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Bounded cohomology of subgroups of mapping class groups

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I give a talk on a joint work with Mladen Bestvina [3].

When G is a discrete group, a *quasi-homomorphism* on G is a function $h: G \to \mathbf{R}$ such that

$$\Delta(h) := \sup_{\gamma_1, \gamma_2 \in G} |h(\gamma_1\gamma_2) - h(\gamma_1) - h(\gamma_2)| < \infty.$$

The number $\Delta(h)$ is the *defect* of h. We denote by $QH(G)$ the vector space of all quasi-homomorphisms $G \to \mathbf{R}$ modulo the subspace of bounded functions, and by $\widetilde{QH}(G)$ the vector space of all quasi-homomorphisms $G \to \mathbf{R}$ modulo the subspace of functions within uniform distance to a homomorphism.

Let S be a compact orientable surface of genus g and p punctures. We consider the associated mapping class group $Mod(S)$ of S. This group acts on the *curve complex* X of S defined by Harvey [7] and successfully used in the study of mapping class groups by Harer [6], [5]. For our purposes, we will restrict to the 1-skeleton of Harvey's complex, so that X is a graph whose vertices are isotopy classes of essential, nonparallel, nonperipheral, pairwise disjoint simple closed curves in S (also called *curve systems*) and two distinct vertices are joined by an edge if the corresponding curve systems can be realized simultaneously by pairwise disjoint curves. In certain sporadic cases X as defined above is 0-dimensional (this happens when there are no curve systems consisting of two curves, i.e. when $g = 0$, $p \leq 4$ and when $g = 1$, $p \leq 1$). In the theorem below these cases are excluded. The mapping class group $Mod(S)$ acts on X by $f \cdot a = f(a)$.

H. Masur and Y. Minsky proved the following remarkable result.

Theorem 1 *[9] The curve complex X is δ-hyperbolic. An element of $Mod(S)$ acts hyperbolically on X if and only if it is pseudo-Anosov .*

Using their result, we show the following theorem. H. Endo and D. Kotschick [2] have shown using 4-manifold topology and Seiberg-Witten invariants that $\widetilde{QH}(Mod(S)) \neq 0$ when S is hyperbolic.

Theorem 2 *[3] Let G be a subgroup of $Mod(S)$ which is not virtually abelian. Then* $\dim \widetilde{QH}(G) = \infty$.

The following is a version of superrigidity for mapping class groups. It was conjectured by N.V. Ivanov and proved by Kaimanovich and Masur [8] in the case when the image group contains independent pseudo-Anosov homeomorphisms and it was extended to the general case by Farb and Masur [4] using the classification of subgroups of $Mod(S)$ as above. Our proof is different in that it does not use random walks on mapping class groups, but instead uses the work of M. Burger and N. Monod [1] on bounded cohomology of lattices.

Corollary 3 *[3] Let Γ be an irreducible lattice in a connected semi-simple Lie group G with no compact factors and of rank > 1. Then every homomorphism $\Gamma \to Mod(S)$ has finite image.*

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