ON ORDERED MONOID RINGS

岡山大学理学部 平野康之 (Yasuyuki Hirano)
Faculty of Science, Okayama University,

A monoid $M$ is said to be ordered if the elements of $M$ are linearly ordered with respect to the relation $<$ and that, for all $x, y, z \in G$, $x < y$ implies $zx < zy$ and $xz < yz$. It is well known that torsion-free nilpotent groups and free groups are ordered groups (see [8, Lemma 13.1.6 and Corollary 13.2.8]). Hence any submonoid of a torsion-free nilpotent group or a free group is an ordered monoid.

Let $R$ be a ring. A left (resp. right) annihilator of a subset $U$ of $R$ is defined by $l_R(U) = \{a \in R \mid aU = 0\}$ (resp. $r_R(U) = \{a \in R \mid Ua = 0\}$). Let $G$ be an ordered monoid. Put $rAnn_R(2^R) = \{r_R(U) \mid U \subseteq R\}$ and $lAnn_R(2^R) = \{l_R(U) \mid U \subseteq R\}$. If $U$ is a subset of $R$, then $r_{RG}(U) = r_R(U)RG$. Hence we have a map $\Phi : rAnn_R(2^R) \rightarrow rAnn_{RG}(2^{RG})$ defined by $\Phi(I) = I(RG)$ for every $I \in rAnn(R)$. For an element $f \in RG$, $C_f$ denotes the set of coefficients of $f$ and for a subset $V$ of $RG$, $C_V$ denotes the set $\cup_{f \in V} C_f$. Then $r_{RG}(V) \cap R = r_R(V) = r_R(C_V)$. Hence we also have a map $\Psi : rAnn_{RG}(2^{RG}) \rightarrow rAnn_R(2^R)$ defined by $\Psi(I) = I \cap R$ for every $I \in \Delta$. Obviously $\Phi$ is injective and $\Psi$ is surjective. Clearly $\Phi$ is surjective if
and only if $\Psi$ is injective, and in this case $\Phi$ and $\Psi$ are the inverses of each other. We consider when $\Phi$ is surjective.

Following Rege and Chhawchharia [10] a ring $R$ is called an Armendariz ring if whenever polynomials $f(x) = \sum_{i=0}^{n} a_i x^i, g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$, satisfy $f(x)g(x) = 0$ we have $a_i b_j = 0$ for every $i$ and $j$. This name is connected with the work of Armendariz [2]. Some results of Armendariz rings can be found in [1], [5], [7] and [10]. Let $G$ be an ordered monoid. A ring $R$ is called a G-Armendariz ring if whenever $p = \sum_{g\in G} a_g g, q = \sum_{h\in G} b_h h \in RG$ satisfy $pq = 0$ we have $a_g b_h = 0$ for every $g$ and $h$ in $G$.

The following proposition shows that $\Phi$ is bijective if and only if $R$ is Armendariz.

**Proposition 1.** Let $R$ be a ring and let $G$ be an ordered monoid. The following statements are equivalent:

1) $R$ is $G$-Armendariz.

2) $r Ann_R(2^R) \rightarrow r Ann_{RG}(2^{RG}); A \rightarrow A(RG)$ is bijective.

3) $l Ann_R(2^R) \rightarrow l Ann_{RG}(2^{RG}); B \rightarrow (RG)B$ is bijective.

Following Kaplansky [6], a ring $R$ is called a Baer ring if the left annihilator of each subset is generated by an idempotent. We note that the definition of Baer rings is left-right symmetric. A ring $R$ is called a left (resp. right) p.p.-ring if the left (resp. right) annihilator of each element of $R$ is generated by an idempotent. A left and right p.p. ring is called a p.p. ring. Using Proposition 1, we can generalize [7, Theorems 9 and 10]) as follows:

**Corollary 2.** Let $G$ be an ordered monoid and let $R$ be a $G$-Armendariz ring. Then $R$ is a Baer ring (resp. p.p. ring) if and only if $RG$ is a Baer ring (resp. p.p. ring).
A ring $R$ is called a $G$-quasi-Armendariz ring if whenever $p = \sum_{i=0}^{m} a_i g_i$, $q = \sum_{j=0}^{n} b_j h_j \in RG$ satisfy $pRGq = 0$, then we have $a_i R b_j = 0$ for every $i$ and $j$. In case $G = \{x^i \mid i = 0, 1, 2, \cdots\}$, a $G$-quasi-Armendariz ring is simply called a quasi-Armendariz ring. In [5], we studied quasi-Armendariz rings. Let $r\text{Ann}_R(id(R))$ (resp. $l\text{Ann}_R(id(R))$) denote the set $\{r_R(U) \mid U$ is an ideal of $R\}$ (resp. $\{l_R(U) \mid U$ is an ideal of $R\}$).

