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Regularized long-wave expansion: a kind of summation method developed for a free-surface problem of a thin liquid layer falling down a solid wall

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1 Introduction

We discuss free-surface flow motion of viscous liquid falling down a solid wall, referred to as falling film flows. This problem has a long history, starting from the study of Nusselt [1] and the pioneering experiments by Kapitza [2]; see the introduction of the paper by Salamon *et al.* [3], as well as the review by Chang [4].

The dynamics of falling liquid films is fairly well described as a free-surface problem of two-dimensional, single-phase Navier-Stokes equation. Of course, this problem with Navier-Stokes equation is quite difficult to treat directly, and therefore a great effort has been made to simplify the problem. By utilizing a fact that the film thickness h is much smaller than the surface wave length, the depthwise freedom of motion is eliminated, so that we can obtain a simplified equation; this is the essential idea of the long-wave expansion initiated by Benney [5]. However, since Benney's long-wave expansion is poor at convergence and therefore sometimes it fails to work very well, we need to replace it by what we call *regularized long-wave expansion method* or *regularization method* [6]. The essential idea consists in the treatment of "poorly convergent" power series solution obtained by the traditional long-wave expansion: this may be understood as kind of summation method, similar to Padé approximation, in regard to differential operators. The surface equation obtained by this method,

$$\partial_t h - \frac{4}{21}R\partial_x\partial_t(h^5) - \partial_x\left(h^2\partial_x\partial_t h\right) + \frac{2}{3}\partial_x\left[h^3 - \partial_x\left(\frac{\cot\alpha}{4}h^4 + \frac{72}{245}Rh^7\right) + Wh^3\partial_x^3 h\right] = 0. \qquad (1)$$

is referred to as *regularized equation* of film flows.

Among recent studies on the problem of simplifying the equation of falling film flows, the improved depth-averaging method by Ruyer-Quil and Manneville [7] is outstandingly remarkable, as well as the center-manifold reduction by Roberts [8]. One of the main differences between their approaches and that of the present study is that they develop two-mode or three-mode equation, while we insist on a "one-mode" equation; Eq. (1) includes only one dependent variable h, and no higher-order derivatives in t (such as $\partial_t^2 h$, $\partial_t^3 h$ etc.) are present. Instead, cross-differential terms $\partial_x^n\partial_t h$ are allowed in Eq. (1).

In Sec. 2 we formulate the problem, describing the basic equation together with the boundary condition. Benney's long-wave expansion method is reviewed in Sec. 3; Gjevik's equation (16) is obtained as a result, but it mispredicts the bifurcation of permanent solutions. To save the long-wave expansion from that failure, we propose the regularization method. A basic idea is shown in Sec. 4 in terms of a model equation. This method is applied to falling film flows in Sec. 5, so that Eq. (1) is obtained. In Sec. 6 we discuss its relation to Benney's long-wave expansion, zero-mode interaction, Whitham's wave hierarchy, and Kuramoto-Sivashinsky equation.

2 Formulation of the problem

2.1 Basic Equations

The wall is sloped by an angle α; it is vertical if $\alpha = \pi/2$. With the x-axis taken downward along the wall and the z-axis perpendicular to it, the components of the velocity field $\mathbf{u}$ and the gravitational acceleration $\mathbf{g}$ are denoted as

$$\mathbf{u} = \begin{bmatrix} u \\ w \end{bmatrix}, \qquad \mathbf{g} = \begin{bmatrix} g_x \\ -g_z \end{bmatrix} = g \begin{bmatrix} \sin\alpha \\ -\cos\alpha \end{bmatrix}.$$

We assume that the liquid, filling the volume between the wall $z = 0$ and the film surface $z = h(x,t)$, is governed by the 2-dimensional Navier-Stokes equation, which consists of the continuity equation,

$$\operatorname{div}\mathbf{u} = 0, \qquad \rho = \text{const.}, \tag{2a}$$

the momentum equation,

$$\partial_t(\rho\mathbf{u}) + \operatorname{div}(\rho\mathbf{u}\mathbf{u} - \overleftrightarrow{\tau}) = \rho\mathbf{g}, \tag{2b}$$

and the constitutive equation prescribing Newtonian viscosity,

$$\overleftrightarrow{\tau} = -p\overleftrightarrow{1} + 2\rho\nu\,\operatorname{sym}\operatorname{grad}\mathbf{u}. \tag{2c}$$

