\textit{CPN Languages and Codes}

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Let $D = (P, X, \delta, \mu_0)$ be a Petri net with a initial marking $\mu_0$ where $P$ is the set of places, $X$ is the set of transitions, $\delta$ is the transition function and $\mu_0 \in N_+^P$ is a positive marking, i.e. $\pi_p(\mu_0) > 0$ for any $p \in P$. Notice that $\pi_p(\mu_0)$ is meant the number of tokens at $p$ of the marking $\mu_0$. A language $C$ is called a CPN language over $X$ generated by $D$ and denoted by $C = \mathcal{L}(D)$ if $C = \{u \in X^+ | \exists p \in P, \pi_p(\delta(\mu_0, u)) = 0, \forall q \in P, \pi_q(\delta(\mu_0, u)) \geq 0, \text{ and } \forall q' \in P, \pi_{q'}(\delta(\mu_0, u')) > 0 \text{ for } u' \in P_r(u) \setminus \{u\}\}$ where $P_r(u)$ is the set of all prefixes of $u$. Then it is obvious that $C = \mathcal{L}(D)$ is a prefix code over $X$. If $C$ is a maximal prefix code over $X$, then $C$ is called an mCPN language over $X$.

\textbf{Theorem 1} Let $A, B \subseteq X^+$ be finite maximal prefix codes over $X$. If $AB$ is an mCPN language over $X$, then $A, B$ are full uniform codes over $X$.

\textbf{Remark 1} In the above theorem, the condition for $A$ and $B$ to be finite is necessary. For instance, let $X = \{a, b\}$ and let $A = B = b^*a$. Then $AB = b^*ab^*a$ is an mCPN language over $X$ but neither $A$ nor $B$ is a full uniform code over $X$.

Now we consider some constructions of mCPN languages.

\textbf{Definition 1} Let $A, B \subseteq X^+$. Then by $A \oplus B$ we denote the language $(\cup_{b \in X} \{(P_r(A) \setminus A) \odot B b^{-1}\} b) \cup (\cup_{a \in X} \{(P_r(B) \setminus B) \odot A a^{-1}\} a)$ where $\odot$ is meant the shuffle operation.

\textbf{Proposition 1} Let $X = Y \cup Z$ where $Y, Z \neq \emptyset, Y \cap Z = \emptyset$. If $A \subseteq Y^+$ is an mCPN language over $Y$ and $B \subseteq Z^+$ is an mCPN language over $Z$, then $A \oplus B$ is an mCPN language over $X$. 
Example 1 Let $X = \{a, b\}$. Consider $A = \{a\}$ and $B = \{bb\}$. Then both $A$ and $B$ are $mCPN$ languages over $\{a\}$ and $\{b\}$, respectively. Hence $A \oplus B = \{a, ba, bb\}$ is an $mCPN$ language over $X$.

Proposition 2 Let $A, B \subseteq X^+$ be finite $mCPN$ languages over $X$. Then $A \oplus B$ is an $mCPN$ language over $X$ if and only if $A = B = X$.

Remark 2 For the class of infinite $mCPN$ languages over $X$, the situation is different. For instance, let $X = \{a, b\}$ and let $A = B = b^*a$. Then $A \oplus B = b^*a$ and $A, B$ and $A \oplus B$ are $mCPN$ languages over $X$.

Proposition 3 Let $A, B \subseteq X^+$ be $mCPN$ languages over $X$. Then there exist an alphabet $Y$, $D \subseteq Y^+$: an $mCPN$ language over $Y$ and a homomorphism $h$ of $Y^*$ onto $X^*$ such that $A \oplus B = h(D)$.

Definition 2 Let $A \subseteq X^+$. By $m(A)$, we denote the language $\{v \in A | \forall u, v \in A, \forall x \in X^*, v = ux \Rightarrow x = 1\}$. Obviously, $m(A)$ is a prefix code over $X$. Let $A, B \subseteq X^+$. By $A \otimes B$, we denote the language $m(A \cup B)$.

Proposition 4 Let $A, B$ be $mCPN$ languages over $X$. Then, $A \otimes B$ is an $mCPN$ language over $X$.

Example 2 It is obvious that $a^*b$ and $(a \cup b)^3$ are $mCPN$ languages over $\{a, b\}$. Hence $a^*b \otimes (a \cup b)^3 = \{b, ab, aaa, aab\}$ is an $mCPN$ language over $\{a, b\}$.

