数理解析研究所講究録 1222 巻 2001 年 170-177

CONSENSUS ON p-BELIEF COMMUNICATION

石川竜一郎 (RYUICHIRO ISHIKAWA)* 一橋大学大学院経済学研究科,
松久隆 (TAKASHI MATSUHISA)† 茨城工業高等専門学校, AND
赤川洋介 (YOH-SUKE AKAGAWA) 茨城大学理学部

ABSTRACT. The communication process in the p-belief system that reaches consensus is presented: The agents communicate the events that they believe more than their own posteriors. We show that in the long run each posterior converges and that any two limiting values of posterior can be same.

1. INTRODUCTION

The purposes of this article are the two points: First to present the communication process according to a protocol with the p-belief system, and secondly to extend the agreement theorem of Aumann [1] to this model in the line of Geanakoplos and Polemarchakis [4] and Krasucki [5].

In his seminal paper Aumann [1] showed his agreement theorem that if all players commonly know their posteriors of an event then their posteriors must be same. Geanakoplos and Polemarchakis [4] investigated a communication process between two players in which the players announce their posteriors to each other. In the process players learn and revise their posteriors and they reached consensus. This is an extension of the agreement theorem of Aumann regardless of common-knowledge. Krasucki [5] extended the result for the communication process among at least three players.

The knowledge of players in the above models are given by partition. However many researchers have been pointing out that the assumption for the partition is problematic in the decision makings among players and that the model should be constructed without such strong assumption.

Monderer and Samet [7] introduced the weaker model, called the p-belief system, that formalizes player's belief of something as he believes it at least probability p. They showed the approximating agreement theorem: If all players commonly p-believe their posteriors of an event then the difference between any two players' posteriors can differ at most $2(1-p)$.

We introduce the communication process with the p-belief system; it is the revision process of the values of players' posterior of an event through the messages. Our result is an extension of the above agreement theorems as follows:

2000 *Mathematics Subject Classification.* Primary 91A35, Secondary 68Q55.
Journal of Economic Literature Classification: C62, C79.
Key words and phrases. Consensus, Communication, Information structure, Knowledge, p-Belief system, Protocol.

This is a preliminary version and the final form will be published elsewhere.

*Corresponding author.

†Supported by the Grand-in-Aid for Scientific Research(C)(2)(No.12640145) in the Japan Society for the Promotion of Sciences.

Theorem 1. *In the p-belief communication consensus on the limiting values of the posteriors of an event X can be guaranteed if the protocol contains no cycle.*

We begin in Section 2 by reviewing the basic model, which consists of the standard model of knowledge, p-belief system and a protocol. The p-belief communication is introduced on the system. In Section 3, a simple example is presented, which shows a p-belief communication process among three persons. In Section 4, we show the results and the proof and state the conclusion in the last section.

2. The Model

Let N be a set of finitely many *players* and i denote an player. A *state-space* is a finitely non-empty set, whose members are called *states*. An *event* is a subset of the state-space. If Ω is a state-space, we denote by 2^Ω the field of all subsets of it. An event F is said to *occur* in a state ω if $\omega \in F$.

2.1. **Partitional Information Structure** [1]. By this we mean a class of mappings $(\Pi_i)_{i\in N}$ such that $\Pi_i : \Omega \to 2^\Omega$ satisfies the three properties, *Reflectivity, Transitivity* and *Symmetry*:

(Ref) $\omega \in \Pi_i(\omega)$;
(Trn) $\xi \in \Pi_i(\omega)$ implies $\Pi_i(\xi) \subseteqq \Pi_i(\omega)$;
(Sym) $\xi \in \Pi_i(\omega)$ implies $\omega \in \Pi_i(\xi)$.

Specifically, an agent i for whom $\Pi_i(\omega) \subseteqq E$ knows in the state ω that some state in the event E has occurred. We call Π_i the i's *information partition* and $\Pi_i(\omega)$ the i's *possibility set* at ω. This is interpreted as the set of states that i thinks are *possible* when ω occurs. By i's *posterior* q_i at ω of a given event X we mean the conditional probability of X under the possibility set at ω; i.e.; $q_i = \mu(X \mid \Pi_i(\omega))$. We will also call it the *initial* posterior of X at ω in later sections.