Eqs. (2) are combined with two boundary conditions at $z = h(x,t)$, namely the dynamical and kinematic conditions at the free surface, as well as with the no-slip condition at the wall,

$$\mathbf{u}|_{z=0} = \mathbf{0}. \tag{3}$$

In terms of the surface normal vector $\mathbf{n}$ defined by

$$\mathbf{n} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}, \qquad \theta = \tan^{-1}\partial_x h,$$

the dynamical boundary condition is described as

$$\overleftrightarrow{\tau} \cdot \mathbf{n}\big|_{z=h} = \tau_* \, \mathbf{n}, \tag{4a}$$

$$\tau_* = -p_{\mathrm{atm}} + \frac{T\partial_x^2 h}{[1 + (\partial_x h)^2]^{3/2}}, \tag{4b}$$

with p_{atm} and T being the atmospheric pressure and the surface tension coefficient, respectively. The kinematic condition concerns the surface motion; with D_t being the Lagrange derivative, this condition is prescribed as

$$D_t(h - z)\big|_{z=h} = \partial_t h + u\big|_{z=h}\partial_x h - w\big|_{z=h} = 0. \tag{4c}$$

It is easily verified that Eq. (4c) is equivalent to the mass conservation equation

$$\partial_t h + \partial_x Q = 0, \tag{5a}$$

where Q is the volume flux, defined by

$$Q \overset{\mathrm{def}}{=} \int_0^h u \, dz. \tag{5b}$$

2.2 Nusselt solution and dimensionless parameters

The solution describing the unperturbed flat-film flow ($h = h_0 = \text{const.}$),

$$\mathbf{u} = \begin{bmatrix} U_{\mathrm{N}}(2\zeta - \zeta^2) \\ 0 \end{bmatrix}, \qquad \zeta = \frac{z}{h_0}, \tag{6a}$$

$$U_{\mathrm{N}} = \frac{g_x h_0^2}{2\nu} = \frac{(g \sin\alpha) h_0^2}{2\nu}, \tag{6b}$$

is referred to as the Nusselt solution [1]. The Nusselt velocity U_{N}, defined by Eq. (6b), stands for the velocity at the surface of the unperturbed film. The viscous stress at the wall for the Nusselt solution is given by $\tau_{\mathrm{N}} = \tau_{xz}\big|_{z=0} = \rho g_x h_0$.

Suppose that now the film is not flat but still we can specify its representative thickness h_0 in some way. Dimensional analysis shows that the system is characterized by two nondimensional parameters besides the inclination angle α. One of the most common styles is to introduce the Reynolds number R and the Weber number W, defined by

$$R \overset{\mathrm{def}}{=} \frac{U_{\mathrm{N}} h_0}{\nu} = \frac{(g \sin\alpha) {h_0}^3}{2\nu^2}, \tag{7a}$$

$$W \overset{\mathrm{def}}{=} \frac{T h_0^{-1}}{\tau_{\mathrm{N}}} = \frac{T}{(\rho g \sin\alpha) {h_0}^2}; \tag{7b}$$

they are indicators of the inertia and of the surface tension, respectively.

The basic equations and the boundary conditions, i.e. Eqs. (2)–(4), are now nondimensionalized. After some rearrangements, they are rewritten as follows:

$$\partial_z^2 u = RD_t u + 2\partial_x p - 2 - \partial_x^2 u, \tag{8a}$$

$$\partial_z p = -\frac{1}{2} RD_t w - \cot\alpha + \frac{1}{2}\left(\partial_x^2 + \partial_z^2\right) w, \tag{8b}$$

