数理解析研究所講究録 1222 巻 2001 年 40-45

Topics on finite and countable infinite BCK

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In this Note, we present some new results on BCK without proofs, in particular, we mainly concern with finite and countable infinite BCK.

First I give a new definition of BCK which is equivalent to an old defintion(for exmple, see [5], [7], [8]). Let P be a partially ordered set with a least element 0 on which a binary operaion * is defined. We assume that the partial order $x \leq y$ is reflexive, antisymmetric and transitive.

We say that P is a **BCK** if a binary operation * on P satisfies the following conditions:

1) $(x * y) * (x * z) \leq z * y$,
2) $x * (x * y) \leq y$,
3) $x * x = 0$,
4) $0 * x = 0$,
5) $x * y = 0$ is equivalent to $x \leq y$.

We know some special classes of BCK which are defined by the following way: Let X be a BCK.

(1) X is **positive implicative**, if $(x * y) * y = x * y$ for any x, y in X.

(2) X is **commutative**, if $x * (x * y) = y * (y * x)$ for any x, y in X.

Then $x * (x * y)$ is the greatest lower bound of x, y. Hence it is denoted by $x \wedge y$. Therefore a commutative BCK is a lower semilattice.

(3) X is **implicative**, if X is positive implicative and commutative.

(4) X is a BCK **with condition** (S), if for any $a, b \in X$, non-empty set

$$\{x : x * a \leq b\}$$

has the gereatest element in X. Its element is denoted by aob.

Then X is a partially ordered commutative semigroup with respect to the just introduced operation o. This is refered as the **associated** (p.o.commutative) **semigroup** of X.

The operation o has the following basic properties:

6) $x, y \leq xoy = yox$,
7) $xo(yoz) = (xoy)oz$,
8) $x * (y * z) = x * (yoz)$,
9) $x \leq y \rightarrow xoz \leq yoz$ for any $z \in X$.

We can find many basic and useful properties of these classes in [7] and [8]. Among them, we only mention some results which hold in BCK.

10) $x * 0 = x$,
11) $(x * y) * z = (x * z) * y$. (permutation rule)
12) $x * y \leq x$.
13) $x * [x * (x * y)] = x * y$.

(5) W.H.Cornish introduced a concept which is called a BCK with supremum ([4], [5]).

Let X be a BCK with conidition (S). X is called a **BCK with supremum,** if the following identity holds for any elements x, y, z in X:

$$xo(y * x) = yo(x * y).$$

$xo(y*x)$ is denoted by $x \vee y$. It is the least upper bound of x, y. Therefore, a BCK with supremum is an upper semilattice.

Remark 1. If the conidtion 4) in the définition of BCK is replaced by a condition: there is no elements less than 0, namely $0 \leq x$ implies $x = 0$, we obtain the concept of **BCI.**

Theorem 1. *There exists at least one BCK structure on any partially ordered set with a least element 0. This BCK structure is given by the following way:*

$$x * y = \begin{cases} 0, & \text{if } x \leq y, \\ x, & \text{otherwise.} \end{cases}$$

Consequently this structure is positive implicative, as easily seen. Then we can not change $x(= x*y)$ into z greater than x. In this sense, the obtained BCK is **maximal,** namely if there is a new BCK structure on X, then $x * y$ can not greater than $x(= x * y)$ in the first given BCK structure.

To define another important class of BCK, we introduce a simple concept on a partially ordered set.

Definition 1. A partially ordered set P with a least element 0 is called the **type Y** if there exists an element $a(\neq 0)$ such that $a \leq x$ for every non-zero element x in P.

Let us denote by 1 such element a.

A very simple BCK is a fan shaped one which has two different kinds. We define two kinds of fans, namely Japanese style and Chinese style. One of them is of the type Y.

Definition 2. A partially ordered set P with a least element 0 is of a **Japanese (style) fan** if it is of type Y and for any $a(\neq 0, 1)$ in P, there are no elements x, y satisfying $1 < x < a, a < y$. P is called **Chinese (style) fan** if it is not a type V and for any $a(\neq 0)$ there are no elements x, y such that $0 < x < a, a < y$.

In the theory of BCK, the distinctions are quite important. It is very easy to introduce at least one BCK (maximal) structure by applying Theorem 1. There exists no other BCK(maximal) structure on any Chinese style fan and all subsets including 0 are subalgebras. On the other hand, any Japanese style fan has some other BCK structures.

Problem 1. Find all(finite and infinite) BCK with only one BCK structures. Are there none in the finite BCK with only one structure except the Chinese style one ? The answer may depend on the order.

Example 1. Let us consider the Japanese style fan of order 5 as an example. Then the *-table of the maximal BCK is given in the following left tables. In this case, there exist more three different BCK strcutures on it.

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	2	0	2	2
3	3	3	3	0	3
4	4	4	4	4	0

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	1	0	1	1
3	3	1	1	0	1
4	4	1	1	1	0

The above right structure is commutative. Other two structures are mentioned in the following tables, but it seems that no these structures have special kinds of properties.

