = w_1w_2w_3$, where $w_1, w_2, w_3 \in (X \cup X^{-1})^+$. Then $w_1w_3 \in \mathrm{TRACE}(S, X, \tau)$ if and only if $w_2 \in \mathrm{LOOP}(S, X, \tau)$.
(5) Let ϕ be a mapping of X into T_G. If w lies in $\mathrm{TRACE}(\mathrm{T}_G, X, \phi)$, then we have $\mathrm{Ran}(\phi_{h_i}) = \mathrm{Dom}(\phi_{h_{i+1}})$ for every $i = 1, 2, \ldots, n-1$. If w lies in $\mathrm{LOOP}(\mathrm{T}_G, X, \phi)$, then $\mathrm{Ran}(\phi_{h_n}) = \mathrm{Dom}(\phi_{h_1})$. □

A word w in $(X \cup X^{-1})^+$ is called *Dyck* if it is the identity in the free group $\mathrm{FG}(X)$. Any Dyck word on X is an idempotent in any inverse semigroup generated by X.

Lemma 2.2 *(1) If $w = h_1h_2\cdots h_n \in (X \cup X^{-1})^+$ lies in $\mathrm{LOOP}(S, X, \tau)$ and $E(S)$ under τ, then w is the identity morphism of the object $\tau(h_1)\tau(h_1)^{-1} = \tau(h_n)^{-1}\tau(h_n)$ in $\mathrm{Gpd}(S)$ under τ.*
(2) Let $d = h_1h_2\cdots h_n$ be a Dyck word on X. If d belongs to $\mathrm{TRACE}(S, X, \tau)$, then $d \in \mathrm{LOOP}(S, X, \tau)$ and d is the identity morphism of the object $\tau(h_1)\tau(h_1)^{-1} = \tau(h_n)^{-1}\tau(h_n)$ in $\mathrm{Gpd}(S)$ under τ. □

Lemma 2.3 *Let S be an inverse semigroup. Suppose a word $w \in (X \cup X^{-1})^+$ belongs to $E(S)$, and $w = w_0w_1\cdots w_n$, where $w_i \in (X \cup X^{-1})^+$. If $e_i \in E(S)$ for every $i = 1, 2, \ldots, n$, then $w_0e_1w_1e_2\cdots e_nw_n \leq w$ and $w_0e_1w_1e_2\cdots e_nw_n \in E(S)$.* □

3 Van Kampen diagrams and trace products

In this section, we investigate the trace product structure on boundary cycles of *van Kampen diagrams* and give a condition for an inverse semigroup coextension of a group to be E-unitary. This result is crucial in the embedding proble for groups (see [12]).

A *1-complex* (or *graph*) Y consists of two sets $\mathrm{Vert}(Y)$ and $\mathrm{Edge}(Y)$, together with the functions $\alpha : \mathrm{Edge(Y)} \to \mathrm{Vert}(Y)$, $\omega : \mathrm{Edge(Y)} \to \mathrm{Vert}(Y)$ and $\eta_1 : \mathrm{Edge(Y)} \to \mathrm{Edge(Y)}$. We call the elements of $\mathrm{Vert}(Y)$ *vertices*, and the elements of $\mathrm{Edge(Y)}$ *edges*. For $y \in \mathrm{Edge}(Y)$, $\alpha(y), \omega(y), \eta_1(y)$ is called the *initial vertex*, the *terminal vertex* and the *inverse edge*, respectively. A *2-complex* (or a *map*) M consists of a 1-complex M', its *1-skeleton*, together with a set $\mathrm{Face}(M)$ and two functions ∂ and η_2 defined on $\mathrm{Face}(M)$. We call the elements of $\mathrm{Face}(M)$ *faces* (or *2-cells*). For each F in $\mathrm{Face}(M)$, ∂F is the boundary cycle of F, and $\eta_2(F)$ is the inverse of F satisfying $\eta_1(\partial F) = \partial(\eta_2(F))$. The geometric realization of a vertex, an edge and a face of a planar 2-complex is a point, a bounded subset homeomorphic to an open unit interval and a bounded subset homeomorphic to an open unit disk in the Euclidean space $\mathbf{E}^2$, respectively. We denote the boundary cycle of a connected and simply connected 2-complex M by ∂M. Here the boundary cycle of M is the word that is read along the contour of M counterclockwise starting and ending at the distinguished vertex. The subgraph consisting of the boundary of M is denoted by ∂M as well. We denote the *interior* and the *closure* of a topological space $Z \subset \mathbf{E}^2$ by Z° and $\overline{Z}$, respectively.