2.2. **p-Belief system.** We assume that the partitional information structure $(\Pi_i)_{i\in N}$ on the given Ω. Let p be a number with $0 \lneqq p \leqq 1$ and fix it. The *p-belief system* associated with the information partitions is the tuple $\langle \Omega, \mu, (\Pi_i)_{i\in N}, (B_i^p)_{i\in N}\rangle$ consisting of the following structures and interpretations: (Ω, μ) is a finite probability space such that $\mu(\omega) \gneqq 0$ for every $\omega \in \Omega$, and the i's *p-belief operator* B_i^p on 2^Ω is defined such that B_i^pE is the set of states of Ω in which i believes that E has occurred with probability at least p ; that is,

$$B_i^pE := \{\omega \in \Omega \mid \mu(E \mid \Pi_i(\omega)) \geqq p\}.$$

We call it the event that player i *p-believes* E. When $\omega \in B_i^p(E)$ we say that i p-believes E at ω. We record the properties of B_i^p as follows[2]: For every E, F of 2^Ω,

N $B_i^p\Omega = \Omega$ and $B_i^p\emptyset = \emptyset$;
M $B_i^pE \subseteqq B_i^pF$ whenever $E \subseteqq F$;
T4 $B_i^pE = B_i^pB_i^pE$;
IE $\mu(E \mid B_i^pE) \geqq p$.

The *mutual* belief operator B_E^p on 2^Ω is defined by $B_E^pE := \bigcap_{i\in N} B_i^pE$, whereas B_E^pE is interpreted as the event that all agents p believe E.

[1]Aumann [1], Binmore [3].

[2]See Proposition 2 in Monderer and Samet [7].

2.3. **Protocol** [3]. We assume that players communicate by sending *messages*. A *protocol* is a mapping Pr of the set of non-negative integers $\mathbb{Z}_+$ into the product set $N \times N$ that assigns to each t a pair of players $(s(t), r(t))$. Here t stands for *time* and $s(t)$ and $r(t)$ are, respectively, the *sender* and the *recipient* of the communication which takes place at time t. We can consider it as the directed graph. A protocol is said to be *fair* if the graph is strongly-connected; in words, every player in this protocol communicates directly or indirectly with every other player infinitely often. It is said to be *acyclic* if the graph contains no cyclic path; that is, there are players $i_1, i_2, \dots, i_k$ with $k \geqq 3$ such that for all $m < k$, i_m communicates directly with i_{m+1}, and such that i_k communicates directly with i_1

2.4. **Communication with p-belief system.** The *p-belief communication process* π of revisions of the posteriors $(q_i^t)_{(i,t)\in N\times\mathbb{Z}_+}$ of an event X is a triple $\langle \mathrm{Pr}, (Q_i^t)_{(i,t)\in N\times\mathbb{Z}_+}, X\rangle$, in which $\mathrm{Pr}(t) = (s(t), r(t))$ is a fair protocol such that for every $t \in \mathbb{Z}_+$, $r(t) = s(t+1)$, communications proceed in *rounds*[4] and Q_i^t is the mapping of Ω into 2^Ω for agent i at time t that is defined inductively as follows:

- We assume given a mapping $Q_i^0 := \Pi_i$ and q_i^o is the initial posterior $q_i = \mu(X|Q_i^0(\omega))$.
- Suppose Q_i^t is defined.
  - Π_i^t is the partition on Ω induced by Q_i^t; that is, $\Pi_i^t(\omega) := \{\xi \in \Omega \mid Q_i^t(\xi) = Q_i^t(\omega)\}$;
  - $q_i^t(X;\omega)$ denotes $q_i^t := \mu(X|\Pi_i^t(\omega))$;
  - W_i^t is the mapping of Ω into 2^Ω which assigns to each state ω the event $W_i^t(\omega)$ that consists of all the states ξ such that $\mu(X|\Pi_i^t(\xi)) \geqq q_i^t(X;\omega)$[5];
- If j send the message $W_j^t(\omega)$ at t to i, then Q_i^{t+1} is defined as follows:
  - If i is not a recipient of a message at time t, then $Q_i^{t+1} = Q_i^t$.
  - If i is a recipient of a message at time t, then $Q_i^{t+1} = Q_i^t \cap W_i^t$ defined by $Q_i^{t+1}(\omega) = Q_i^t(\omega) \cap W_j^t(\omega)$.