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	1	0	1	1
3	3	3	3	0	3
4	4	4	4	4	0

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	1	0	1	0
3	3	1	1	0	1
4	4	4	4	4	0

All elements in a BCK X appear in the column containing 0 in the *-table of X. All elements which appear in the diagonal of the *-table are 0. Let us consider the part of the triangle of the left side of the diagonal, and eliminate the first column(the 0-column). For example, from the above first two *tables, we have the following parts:

2			1		
3	3		1	1	
4	4	4	1	1	1

Such a triangle is called the **essential part** of X. Similarly, we can define the quasiessential part of X. Let us consider the triangle formed through the right side of the diagonal. We elminate the first row(0-row) of the triangle. The triangle obtained is called the **quasiesential part** of X.

The quasiessential parts of the first two in Example 1 are as follows:

$$\begin{array}{ccc} 0 & 0 & 0 \\ & 2 & 2 \\ & & 3 \end{array} \qquad \begin{array}{ccc} 0 & 0 & 0 \\ & 1 & 1 \\ & & 1 \end{array}$$

For any BCK linearly ordered, all elements of its quasiessential part are 0, as easily seen.

Definition 3. A BCK is called **minimal**, if all elements of the essential and quasiessential parts are 0 or 1.

The above Japanese style fan has a minmal structure, but for a Chinese style fan, it is not always true. Then we have the following important

Theorem 2. *A BCK is the type Y if and only if it is minimal.*

Hence any linear(finite or infinite) BCK has a minimal structure. The minimal linear BCK of order 3 is commutative, but such a BCK of order 4 is not commutative.

The essential parts of the later two structures are

$$\begin{array}{ccc} 1 & & \\ 3 & 3 & \\ 4 & 4 & 4 \end{array} \qquad \begin{array}{ccc} 1 & & \\ 1 & 1 & \\ 4 & 4 & 4 \end{array}$$

The first element of these essential parts are 1. They are smaller than the first element 3 of the maximal structure, and greater than the first element 1 of the minimal structure. There are the same situations among the corresponding elements. The same fact holds for the quasiessential part. Therefore we can say that each structure is between the maximal and minimal structures(This is valid for a non minimal case). For every finite BCK, we can theoretically find all BCK structures, but the calculation is not easy. This is very tedious work.

Problem 2. Find an algorithm to determine all finite BCK structures. Is there a Turing machine to describe the structures of all finite BCK ?

Next we consider a factor(branch) of a partially ordered set P with a least element 0.

Definition 4. Each member of family $\{P_n\}$ of subsets of a BCK X is called **factor(branch)** if it satisfies the following conditions:

(F-1) Each P_n is a partially ordered set with 0,

(F-2) X is the union of $\{P_n\}$,

(F-3) $P_m \cap P_n = \{0\}$ for $m \neq n$.

Then each P_n is a BCK, and for $x \in P_m, y \in P_n (m \neq n)$ we have $x*y = x$.

Conversely, let $\{P_n\}$ be a family of partially ordered sets. Let us suppose that the least element 0 is common in all P_n, and $\{P_n\}$ is a disjoint family except the element 0. Moreover, we suppose each of $\{P_n\}$ has a BCK structure. Then the following result holds true.

Theorem 3. *On the union P of P_n, a BCK structure is uniquely introduced, and each P_n is subalgebra under the structure on P.*

For $x \in P_m$ and $y \in P_n(m \neq n)$,

$$x * y = x.$$

The BCK structure of each P_n is preserved in P.

For example, the union of several two elements BCK is a Chinese style fan and the BCK structure is given by the above equation. The uniqueness of the BCK structure on a Chinese style fan also follow from Theorem 3.

Moreover, X is positive implicative(commutative) if and if so is each P_m So, each P_m is implicative if and if P is implicative. Many interesting commutative BCK are constructed by the above method. But a commutative BCK is very strange in BCK theory. Its structure has not completely known until now.

Finally, as easily sen, there exist a maximal BCK, a minimal BCK, a positive implicative BCK, a commutative BCK, an imlicative BCK with any cardinality. I wold like to mention this fact is very useful to study various classes of BCK. In this Note, we considered the case of the union of BCK such that only 0 is identified. An important researche program is to develop general identification theory of BCK.

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Correction of my paper **On finite BCK with condition (S)** in Proc. 4th Symposium on Algebraic Langueges and Computation(2000). 1. p.16. 15 $x * (y * x)$ should be read as $x * (y * z)$.

2. p.18. In the right table in 7, $x_{21} = 2$ should be read $x_{21} = 3$.

3. p.18. In the right table in 10, $x_{11} = 2$ and $x_{12} = 3$ should be respectively read as $x_{11} = 1$ and $x_{12} = 2$.

4. p.18. In the right table in 6, $x_{11} = 2$ should be read as $x_{11} = 1$.

5. p.18. in the right table in 13, $x_{22} = 3$ should be read as $x_{22} = 2$