Let H be a group presented by $\mathrm{Gp}(X \mid R)$, where R is cyclically reduced. Suppose w is a freely reduced word in $(X \cup X^{-1})^+$ and $w = 1$ in H. A *van Kampen diagram* for w over H is a finite planar connected and simply connected 2-complex M such that the boundary cycle ∂F of any face F of M is a cyclic conjugate of a relator in R and the boundary cycle ∂M of M starting and ending at the distinguished vertex v_0 is equal to w as words. The reader is referred to [4, 7] for van Kampen diagrams and van Kampen's lemma. We shall call a maximal subcomplex homeomorphic to a closed disk a *generalized face* in this paper. A *stalk* of a van Kampen diagram M is a subpath consisting of edges $y_1, y_2, \ldots, y_k$ and their initial and terminal vertices such that every y_i is disjoint from $\overline{M^\circ}$. Note that $\overline{M^\circ}$ is a union of generalized faces of M, and $\overline{M \setminus \overline{M^\circ}}$ is a union of stalks of M. Thus, any van Kampen diagram is a union of finitely many generalized faces and stalks.

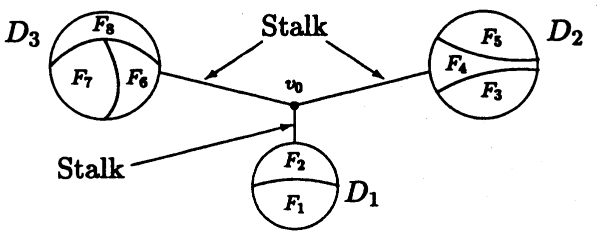
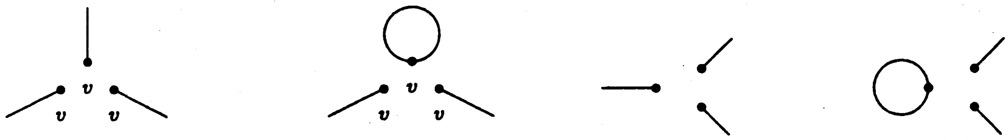


Figure 2: Van Kampen diagram with 8 faces and 3 generalized faces

The *ramification number* of a vertex v (or a generalized face D) of a finite planar connected and simply connected 2-complex M is the number of connected components of the topological space $M \setminus \{v\}$ (or $M \setminus D$). Let C be a connected component of $M \setminus \{v\}$. Then $M_1 = C \cup \{v\}$ is a connected and simply connected 2-complex and called a *branch* of M at v. A *branch* at a generalized face D is similarly defined (see Figure 3 and 4). A finite planar connected and simply connected 2-complex M is called *chain-shaped* if any vertex and any generalized face has ramification number at most two. In Figure 5, a chain-shaped 2-complex with seven generalized faces, in which the path P_i may be empty, is illustrated.

(a) (b) (c) (d)

Figure 3: Several vertices and generalized faces having ramification number 3



(a) (b) (c) (d)

Figure 4: Branches in Figure 3

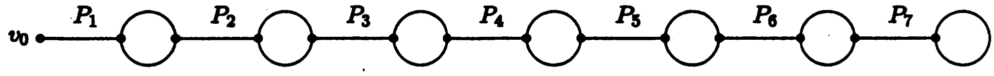
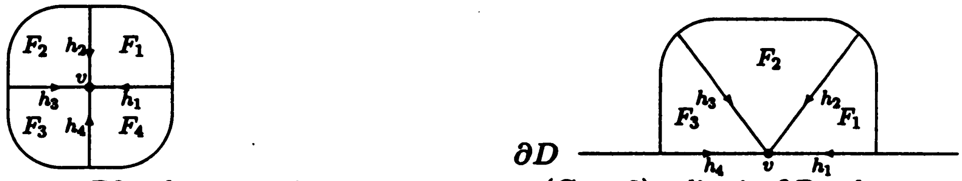


Figure 5: Chain-shaped 2-complex

Lemma 3.1 **(Van Kampen [11])** *Let $w \in (X \cup X^{-1})^+$ be a freely reduced word such that $w = 1$ in H. There exists a van Kampen diagram for w over H.* □

Lemma 3.2 *Let τ be a mapping of X into an inverse semigroup S such that $R \subset \mathrm{LOOP}(S, X, \tau)$. Suppose M is a van Kampen diagram over H for a freely reduced word $w \in (X \cup X^{-1})^+$ representing* 1 *in H.*
(1) Let F be a face of M. Then ∂F belongs to $\mathrm{LOOP}(S, X, \tau)$. *If $R \subset E(S)$ under τ, then ∂F belongs to $E(S)$ under τ.*
(2) Let D be a generalized face of M. Any word in $X \cup X^{-1}$ labeling a path in D belongs to $\mathrm{TRACE}(S, X, \tau)$. *In particular, ∂D belongs to* $\mathrm{LOOP}(S, X, \tau)$.
(3) If $R \subset E(S)$ under τ, then ∂D belongs to $E(S)$ under τ for every generalized face D of M.

