The ideal transforms of semigroups

Mitsuo Kanemitsu
Aichi University of Education

By a semigroup we mean a submonoid of a torsion-free abelian (additive) group in this paper. Let $S$ be a semigroup with the quotient group $q(S)$, that is, $q(S) = \{ s - s' \mid s, s' \in S \}$. Any semigroup $T$ between $S$ and $q(S)$ is called an oversemigroup of $S$.

Moreover, let $\mathbb{Z}$ be the set of all integers and let $\mathbb{Z}_n = \{ a \in \mathbb{Z} \mid a \geq n \}$ and $X \cdot \mathbb{Z}_m = \{ aX \mid a \in \mathbb{Z}_m \}$. And $S[X] = S + \mathbb{Z}_0 X = \{ s + nX \mid s \in S, n \in \mathbb{Z}_0 \}$ is called a polynomial semigroup over $S$ (cf. [KOM]).

Let $I$ be a subset of $S$. $I$ is called an ideal of $S$ if $I + S = I$, that is, $a + s \in I$ for each $a \in I$ and each $s \in S$. For any $a \in S$, put $(a) = a + S = \{ a + s \mid s \in S \}$. Then $(a)$ is an ideal of $S$ and it is called a principal ideal of $S$. For $a_1, a_2, \ldots, a_n \in S$, we set $I = (a_1, a_2, \ldots, a_n) = \cup_{i=1}^n (a_i) = \cup_{i=1}^n (a_i + S)$. The $(a_1, a_2, \ldots, a_n)$ is an ideal of $S$ and it is called an ideal generated by $a_1, a_2, \ldots, a_n$ and $\{ a_1, a_2, \ldots, a_n \}$ is called a basis of $I$.

An element $u$ of $S$ is called a unit if $u + v = 0$ for some $v \in S$. Let $U(S)$ be the set of units in $S$. Note that $U(S)$ be a subgroup of $q(S)$.

If we put $M = S - U(S)$, then $M$ is an ideal of $S$. Moreover, if $I$ is an ideal of $S$ such that $M \subset I$, then $M = I$ or $I = S$. $M$ is called the maximal ideal of $S$. A proper ideal $P$ of $S$ is called a prime ideal of $S$ if $a + b \in P$ with $a, b \in S$ implies either $a \in P$ or $b \in P$. We note that the maximal ideal of $S$ is a prime ideal, $\phi$ is a prime ideal of $S$ and $S$ has the only one maximal ideal.

We give semigroup versions of some results in [F].

Let $S$ be a semigroup. Also, let Spec($S$) be the set of all prime ideals of $S$. For an ideal $I$ of $S$, we put $V(I) = \{ P \in \text{Spec}(S) \mid P \supset I \}$ and $D(I) = \{ P \in \text{Spec}(S) \mid P \not\supset I \} = \text{Spec}(S) - V(I)$. In particular, put $D((a)) = D(a)$ for $a \in S$.

**Lemma 1.** Let $\{ I_\lambda \}_{\lambda \in \Lambda}$ is a family of ideals of $S$ and let $I$ and $J$ are ideals of $S$. Then we have the following statements.
(1) $\cap \{ I_{\lambda} \mid \lambda \in \Lambda \}$ is an ideal of $S$.
(2) $\cup \{ I_{\lambda} \mid \lambda \in \Lambda \}$ is an ideal of $S$.
(3) $I + J = \{ a + b \mid a \in I, b \in J \}$ is an ideal of $S$ such that $I + J \subset I \cap J$.
(4) If $P = I \cap J$ is a prime ideal of $S$, then $I = P$ or $J = P$.
(5) If $P$ and $Q$ are two prime ideals of $S$, then $P \cup Q$ is also a prime ideal of $S$.

Lemma 2. Let $S$ be a semigroup. Then the following statements hold.
(1) $V(\phi) = \text{Spec}(S)$, $V(S) = \phi$.
(2) If $I \subset J$, then $V(I) \supset V(J)$.
(3) $V(I_1) \cap V(I_2) = V(I_1) \cup V(I_2)$.
(4) $V(\cup \{ I_{\lambda} \mid \lambda \in \Lambda \}) = \cap \{ V(I_{\lambda}) \mid \lambda \in \Lambda \}$.