Remark 1. The sequence of sets $\{Q_i^t \,|\, t = 0, 1, 2, \dots\}$ is stationary in finitely many rounds because it is a descending chain by definition and Ω is finite. That is, there is a sufficiently large time $\tau \in T$ such that for every i, for all $\omega \in \Omega$ and for all $t \geqq \tau$, $Q_i^t(\omega) = Q_i^\tau(\omega)$, and therefore $q_i^t(X;\omega) = q_i^\tau(X;\omega)$

2.5. **Consensus.** We note that the limit Q_i^∞ exists in each state by Remark 1, and thus Π_i^∞ can be defined. We denote $q_i^\infty(\omega) = \mu(X|\Pi_i^\infty(\omega))$ called the i's *limiting value* of X at ω. We say that *consensus* on the limiting values of the posteriors about X can be guaranteed in the communication process if $q_i^\infty(\omega) = q_j^\infty(\omega)$ for each player i, j and in all the states ω.

3. Example

We consider a comprehensive example as follows. Players are Alice, Bob, and Nanny. They communicate about p-belief of some event X to each other according the protocol

[3]Krasucki [5].

[4]That is, there exists a natural number m such that for all t, $s(t) = s(t+m)$.

[5]The specification of the message sent by i at t allows that the recipient j receives the information that i believes X with probability at least $q_i^t(X;\omega)$.

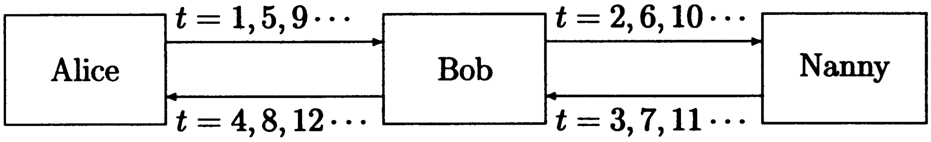
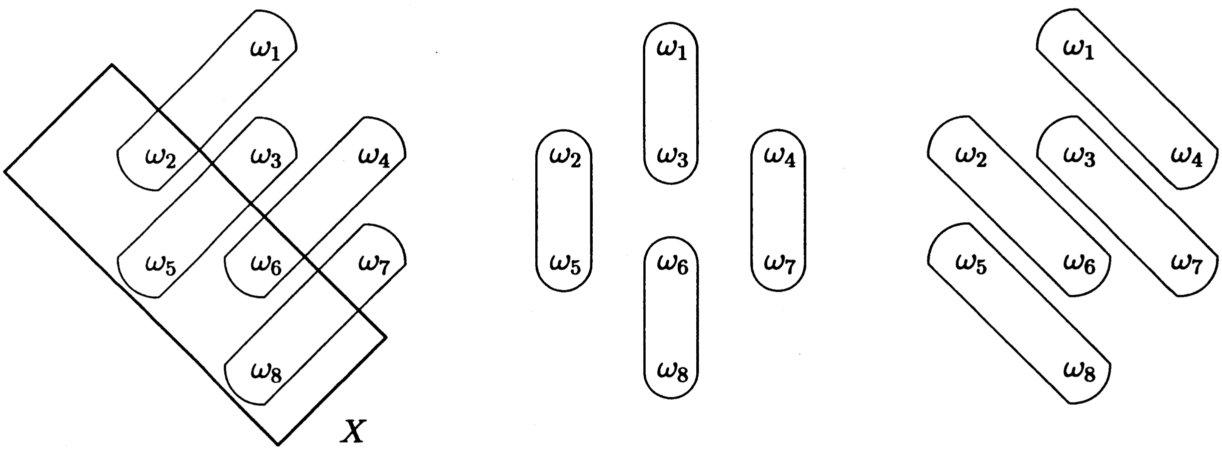


FIGURE 1. Communication protocol among three players.

with no cycle in Figure 1. Now let Ω be the state space $\{\omega_1, \cdots, \omega_8\}$ and μ the common-prior on Ω defined by $\mu(\omega) = 1/8$ for each ω.