Proof. (1) Since the boundary cycle ∂F of any face F of M is a cyclic conjugate of a relator in R, $\partial F \in \mathrm{LOOP}(S, X, \tau)$ by Lemma 2.1 (2). If $R \subset E(S)$, then $\partial F \in E(S)$ by Lemma 2.1 (3).
(2) Let v be a vertex in D. Suppose $y_1, y_2, \ldots, y_n$ are edges in D entering v and each y_i is labeled by $h_i \in X \cup X^{-1}$. It suffices to show that $\tau(h_i)\tau(h_j)^{-1}$ is a trace product in $\mathrm{Gpd}(S)$, equivalently, $\tau(h_i)^{-1}\tau(h_i) = \tau(h_j)^{-1}\tau(h_j)$ in S for all $1 \le i, j \le n$. Since D is a planar diagram, we may assume $y_1, y_2, \ldots, y_n$ are enumerated counterclockwise around v in this order. Since D is homeomorphic to a closed disc, there are two possible cases: (Case 1) v lies in the interior of D, and (Case 2) v lies on the boundary of D (see Figure 6).



(Case 1) v is in D°, where $n = 4$ (Case 2) v lies in ∂D, where $n = 4$

Figure 6: Vertex v and edges entering it

(Case 1) The edges y_{i+1} and $\eta_1(y_i)$ with $1 \le i \le n-1$ (and y_1 and $\eta_1(y_n)$) form a subpath of the boundary cycle of a certain face F_i in D. By (1), ∂F_i belongs to $\mathrm{LOOP}(S, X, \tau)$, and hence, $\tau(h_{i+1})\tau(h_i)^{-1}$ belongs to $\mathrm{TRACE}(S, X, \tau)$ for every i as $\tau(h_{i+1})\tau(h_i)^{-1}$ is a subword of ∂F_i. Hence, $\tau(h_{i+1})^{-1}\tau(h_{i+1}) = \tau(h_i)^{-1}\tau(h_i)$ for every i. It follows that $\tau(h_i)^{-1}\tau(h_i) = \tau(h_j)^{-1}\tau(h_j)$ for all i, j. (Case 2) We may assume without loss of generality that y_1 and y_n are on the boundary of D. For each consecutive pair y_i and y_{i+1} $(1 \le i \le n-1)$, there exists a face in which y_i and y_{i+1} form a subpath of the boundary cycle, and hence, we can prove $\tau(h_i)^{-1}\tau(h_i) = \tau(h_j)^{-1}\tau(h_j)$ for all i, j as in (Case 1). It follows that any word labeling a path in D belongs to $\mathrm{TRACE}(S, X, \tau)$. Since any cyclic conjugate of ∂D is a path in D, ∂D belongs to $\mathrm{LOOP}(S, X, \tau)$.
(3) Suppose $R \subset E(S)$ under τ. Let D be a generalized face of M. We shall show that a boundary cycle of any connected and simply connected subcomplex L of D belongs to $E(S)$ under τ using induction on the number l of faces in L. First we note that a connected and simply connected subcomplex L with no faces is a tree. Hence, the boundary cycle ∂L is a Dyck word and belongs to $E(S)$. Second, we suppose L has only one face F_0. Then L is obtained from F_0 by attaching finitely many trees to F_0. Note that a boundary cycle of any tree is

a Dyck word. Therefore, the boundary cycle of L is obtained from ∂F_0 by inserting finitely many Dyck words, and hence, written as $w_1 z_1(\partial T_1) z_2(T_2) \cdots z_n(\partial T_n) w_2$, where $w_1 w_2 = \partial T_0$, $\partial F_0 = z_1 z_2 \cdots z_n$ and $T_0, T_1, \ldots, T_n$ are (maybe empty) trees. By Lemma 2.3, $\partial L \in E(S)$. Now we suppose the claim is true for any positive integer less than l and a subcomplex L of D has l faces. Suppose L has at least two generalized faces. There exists a vertex v_1 having ramification number at least two and there exist at least two branches at v_1 such that each of them contains at least one generalized face. Then the boundary cycle of L starting and ending at v_1 can be written as $(\partial L_1)(\partial L_2)$ for some subcomplexes L_1 and L_2 of L having at least one generalized face, where ∂L_1 and ∂L_2 are the boundary cycles of L_1 and L_2 starting and ending at v_1, respectively (see Figure 7).