We make $\text{Spec}(S)$ into a topological space; the topology is called the Zariski topology. The closed sets are defined by the $V(I) = \{ P \in \text{Spec}(S) \mid P \supset I \}$.

Then $D(I)$ is an open set of $\text{Spec}(S)$ and the $D(f) = \{ P \in \text{Spec}(S) \mid f \in P \}$ is an open basis of $\text{Spec}(S)$. For this topology, we give the following statement.

Proposition 3. $\text{Spec}(S)$ is a Kolmogoroff space ($T_0$-space) and a quasi-compact space.

Definition 1. We call the ideal transform of $S$ with respect to an ideal $I$ of $S$ the following oversemigroup of $S$:

$$T_S(I) := \{ z \in q(S) \mid (S :_S z + S) \supset nI \text{ for some } n \geq 1 \}$$

Also, we call the Kaplansky ideal transform of $S$ with respect to an ideal $I$ of $S$ the following oversemigroup of $S$:

$$\Omega_S(I) := \{ z \in q(S) \mid \text{rad}(S :_S z + S) \supset I \}.$$  

where $\text{rad}(J) = \{ a \in S \mid na \in J \text{ for some positive integer } n \}$. Note that $\Omega_S(I)$ is an oversemigroup of $T_S(I)$ and note that if $I$ is finitely generated, then $\Omega_S(I) = T_S(I)$. For a principal ideal $I$, we have that $I + T_S(I) = T_S(I)$.  

Proposition 4. Let $I$ be a principal ideal of $S$ and $P$ be a prime ideal of $S$. Then the following results are hold.

1. $I + T_S(I) = T_S(I)$, $I + \Omega_S(I) = \Omega_S(I)$.
2. $P \in V(I)$ if and only if $P + T_S(I) = T_S(I)$ if and only if $P + \Omega_S(I) = \Omega_S(I)$.

Definition 2 ([K],[KB],[KM] and [MK]). A semigroup $S$ is a valuation semigroup if $\alpha \in q(S)$ then $\alpha \in S$ or $-\alpha \in S$.

Also, we say that $S$ is a seminormal semigroup if $2\alpha, 3\alpha \in S$ for $\alpha \in q(S)$, we have $\alpha \in S$.

It is clear that valuation semigroups are seminormal.

Definition 3. A non-empty subset $N$ of a semigroup $S$ is called an additive system of $S$ if $a, b \in N$ implies $a + b \in N$ and $0 \in N$.

Put $S_N = \{ s - t \mid s \in S, t \in N \}$. Then $S_N$ is an oversemigroup of $S$ and is called the quotient semigroup of $S$. If $P$ is a prime ideal of $S$, then $T = S - P$ is an additive system of $S$ and the quotient semigroup $S_T$ is denoted by $S_P$.

Definition 4 ([OK]). Let $T$ be an oversemigroup of $S$. Then $T$ is said to be flat over $S$ if for any prime ideal $P$ of $S$, either $P + T = T$ or $T \subset S_P$. Put $\text{Flat}(T) = \{ P \in \text{Spec}(S) \mid P + T = T \text{ or } T \subset S_P \}$.

Example 1. Let $S = (Z_1 + Z_1X) \cup \{0\}$. Then $U(S) = \{0\}$ and $M = Z_1 + Z_1X = S - U(S)$ is the maximal ideal of $S$. Also, Krull dim $S = 1$ and $S$ is not valuation semigroup.

Also, let $T = (Z_1 + Z_1X) \cup Z_0$. Then $T$ is not a valuation semigroup and $U(T) = \{0\}$. Put $N = Z_1 + Z_1X$. Then $N \notin \text{Flat}(T)$.

Theorem 5 ([OK]). Let $T$ be an oversemigroup of $S$. Then the following statements are equivalent.

1. $T$ is flat over $S$.
2. $T = S_{N \cap S}$ for the maximal ideal $N$ of $T$.
3. For any two ideals $I, J$ of $S$, $(I \cap J) + T = (I + T) \cap (J + T)$.

Definition 5. Let $S$ be a semigroup and let $T$ be an oversemigroup of $S$. Then $T$ is said to be LCM-stable over $S$ if $((a + S) \cap (b + S)) + T = (a + T) \cap (b + T)$ for each $a, b \in S$. 
A flat oversemigroup $T$ over $S$ is LCM-stable over $S$.