$$\partial_z w = -\partial_x u, \tag{8c}$$

$$u\big|_{z=0} = w\big|_{z=0} = 0, \tag{8d}$$

$$\partial_z u\big|_{z=h} = -\partial_x w\big|_{z=h} + 4\partial_x u\big|_{z=h}\partial_x h + \left[\partial_z u + \partial_x w\right]\big|_{z=h}(\partial_x h)^2, \tag{8e}$$

$$p\big|_{z=h} = -\left[1 + (\partial_x h)^2\right]\partial_x u\big|_{z=h} + p\big|_{z=h}(\partial_x h)^2 - W\Sigma, \tag{8f}$$

$$\partial_t h = -\partial_x Q, \tag{8g}$$

where Σ is the surface tension term defined by

$$\Sigma = \frac{\left[1 - (\partial_x h)^2\right]\partial_x^2 h}{\left[1 + (\partial_x h)^2\right]^{3/2}},$$

and Q is the volume flux defined in Eq. (5b).

Instead of the Weber number W, some researchers prefer to use the Kapitza number K, which is essentially a "Weber number without h_0" (defined by material properties alone) and is identical to $WR^{2/3}$ except for a numerical factor. For water films at room temperature on a vertical wall we have $WR^{2/3} \approx 3000$.

3 Long-wave expansion

3.1 Linear analyses and the long-wave parameter

Benjamin [9] and Yih [10] studied the linear stability of uniform film flows described by the the Nusselt solution (6). According to their analyses, the growth rate of long-wave disturbances (in the time-evolutional picture) is given by

$$c_i k = \frac{8}{15}\left(R - R_c\right)k^2 - \left(\frac{2}{3}Wk^3 + O(R, R^3)\right)k^4 + O(k^6) \tag{9}$$

for long-wave disturbances. Eq. (9), which asserts that the modes with positive growth rate ($c_i > 0$) are limited to a narrow range $0 < k < k_0$, provides with a basis for the long-wave expansion [11, 12, 13]: the governing equations can be expanded by the long-wave parameter μ, whose smallness comes from the smallness of the neutral wave number k_0. If W is large, Eq. (9) allows us to estimate μ to be

$$\mu \sim k_0 = \sqrt{\frac{4}{5}\left(\frac{R - R_c}{W}\right)} \sim \sqrt{\frac{R}{W}}; \tag{10}$$

this assures $\mu \ll 1$ if $W \gg R$, i.e. if the surface tension is large. Most liquids, including water and alcohol, have so large surface tension that $\mu \propto W^{-1/2} \ll 1$ safely

3.2 Benney's long-wave expansion

After Benney [5] we expand Eqs. (8) by the long-wave parameter μ, introduced by

$$\mu \sim \partial_x \stackrel{\text{def}}{=} \mu \partial_{x_1}. \tag{11}$$

It is also necessary to introduce multiple-scale expansion,

$$\partial_t = \mu \partial_{t_1} + \mu^2 \partial_{t_2} + \cdots ,$$

while ∂_z is left unexpanded. The long-wave expansion allows to eliminate $\mathbf{u}$ from the governing equations and leads to a reduced evolution equation which includes the surface profile $h = h(x,t)$ as the sole dependent variable.

During the the long-wave expansion, *provisionally* we assume

$$R \sim |R - R_c| \sim O(1), \tag{12}$$

so that the proper ordering for W is

$$W = \mu^{-2}\tilde{W} \tag{13}$$

where $\tilde{W} \sim O(1)$. Thus we perform the same long-wave expansion as was performed by Gjevik [14] and by Lin [15]. As a result, we obtain a power-series expression for the flux $Q = Q[h]$, namely

$$Q = \int_0^h u\,dz = Q_0[h] + \mu Q_1[h] + \mu^2 Q_2[h] + \cdots , \tag{14a}$$

where

$$Q_0 = \int_0^h u_0 dz = \frac{2}{3}h^3, \tag{14b}$$

$$Q_1 = \int_0^h u_1 dz = \frac{8}{15}Rh^6\partial_{x_1}h - \frac{2}{3}(\cot\alpha)h^3\partial_{x_1}h + \frac{2}{3}\tilde{W}h^3\partial_{x_1}^3 h, \tag{14c}$$