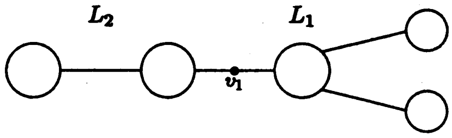
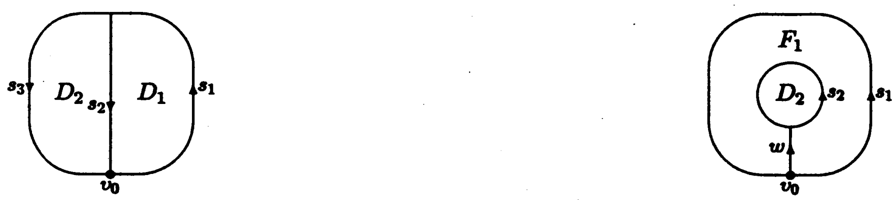


Figure 7: L has at least two generalized faces

By the inductive hypothesis, ∂L_1 and ∂L_2 belong to $E(S)$. Thus, the boundary cycle of L starting and ending at v_1 is written as $(\partial L_1)(\partial L_2)$. By Lemma 2.1 (3), $\partial L \in E(S)$. Next we suppose L has only one generalized face with no stalks. Then we show that ∂L lies in $E(S)$. There are two cases. (Case 1) At least two faces of L have edges on the boundary of L. In such a case, L is a union of connected and simply connected subcomplexes D_1 and D_2 with more than one generalized faces that have edges on the boundary ∂L and $D_1^\circ \cap D_2^\circ$ is empty. (Case 2) Only one face F_1 has the edges in ∂L (see Figure 8).



(Case 1) D_1 and D_2 have edges in ∂L. (Case 2) Only F_1 has edges in ∂L

Figure 8: Decomposition of L

(Case 1) We can write $\partial L = s_1 s_3$, $\partial D_1 = s_1 s_2$, $\partial D_2 = s_2^{-1} s_3$. Since $s_2 s_2^{-1}$ is a path in D, $s_2 s_2^{-1}$ is a Dyck word belonging to TRACE(S, X, τ) by the part (2). Hence, $s_2 s_2^{-1}$ is the identity morphism in Gpd(S) by Lemma 2.2 (2). Then we have $s_1 s_3 = s_1 s_2 s_2^{-1} s_3$ under τ. On the other hand, $\partial D_1, \partial D_2 \in E(S)$ under τ by the inductive hypothesis. It follows that $\partial L = s_1 s_3 = s_1 s_2 s_2^{-1} s_3 = (\partial D_1)(\partial D_2) \in E(S)$. (Case 2) We can write $\partial L = s_1$, $\partial F_1 = s_1 w s_2^{-1} w^{-1}$, $\partial D_2 = s_2$, where D_2 is a connected and simply connected subcomplex of D. Note that D_2 has $l-1$ faces. By the inductive hypothesis, ∂D_2 belongs to $E(S)$ under τ. Moreover, ∂F_1 belongs to $E(S)$ under τ as $R \subset E(S)$. Since $w s_2^{-1} w^{-1} (w s_2^{-1} w^{-1})^{-1}$ is a Dyck word and lies in TRACE(S, X, τ) by the part (2), it is the identity morphism in Gpd(S) by Lemma 2.2 (2). It follows that $s_1 = s_1 w s_2^{-1} w^{-1} (w s_2^{-1} w^{-1})^{-1} = (\partial F_1)(\partial D_2) \in E(S)$ and ∂L is an idempotent in S. Hence, if L has no stalks, then ∂L lies in $E(S)$. We now suppose L consists of a generalized face D and several stalks. By the argument above, we have $\partial D \in E(S)$. If $\partial D = w_0 w_1 \cdots w_n$, then $\partial L = z_1 w_0 e_1 w_1 e_2 \cdots e_n w_n z_2$, where each e_i is the boundary cycle of a stalk and $z_1 z_2$ is a boundary cycle of a stalk. Since a stalk is a tree, its boundary cycle is a Dyck word. By Lemma 2.3, $\partial L \in E(S)$ under τ. This completes the induction, and hence, $\partial D \in E(S)$. $\square$