**Theorem 6.** Assume that $S$ be a Noetherian semigroup. Let $T$ be an oversemigroup of $S$. Then $T$ is flat over $S$ if and only if $T$ is LCM-stable over $S$.

**Corollary 7.** If $S$ is a valuation semigroup and a proper principal ideal $I = (a)$ of $S$, then $P \in D(I)$ if and only if $T_S(I) = \Omega_S(I) \subset S_P$.

**Proposition 8.** The following statements are hold.
(1) $S_a = \Omega_S((a))$ for a non-unit $a \in S$.
(2) If $I$ and $J$ are ideals of $S$ such that $I \subset J$, then $\Omega_S(I) \supset \Omega_S(J)$.
(3) $\Omega_S(I) = \cap \{S_P \mid P \in D(I)\} = \cap \{\Omega_S(I + S_P) \mid P \in \text{Spec}(S)\}$.
(4) If $I$ is a proper ideal of $S$, then $\Omega_S(I) = \cap \{\Omega_S(a + S) \mid a \in I\} = \cap \{S_a \mid a \in S_a\}$.

**Definition 6.** $x \in G$ is called an almost integral element of $S$ if there exists an element $a \in S$ such that $a + nx \in S$ for each positive integer $n$. Also, $S$ is a completely integrally closed if $x$ is almost integral over $S$ then $x \in S$.

**Theorem 9 ([K]).** Let $S$ be a valuation semigroup such that $S \neq q(S)$. Then Krull dim $S = 1$ if and only if $S$ is a completely integrally closed semigroup.

**Theorem 10 ([KHF]).** (1) Spec($\mathbb{Z}_0[X]$) = \{ $(X), (1, X), \phi$ \}.
(2) The primary ideals of $\mathbb{Z}_0[X]$ are the following:
(i) All the ideals that contains both elements of $\mathbb{Z}_0$ and $\mathbb{Z}_0X$.
(ii) $\mathbb{Z}_k + \mathbb{Z}_0X = (k)$ with $k \in \mathbb{Z}_0$.
(iii) $\mathbb{Z}_0 + \mathbb{Z}_kX = (kX)$ with $k \in \mathbb{Z}_0$.

**Example 2.** Let $S = \mathbb{Z}_0 \cup (Z + Z_1X)$. Put $P = Z + Z_1X$ and $M = (1) = 1 + S = P \cup Z_1$. Then Spec($S$) = \{ $\phi, P, M$ \}. Since $\phi \subset P \subset M$, we have that Krull dim $S = 2$. It is clear that $S$ is a valuation semigroup. Since $P$ is not finitely generated, $S$ is not Noetherian semigroup.

**Theorem 11.** Let the notation be as in Example 2 and let $I$ be an ideal of $S$. Then the following statements hold.
(1) Let $I = (f)$ be a principal ideal of $S$. If $f = 0$, then $T_S(I) = \Omega_S(I) = S$. Also, if $f \in M - P$, then $T_S(I) = \Omega_S(I) = S_f = \mathbb{Z}[X] = \mathbb{Z} + \mathbb{Z}_0 X$. Next, if $f \in P$, then $T_S(I) = \Omega_S(I) = S_f = q(S).

(2) If $I$ is not a finitely generated ideal of $S$, then $I = \mathbb{Z} + \mathbb{Z}_n X (n \geq 1)$ and $\Omega_S(I) = q(S).

(3) Let $I \neq S$. Then $\text{Spec}(\Omega_S(I)) \cong D(I)$ if and only if $I + \Omega_S(I) = \Omega_S(I).

(4) $S$ is not a completely integrally closed and each oversemigroup of $S$ is a flat semigroup over $S$, and so $\text{Flat}(T) = \text{Spec}(S)$ for each oversemigroup $T$ of $S$.

**Theorem 12.** Let $S$ be a semigroup and $I$ an ideal of $S$. Then the following statements are equivalent.

1. $D(I)$ is an affine open subspace of $\text{Spec}(S)$.
2. $\Omega_S(I)$ is flat over $S$ and, for each $P \in \text{Spec}(S)$ with $P \supset I$, $P + \Omega_S(I) = \Omega_S(I)$.
3. $I + \Omega_S(I) = \Omega_S(I)$.

**References**