Each player has his initial knowledge defined by the partition on Ω. Figure 2 shows this situation. Now suppose an event $X = \{\omega_2, \omega_5, \omega_6, \omega_8\}$ is occurred. Then their posteriors will reach consensus: the values of the posteriors are all one in each state in X and the values are zero in other states.



Π_A Π_B Π_N

FIGURE 2. Information partitions for each of players.

We show the revision process about their knowledge and posteriors. First, Alice sends her message to Bob at time $t = 1$ as Table 1. He receives it and constitutes his own partition. However the message at time $t = 1$ is not useful for him because his initial knowledge is finer. Therefore his information partition is not changed, that is $\Pi_B^1 = \Pi_B$. As result, the information structure at time $t = 1$ for each player is the same as the initial.

In a similar way, Bob communicates with the protocol. Table 2 shows his message to Nanny at time $t = 2$. Though she receives it, it is not useful for her as well. Therefore her information structure is the same as the initial one. The situation continues at time $t = 3$.

	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8
q_A^0	1/2	1/2	1/2	1/2	1/2	1/2	1/2	1/2
W_A^0	Ω	Ω	Ω	Ω	Ω	Ω	Ω	Ω

TABLE 1. Alice's message at time $t = 1$.

	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8
q_B^1	0	1	0	0	1	1	0	1
W_B^1	Ω	$\{\omega_2, \omega_5, \omega_6, \omega_8\} = D$	Ω	Ω	D	D	Ω	D

TABLE 2. Bob's message at time $t = 2$.

At time $t = 4$, Bob sends a useful message to Alice. Table 3 denotes Bob's at the time. That is, his message is finer than her knowledge held until now. Consequently her knowledge is revised as Q_A^4 in Figure 3. Therefore her message to Bob is finer as denoted in Table 4.

	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8
q_B^3	0	1	0	0	1	1	0	1
W_B^3	Ω	$\{\omega_2, \omega_5, \omega_6, \omega_8\} = D$	Ω	Ω	D	D	Ω	D

TABLE 3. Bob's message at time $t = 4$.

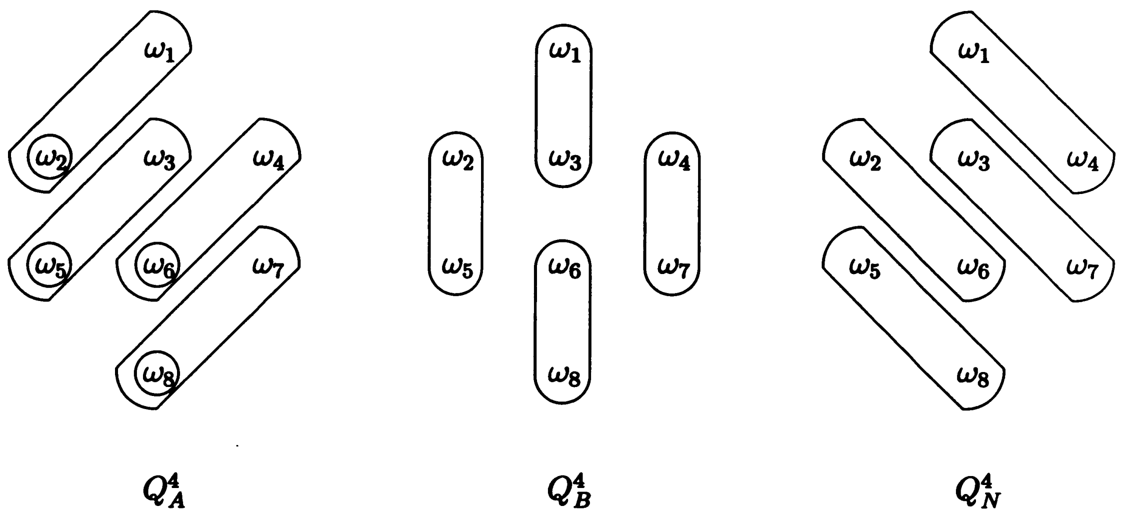


FIGURE 3. Information structures at time $t = 4$.