$$\begin{aligned} Q_2 = R^2 &\left[\frac{1016}{315}h^9(\partial_{x_1}h)^2 + \frac{32}{63}h^{10}\partial_{x_1}^2 h\right] \\ &- (R\cot\alpha)\left[\frac{32}{15}h^6(\partial_{x_1}h)^2 + \frac{40}{63}h^7\partial_{x_1}^2 h\right] + \frac{14}{3}h^3(\partial_{x_1}h)^2 + 2h^4\partial_{x_1}^2 h \\ &+ \tilde{W}R\left[\frac{40}{63}h^7\partial_{x_1}^4 h + \frac{16}{3}h^6(\partial_{x_1}h)\partial_{x_1}^3 h + \frac{16}{5}h^6(\partial_{x_1}^2 h)^2 + \frac{32}{5}h^5(\partial_{x_1}h)^2\partial_{x_1}^2 h\right]. \end{aligned} \tag{14d}$$

It is easy to proceed to higher order with the aid of symbolic manipulations such as MATHEMATICA, as Eqs. (8) are in a form allowing recursive substitution.

The results of the long-wave expansion should be combined with the mass conservation, i.e. Eq. (5a) or (8g), so that the reduced equation has the form

$$\partial_t h + \partial_x \left\{Q_0[h] + \mu Q_1[h] + \mu^2 Q_2[h] + \cdots\right\} = 0. \tag{15}$$

Note that we can rewrite Eq. (15) in such form that does not include μ explicitly, by remembering $\mu\partial_{x_1} = \partial_x$ and $\mu^{-2}\tilde{W} = W$.

The traditional approach is to truncate the infinite-order equation (15) at a certain order. In particular, taking the results up to Q_1 leads to the long-wave equation of Gjevik [11]:

$$\partial_t h + \frac{2}{3}\partial_x \left[h^3 + \left(\frac{4}{5}Rh^6 - h^3 \cot\alpha \right) \partial_x h + Wh^3 \partial_x^3 h \right] = 0. \qquad (16)$$

3.3 Failure of Benney's long-wave expansion

Pumir *et al.* [13] numerically studied Gjevik's long-wave equation (16). By assuming a permanent solution (i.e. a solution which is steady in a moving frame), they obtained its solitary-wave solutions, which were found to resemble the waves experimentally observed by Kapitza [2]. It was also shown that such solitary waves can be realized by time-evolutional calculation of Eq. (16), provided that the Reynolds number R is less than some limiting value R_*. According to the bifurcation diagram shown as Fig. 5 in their paper [13], permanent solitary waves can be realized only for a limited range of the Reynolds number, $R_c < R < R_*$. In time-evolutional problems for $R > R_*$, a self-focusing of a solitary wave was observed, leading to divergence of the wave amplitude in a finite time.

The question is how this singular behavior is related to the real film flows: this singularity may reflect some unusual phenomena which really occur in film flows, or it may have nothing to do with the reality, being merely a failure of the long-wave equations. This question was answered by Salamon *et al.* [3]. They directly applied the finite-element method to the Navier-Stokes equation (2) to obtain its permanent wave solutions, and compared them with the corresponding results of the long-wave expansion. Although their solutions agreed with the results of long-wave equations when R is close to R_c, their solutions exhibited no limiting point such as R_*. They also found that including more terms (up to Q_2) does not save the truncated long-wave equation from the failure. It was concluded that there is a serious limitation in the range of validity of the long-wave equations.

4 Basic idea of regularized long-wave expansion

4.1 Model equation as a mathematical example

Though Benney's long-wave expansion is expected to converge for $W \to +\infty$ with R kept finite, it is quite doubtful whether it converges for finite W and finite R. The failure of Eq. (16), as well as its higher-order version, seems to suggest that the long-wave expansion is poorly convergent.