Remark 3 Proposition 4 does not hold for the classe of $CPN$ languages over $X$. The reason is the following: Suppose that $A \otimes B$ is a $CPN$ language over $X$ for any two $CPN$ languages $A$ and $B$ over $X$. Then we can show that, for a given finite $CPN$ language $A$ over $X$, there exists a finite $mCPN$ language $B$ over $X$ such that $A \subseteq B$ as follows. Let $A \subseteq X^+$ be a finite $CPN$ language over $X$ which is not an $mCPN$ language. Let $n = \max\{|u||u \in A\}$. Consider $X^n$ which is an $mCPN$ language over $X$. By assumption, $A \otimes X^n$ becomes a $CPN$ language (in fact, an $mCPN$ language) over $X$. By the definition of the operation $\otimes$, it can be also proved that $A \subseteq A \otimes X^n$. Notice that there exists a finite $CPN$ language $A$ over $X$ such that there exists no $mCPN$ language $B$ over $X$ with $A \subseteq B$. Hence, Proposition 4 does not hold for the class of all $CPN$ languages over $X$. 
Remark 4 The set of all mCPN languages over $X$ forms a semigroup under $\otimes$. Moreover, the operation $\otimes$ has the following properties:

1. $A \otimes B = B \otimes A$
2. $A \otimes A = A$
3. $A \otimes X = X$

Consequently, the set of all mCPN languages over $X$ forms a commutative band with zero under $\otimes$.

Definition 3 Let $A \subseteq X^+$ be a CPN language over $X$. By $r(A)$ we denote the value $\min\{|P| | D = (P, X, \delta, \mu_0), \mathcal{L}(D) = A\}$.

Remark 5 Let $A \subseteq X^+$ be a finite CPN language over $X$. Then $r(A) \leq |A|$. Moreover, let $A, B \subseteq X^+$ be mCPN languages over $X$. Then $r(A \otimes B) \leq r(A) + r(B)$. In the above, if $|A|, |B|$ are finite, then $|A \otimes B| \leq \max(|A|, |B|)$.

We define three language classes as follows: $\mathcal{L}_{CPN} = \{A \subseteq X^+ | A$ is a CPN language over $X\}$, $\mathcal{L}_{mCPN} = \{A \subseteq X^+ | A$ is an mCPN language over $X\}$, $\mathcal{L}_{NmCPN} = \{A \subseteq X^+ | A: an mCPN language over X, \exists D = (P, X, \delta, \mu_0), \forall p \in P, \forall a \in X, \#(parrow a) \leq 1, \mathcal{L}(D) = A\}$. Then it is obvious that we have the following inclusion relations: $\mathcal{L}_{NmCPN} \subseteq \mathcal{L}_{mCPN} \subseteq \mathcal{L}_{CPN}$. It is also obvious that $\mathcal{L}_{mCPN} \neq \mathcal{L}_{NmCPN}$.

Problem 1 $\mathcal{L}_{mCPN} \neq \mathcal{L}_{NmCPN}$?

Proposition 5 Let $A \in \mathcal{L}_{CPN}$ and let $r(A) = k$. Then there exist $A_1, A_2, \ldots, A_k \in \mathcal{L}_{CPN}$ such that $r(A_i) = 1, i = 1, 2, \ldots, k$ and $A = A_1 \otimes A_2 \otimes \ldots \otimes A_k$. Moreover, in the above, if $A \in \mathcal{L}_{NmCPN}$, then $A_1, A_2, \ldots, A_k$ are in $\mathcal{L}_{NmCPN}$ and context-free.

For mCPN languages with rank 1, we have the following:

Proposition 6 Let $A \subseteq X^+$ be a finite mCPN language with $r(A) = 1$ over $X$. Then $A$ is a full uniform code over $X$.

Proposition 7 Let $A \subseteq X^+$ be an mCPN language with $r(A) = 1$ over $X$ and let $k$ be a positive integer. Then $A^k$ is an mCPN language with $r(A^k) = 1$ over $X$.

Proposition 8 Let $A \in \mathcal{L}_{NmCPN}$ and let $r(A) = k$. Then there exist $A_1, A_2, \ldots, A_k \in \mathcal{L}_{NmCPN}$ such that $r(A_i) = 1, i = 1, 2, \ldots, k$ and $A = \ldots \otimes A_k$.\)
$A_1 \otimes A_2 \otimes \ldots \otimes A_k$. Let $n_1, n_2, \ldots, n_k$ be positive integers. Then $A_1^{n_1} \otimes A_2^{n_2} \otimes \ldots, \otimes A_k^{n_k} \in \mathcal{L}_{NmCPN}$.

Finally, we can prove the following main theorem by two different ways, i.e. the first one is an indirect proof and the second one is a direct proof.

**Theorem 2** Let $C \subseteq X^+$ be a CPN language over $X$. Then $C$ is a context-sensitive language over $X$. 