Seeing the Table 5, we say that consensus on the limiting values of the posteriors about X can be guaranteed in the example. The information structures on the limiting values for each players is stable after time $t = 5$. Therefore the structures is the same as the Figure 3.

	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8
q_A^4	0	1	0	0	1	1	0	1
W_A^4	Ω	$\{\omega_2, \omega_5, \omega_6, \omega_8\} = D$	Ω	Ω	D	D	Ω	D

TABLE 4. Alice's message at time $t = 5$.

	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8
q_A^∞	0	1	0	0	1	1	0	1
q_B^∞	0	1	0	0	1	1	0	1
q_N^∞	0	1	0	0	1	1	0	1
W_A^∞	Ω	$\{\omega_2, \omega_5, \omega_6, \omega_8\} = D$	Ω	Ω	D	D	Ω	D
W_B^∞	Ω	$\{\omega_2, \omega_5, \omega_6, \omega_8\} = D$	Ω	Ω	D	D	Ω	D
W_N^∞	Ω	$\{\omega_2, \omega_5, \omega_6, \omega_8\} = D$	Ω	Ω	D	D	Ω	D

TABLE 5. The travelers' message at time $t \geqq 5$.

4. REACHING CONSENSUS

We show a proof of Theorem 1 presented in Introduction. For the purpose, we prove Theorem 2.

4.1. **Proof of Theorem 1.** Follows immediately from Theorem 2 below.

Theorem 2. *In the p-belief communication π, if player i communicates his/her message directly to another player j then $q_i^\infty(\omega) = q_j^\infty(\omega)$ for every $\omega \in \Omega$.*

In fact we can observe that the consensus in Theorem 1 follows from Theorem 2 by inductive steps in viewing that the protocol is acyclic.

4.2. **Proof of Theorem 2.** The following lemma is a key to proving the theorem:

Lemma 1. *Let ω be a state in Ω, Π a partition of Ω, and let $q(X;\omega)$ denote $\mu(X|\ \Pi(\omega))$. If there exists a non-empty event H of 2^Ω such that for every ξ of H, the two conditions are true:*

(a) *$\Pi(\xi)$ is contained in H, and*
(b) *$\mu(X|\ \Pi(\xi)) = q(X;\omega)$;*

then we obtain that

$$q(X;\omega) = \mu(X|\ H).^6$$

In fact, H can be decomposed into the disjoint union of components $\Pi(\xi)$ for $\xi \in H$ in view of (a), and thus the result follows from (b).

Proof of Theorem 2: For each state ω denote $[q_i^\infty] = [q_i^\infty(X;\omega)] := \{\xi \mid \mu(X|\Pi_i^\infty(\xi)) = q_i^\infty\}$. Set $H = [q_i^\infty] \cap [q_j^\infty]$; this is non-empty because $\omega \in H$. We can verify the two properties: For $l = i, j$,

(a) $\Pi_l^\infty(\xi) \subseteqq H$ for every $\xi \in H$, and

[6]This lemma plays an essential role in the proof of the agreement theorem of Aumann (Proposition

(b) $H \subseteqq [q_l^\infty]$.

In fact, Part (b) is obviously observed. For (a) it suffices to show that

(c) $\Pi_l^\infty(\xi) \subseteqq [q_{l'}^\infty(X;\omega)]$ for every $\xi \in [q_{l'}^\infty(X;\omega)]$,

where $l' \in \{i,j\} \setminus \{l\}$. Indeed, it is easily verified that $[q_{l'}^\infty(X;\xi)] = [q_{l'}^\infty(X;\omega)]$ for every $\xi \in [q_{l'}^\infty(X;\omega)]$, and thus (a) immediately follows.

We will prove (c): Suppose $\zeta \in \Pi_l^\infty(\xi)$. We note that there exist m such that $s(t) = r(t+m)$ and $r(t) = s(t+m)$ since the protocol is acyclic, and on noting that $Q_l^\infty(\zeta) = Q_l^\infty(\xi)$ it follows from the definition of $\{Q_l^t\}_{t\in \mathbf{Z}_+}$ that $\zeta \in W_{l'}^\infty(\xi)$ and $\xi \in W_{l'}^\infty(\zeta)$. Therefore it can be observed that $q_{l'}^\infty(X;\zeta) = q_{l'}^\infty(X;\xi)$, and thus $\zeta \in [q_{l'}^\infty(X;\xi)]$ as claimed.