As a model to demonstrate how to deal with such kind of "poor convergence", let us consider a (formally) infinite-order partial differential equation

$$\partial_t h + \partial_x h + \partial_x^2 h + \partial_x^3 h + \partial_x^4 h + \cdots + \partial_x^n h + \cdots = 0, \qquad (17)$$

together with the initial condition

$$h|_{t=0} = f(x). \tag{18}$$

Since ∂_x is not a bounded operator, the left-hand side of Eq. (17) diverges and therefore the initial-value problem, Eqs. (17)–(18), seems hardly meaningful. In fact, truncation of Eq. (17) at finite n (≥ 2) yields a parabolic equation with n-th order derivative and can lead to strange results, such as ill-posedness of the initial-value problem, because the growth rate for the Fourier component with wave-number $k \to \infty$ behaves like k^n.

It is possible, however, to rewrite Eq. (17) into a more tractable form. The idea is to let ∂_x operate upon Eq. (17) and subtract the result from the original equation:

$$\begin{array}{rlr} & \partial_t h + \partial_x h + \partial_x^2 h + \partial_x^3 h + \cdots + \partial_x^n h + \cdots & = 0 \\ -) & \partial_t \partial_x h \qquad\quad + \partial_x^2 h + \partial_x^3 h + \cdots + \partial_x^n h + \cdots & = 0 \\ \hline & (1 - \partial_x)\partial_t h + \partial_x h & = 0. \end{array} \tag{19}$$

Instead of the infinite-order equation (17), now we have Eq. (19) which contains only three terms!

By substituting into Eq. (19) an elementary solution of the form

$$h \propto e^{ikx+\sigma t},$$

we find the complex dispersive relation

$$\sigma = \frac{-ik}{1-ik} = \frac{k^2}{1+k^2} - \frac{ik}{1+k^2}. \tag{20}$$

Note that

$$|\sigma| \leq 1 \qquad \text{for } \forall k; \tag{21}$$

this means that the initial value problem of Eq. (19) is well-posed in the sense that $\mathrm{Re}\,\sigma$ is bounded for $k \to \infty$ [16, p.84], in contrast with some of the truncated equation obtained from Eq. (17). We also note, defining $\omega = -\mathrm{Im}\,\sigma$, that the phase velocity ω/k is always positive, while the group velocity $d\omega/dk$ is negative for $1 < k < +\infty$.

If we introduce H by $h = e^{x+t}H$, then Eq. (19) is equivalent to

$$\partial_x \partial_t H = H. \tag{22}$$

This is Klein-Gordon equation (see Appendix A), though not in the form that is familiar to the physicists. The characteristics of Eq. (22) are given by two sets of lines, namely $x =$ const. and $t =$ const.; they are also the characteristics of Eq. (19).

Considering that Eq. (19) as well as Eq. (22) is hyperbolic, we emphasize that the initial value problem treated here is of quite unusual type. The initial data (18) is given on the line $t = 0$ in the space-time, which actually coincides with one of the characteristics. This is quite different from the usual way of providing a hyperbolic equation with initial data; usually we assume that the curve on which initial data is given does *not* coincide with any characteristics of the equation. We may understand this unfamiliar setting of problem by remarking that the hyperbolic equation (19) actually comes from the summation of the infinitely high-order parabolic equation (17), for which, in fact, the initial data (18) seems quite natural.