We remark that Lemma 3.2 (2) implies that TRACE(S, X, τ) includes the language accepted by the 1-skeleton D' of any generalized face D considered as an automaton in which every vertex of D is both an initial state and a terminal state. Furthermore, any word in $X \cup X^{-1}$ labeling a closed path in D belongs to $E(S)$ under τ. We briefly sketch the proof. Choose an arbitrary maximal subtree (spanning tree) T of the 1-skeleton D'. Suppose $y_1, y_2, \ldots, y_n$ is a closed path in D starting and ending at the vertex v_0 and labeled by the word $w = h_1 h_2 \ldots h_n$. There exists a unique geodesic from v_0 to any vertex v in T. We denote the geodesic by $\mathrm{P}(v)$. Take an edge y in D. The subcomplex encompassed by $\mathrm{P}(\alpha(y))$, y, $\eta_1(\mathrm{P}(\omega(y)))$ is denoted by $\mathrm{Tri}(y)$, where $\eta_1(\mathrm{P}(\omega(y)))$ is the inverse path of $\mathrm{P}(\omega(y))$. By the argument above, the boundary cycle $\partial\mathrm{Tri}(y)$ of $\mathrm{Tri}(y)$ starting and ending at v_0 lies in $E(S)$. Each $\eta_1(\mathrm{P}(\omega(y_i)))\mathrm{P}(\alpha(y_{i+1}))$ is a Dyck word (as $\omega(y_i) = \alpha(y_{i+1})$) and belongs to $\mathrm{TRACE}(S, X, \tau)$ by Lemma 3.2 (2), and hence, it is an identity morphism in $\mathrm{Gpd}(S)$ by Lemma 2.2 (2) under τ. Then, $h_1 h_2 \ldots h_n$ is equal to $h_1 \eta_1(\mathrm{P}(\omega(y_1)))\mathrm{P}(\alpha(y_2))h_2 \cdots \eta_1(\mathrm{P}(\omega(y_{n-1})))\mathrm{P}(\alpha(y_n))h_n = (\partial\mathrm{Tri}(y_1))(\partial\mathrm{Tri}(y_2)) \cdots (\partial\mathrm{Tri}(y_n))$, and hence, it belongs to $E(S)$. Therefore, $E(S)$ includes the language accepted by the automaton D' in which the initial state and the terminal state is the same fixed vertex of D.

A finite planar connected and simply connected 2-complex M is said to be *good with respect to a mapping τ of X into an inverse semigroup S* if $\partial D \in \mathrm{LOOP}(S, X, \tau)$ and $\partial D \in E(S)$ under τ for every generalized face D of M. In such a case, any cyclic conjugate of ∂D belongs to $E(S)$ under τ by Lemma 2.1 (3). Lemma 3.2 claims that any van Kampen diagram for a freely reduced word representing 1 in H is good with respect to τ if $R \subset \mathrm{LOOP}(S, X, \tau)$ and $R \subset E(S)$ under τ.

Theorem 3.3 *Let H be a group presented by $\mathrm{Gp}(X \mid R)$, where R is cyclically reduced. Suppose S is an inverse semigroup coextension of H having the same set X of generators as H. If $R \subset \mathrm{LOOP}(S, X, \iota_S)$ and $R \subset E(S)$, then S is E-unitary.*

Proof. We first note that $w = 1$ in H if and only if $\mathrm{NF}(w) = 1$ in H for $w \in (X \cup X^{-1})^+$, where $\mathrm{NF}(w)$ is the normal form for w in $\mathrm{FG}(X)$. If we can prove $\mathrm{NF}(w) \in E(S)$ for w representing 1 in H, then $w \in E(S)$ by Lemma 2.3. Therefore, we have only to show that $u \in E(S)$ for every freely reduced word u representing 1 in H. This is equivalent to show that the boundary cycle of a van Kampen diagram M for a freely reduced word u over H, where $u = 1$ in H, is an idempotent in S by Lemma 3.1. Since $R \subset \mathrm{LOOP}(S, X, \iota_S)$ and $R \subset E(S)$, any van Kampen diagram over H is good with respect to $\iota_S : X \to S$ by Lemma 3.2. We shall show that ∂M is an idempotent in S for every good (finite planar connected and simply connected) 2-complex M using induction on the number of generalized faces of M.