Now, viewing Lemma 1 we obtain that $q_l^\infty = \mu(X|\Pi_l^\infty(\xi)) = \mu(X|H)$ for each $l \in \{i,j\}$, and thus $q_i^\infty(\omega) = q_j^\infty(\omega)$ as required. □

5. Concluding Remarks

This article investigates how players reach consensus through p-belief communication. This research is in the line on the investigations on a communication process in the standard model of knowledge. As stated in Introduction, Geanakoplos and Polemarchakis [4] and Krasucki [5] studied how players reach consensus on the knowledge model. Correspondingly Monderer and Samet [7] showed the approximating agreement theorem that does not guarantee consensus on the p-belief model. The reason the players cannot reach consensus in Monderer and Samet [7] is that the information obtained from the players' ability is ambiguous; the players's knowledge are given not by the partitional structure but the RT-information structure similar to **S4** model in Bacharach [2]. This means that the players' information structures satisfy reflexive and transitive properties but not symmetric property.

To overcome the difficulty, we introduce the model of p-belief communication in which players can generate the partitional information structures by receiving messages though the messages are not given by partitions, and thus the players can keep their knowledge partitional. Consequently we can obtain the result that players can reach consensus on the p-belief communication. In the p-belief communication as above the requirement that players must generate the partitional information by receiving messages is not a strong assumption with respect to the requirement of the standard model of knowledge.

It plays the essential role for reaching consensus through the communication on which players' information structures are given by partitions. When this requirement fails, how we can reach consensus? This problem remains open.

REFERENCES

[1] Robert J. Aumann, *Agreeing to disagree*, Annals of Statistics **4** (1976), 1236-1239.
[2] Michael Bacharach, *Some extensions of a claim of Aumann in an axiomatic model of knowledge*, Journal of Economic Theory **37** (1985),167–190.
[3] Ken Binmore, *Fun and Games*. D. C. Heath and Company, Lexington, Massachusetts USA, (1990), xxx+642pp.
[4] John D. Geanakoplos, and Heraklis M. Polemarchakis, *We can't disagree forever*, Journal of Economic Theory **28** (1982), 192–200.
[5] Paul Krasucki, *Protocol forcing consensus*, Journal of Economic Theory **70** (1996), 266–272.
[6] Takashi Matsuhisa and Shun-Suke Usami, *Awareness, belief and agreeing to disagree*, Far East Journal of Mathematical Sciences **2** No.6 (2000), 833–844.
[7] Dov Monderer and Dov Samet, *Approximating common knowledge with common belief*, Games and Economic Behavior **1** (1989), 170–190.
[8] Paul Milgrom and Nancy Stokey, *Information, trade and common knowledge*, Journal of Economic Theory **26** (1982), 17–27.
[9] Dov Samet, *Ignoring ignorance and agreeing to disagree*, Journal of Economic Theory **52** (1990), 190–207.

(Ryuichiro Ishikawa) GRADUATE SCHOOL OF ECONOMICS, HITOTSUBASHI UNIVERSITY, NAKA 2-1, NISHI, KUNITACHI-SHI, TOKYO 186-8601, JAPAN
E-mail address: `gem7103@@srv.cc.hit-u.ac.jp`

(Takashi Matsuhisa) DEPARTMENT OF LIBERAL ARTS AND SCIENCES, IBARAKI NATIONAL COLLEGE OF TECHNOLOGY, 866 NAKANE, HITACHINAKA-SHI, IBARAKI 312-8508, JAPAN
E-mail address: `mathisa@ge.ibaraki-ct.ac.jp`

(Yoh-Suke Akagawa) UNDERGRADUATE COURSE OF ELECTRIC ENGINEERING, IBARAKI NATIONAL COLLEGE OF TECHNOLOGY. *Current Address*: UNDERGRADUATE COURSE OF MATHEMATICAL SCIENCES, IBARAKI UNIVERSITY, BUNKYO 2-1-1, MITO-SHI, IBARAKI 310-8512, JAPAN.