**Proposition 3.** Let $R$ be a ring and let $G$ be an ordered monoid. Then the following statements are equivalent:

1) $R$ is $G$-quasi-Armendariz.
2) $r\text{Ann}_R(id(R)) \rightarrow r\text{Ann}_{RG}(id(RG)); A \rightarrow A(RG)$ is bijective.
3) $l\text{Ann}_R(id(R)) \rightarrow l\text{Ann}_{RG}(id(RG)); B \rightarrow (RG)B$ is bijective.

A submodule $N$ of a left $R$-module $M$ is called a pure submodule if $L \otimes_R N \rightarrow L \otimes_R M$ is a monomorphism for every right $R$-module $L$.

**Theorem 4.** Let $G$ be an ordered monoid. Then the following are equivalent;

1) $l_R(Ra)$ is pure as a left ideal in $R$ for any element $a \in R$;
2) $l_R(RGz)$ is pure as a left ideal in $RG$ for any element $z \in RG$;

In this case, $R$ is a $G$-quasi-Armendariz ring.

**Corollary 5.** Let $R$ be a commutative ring and let $G$ be an abelian ordered monoid. Then each principal ideal of $R$ is flat if and only if each principal ideal of $RG$ is flat. In this case $R$ is a $G$-Armendariz ring.

A ring $R$ is called quasi-Baer if the left annihilator of every left ideal of $R$ is generated by an idempotent. Note that this definition is left-right symmetric. Some results of a quasi-Baer ring can be found in [3] and [9].
Let \( R \) be a quasi-Baer ring and let \( a \in R \). Then \( l_R(aR) = Re \) for some idempotent \( e \in R \), and so \( l_R(aR) \) is pure as a left ideal in \( R \). Therefore a quasi-Baer ring satisfies the hypothesis of Theorem 4. Hence we obtain the following.

**Corollary 6** ([4, Theorem 1]). Let \( G \) be an ordered monoid. A ring \( R \) is a quasi-Baer ring if and only if \( RG \) is a quasi-Baer ring. In this case, \( R \) is a quasi-Armendariz ring.

Now we consider some extensions of \( G \)-quasi-Armendariz rings. Let \( R \) be a ring and let \( n \) be a positive integer. Let \( M_n(R) \) denote the ring of \( n \times n \) matrices over \( R \) and \( e_{ij} \) denote the \((i,j)\)-matrix unit. \( T_n(R) \) denotes the ring of all \( n \times n \) upper triangular matrices over \( R \).

**Theorem 7.** Let \( G \) be an ordered monoid. If \( R \) a \( G \)-quasi-Armendariz ring and let \( S \) be a subring of \( M_n(R) \) such that \( e_{ii} Se_{jj} \subseteq S \) for all \( i, j \in \{1, \cdots, n\} \). Then \( S \) is also a \( G \)-quasi-Armendariz ring.

**Corollary 8.** Let \( G \) be an ordered monoid. If \( R \) a \( G \)-quasi-Armendariz ring, then, for any positive integer \( n \), \( T_n(R) \) is also a \( G \)-quasi-Armendariz ring.

For \( f \in RG \), the content \( A_f \) of \( f \) is the ideal of \( R \) generated by the coefficients of \( f \). For any subset \( S \) of \( RG \), \( A_S \) denotes the ideal \( \sum_{f \in S} A_f \). In case \( G = \{ x^i \mid i = 0, 1, 2, \cdots \} \), a commutative ring \( R \) is Gaussian if \( A_{fg} = A_f A_g \) for all \( f, g \in R[x] \) (See [1]). We extend this notion to noncommutative rings as follows. A ring \( R \) is said to be \( G \)-quasi-Gaussian if \( A_{fRg} = A_f A_g \) for all \( f, g \in RG \).

**Theorem 9.** A ring \( R \) is \( G \)-quasi-Gaussian if and only if every homomorphism
image of $R$ is quasi-Armendariz.

Example 10. A ring $R$ is fully idempotent if $I^2 = I$ for every twosided ideal $I$ of $R$. Obviously a ring $R$ is fully idempotent if and only if every homomorphic image of $R$ is semiprime. Von Neumann regular rings are fully idempotent. We can easily see that a semiprime ring is $G$-quasi-Armendariz. Therefore by Theorem 9, a fully idempotent ring is a $G$-quasi-Gaussian ring.

References