Suppose M consists of only one generalized face D and finitely many trees attached. There are two possible cases: (Case 1) v_0 is in ∂D and (Case 2) v_0 is not in ∂D (see Figure 9).

(Case 1) v_0 is in D (Case 2) v_0 is not in D

Figure 9: M consists of one generalized face and trees

(Case 1) Suppose v_0 is in ∂D and trees $T_1, T_2, \ldots, T_k$ are attached to D. Then ∂M is written as $w_0(\partial T_1)w_1(\partial T_2)w_2 \cdots (\partial T_k)w_k$, where $\partial D = w_0 w_1 w_2 \cdots w_k$. Note that $\partial T_j \in E(S)$ for every T_j as any boundary cycle of a tree is a Dyck word. Since $\partial D \in E(S)$ (as M is good), $w_0(\partial T_1)w_1(\partial T_2)w_2 \cdots (\partial T_k)w_k \in E(S)$ by Lemma 2.3. Thus, $\partial M \in E(S)$. (Case 2) Suppose v_0 is not in ∂D and trees $T_1, T_2, \ldots, T_k$ are attached to D. We may assume v_0 is in T_1. Suppose

T_1 meets D at the vertex v_1. Let M_1 be the subcomplex consisting of $(M \setminus T_1) \cup \{v_1\}$. By (Case 1), the boundary cycle ∂M_1 starting and ending at v_1 lies in $E(S)$. Then we can write $\partial M = w_1(\partial M_1)w_2$, where $\partial T_1 = w_1 w_2$. Since ∂T_1 is a Dyck word, $\partial T_1 \in E(S)$. Hence, $\partial M \in E(S)$ by Lemma 2.3. Therefore, the claim is true for any 2-complex with only one generalized face. We now suppose the claim is true for any positive integer less than n and the number of generalized faces of M is $n \geq 2$. There are two possible cases: (Case 1) v_0 is in all generalized faces of M and (Case 2) v_0 is not in a certain generalized face of M (see Figure 10).

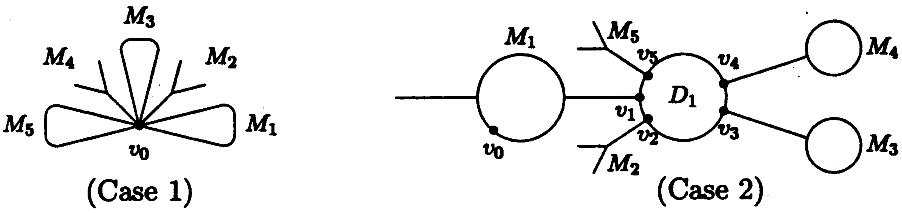


Figure 10: M has more than one generalized faces

(Case 1) Suppose v_0 is in every generalized face of M. All generalized faces are adjacent each other at v_0. Suppose $M_1, M_2, \ldots, M_s$ are the branches at v_0 and enumerated counterclockwise around v_0 in this order and $\partial M = (\partial M_1)(\partial M_2)\cdots(\partial M_s)$. Note that each M_i is a connected and simply connected 2-complex. Since $n \geq 2$ and every generalized face has v_0, each M_i has at most one generalized faces. Then $\partial M_i \in E(S)$ by the inductive hypothesis for every i, where ∂M_i is the boundary cycle of M_i starting and ending at v_0. Therefore, $\partial M \in E(S)$. (Case 2) Suppose v_0 is not in a generalized face D_1. Since the number of generalized faces of M is $n \geq 2$, there exists at least one branch at D_1. Suppose $M_1, M_2, \ldots, M_s$ are branches at D_1 enumerated counterclockwise around D_1 in this order. We may assume M_1 contains v_0. Let $v_1, v_2, \ldots, v_s$ be the vertices at which each M_i is adjacent to D_1. We note that each consecutive vertices v_i and v_{i+1} may be equal. Then $\partial M = w_1 z_1(\partial M_2)z_2(\partial M_3)z_3 \cdots z_{s-1}(\partial M_s)z_s w_2$, where ∂M_i is the boundary cycle of M_i starting and ending at v_i for $2 \leq i \leq s$, $\partial M_1 = w_1 w_2$ and $\partial D_1 = z_1 z_2 \cdots z_s$. By the inductive hypothesis, $\partial M_i \in E(S)$ for every $1 \leq i \leq s$ and $\partial D_1 \in E(S)$. Therefore, $\partial M \in E(S)$ by Lemma 2.3. Consequently S is E-unitary. □

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