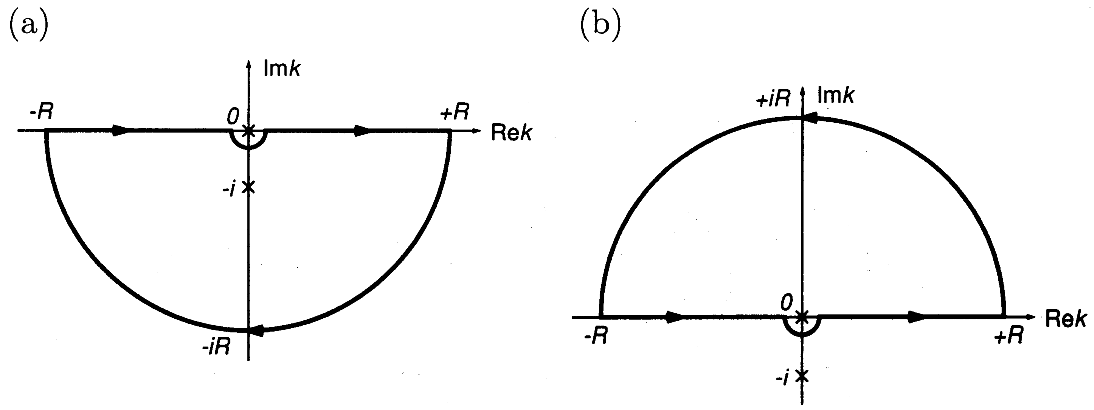

Figure 1: Integral paths for evaluation of Eq. (24): (a) $x > 0$, (b) $x < 0$.

4.2 Solution to the model equation

Now let us solve the initial-value problem of Eq. (19). Without loss of generality, we may assume $h|_{t=0} = \theta(x)$ instead of Eq. (18), where $\theta(x)$ is the step function. It is useful to express this initial condition as

$$h|_{t=0} = \theta(x) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{dk}{k - i\epsilon} e^{ikx} \qquad (\epsilon \to +0) \tag{23}$$

in terms of Fourier transform [17]. Then, taking Eq. (20) into account, we have

$$\begin{aligned} h &= \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{dk}{k - i\epsilon} e^{ikx+\sigma t} \\ &= \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{dk}{k - i\epsilon} \exp\left[(ik-1)x + \frac{t}{ik-1}\right]. \end{aligned} \tag{24}$$

For $x > 0$, the integral path can be closed in the upper half plane, as in Fig. 1(a), and then deformed into a small circle around $k = 0$. Then we find

$$h = \frac{e^{x+t}}{2\pi i}\oint \frac{dk}{k} \exp\left[(ik-1)x + \frac{t}{ik-1}\right] = e^{x+t}e^{-x-t} = 1, \tag{25}$$

i.e. h remains unchanged in the domain $x > 0$.

For $x < 0$, on the other hand, the integral path is closed in the lower half plane, as shown in Fig. 1(b), so that we evaluate the integral (24) on the anticlockwise path around $k = -i$. It is convenient to change the variable from k into

$$\zeta \stackrel{\text{def}}{=} \sqrt{\frac{-x}{t}}(-ik+1),$$

$$
\begin{aligned}
h &= -\frac{e^{x+t}}{2\pi i}\oint \frac{-\sqrt{\frac{t}{-x}}}{1-\sqrt{\frac{t}{-x}}\zeta}\exp\left[\sqrt{-xt}\,(\zeta-\zeta^{-1})\right] \\
&= \frac{e^{x+t}}{2\pi i}\oint d\zeta\left\{\sum_{n=0}^{\infty}\left(\frac{t}{-x}\right)^{\frac{n+1}{2}}\zeta^{n}\right\}\sum_{m=-\infty}^{+\infty} J_m\left(2\sqrt{-xt}\right)\zeta^{m} \\
&= e^{x+t}\sum_{n=0}^{\infty}\left(\frac{t}{-x}\right)^{\frac{n+1}{2}} J_{-n-1}\left(2\sqrt{-xt}\right) \\
&= e^{x+t}\sum_{n=1}^{\infty}\left(\frac{t}{-x}\right)^{n/2}(-)^{n} J_n\left(2\sqrt{-xt}\right). \qquad (26)
\end{aligned}
$$

The series in Eq. (26) contains $(t/|x|)^{n/2}$ and therefore rapidly converges for $-\infty < x \ll -t < 0$, i.e. near the axis $t = 0$ and far from the axis $x = 0$. For the opposite cases, i.e. $-t \ll x < 0$, we utilize the summation formula

$$
\sum_{n=-\infty}^{+\infty}\left(\frac{-x}{t}\right)^{n/2} J_n\left(2\sqrt{-xt}\right) = \exp\left[\sqrt{-xt}\left(\sqrt{\frac{-x}{t}}-\sqrt{\frac{t}{-x}}\right)\right] = e^{-x-t}
$$

to rewrite Eq. (26) as

$$
\begin{aligned}
h &= e^{x+t}\sum_{n=1}^{\infty}\left(\frac{t}{-x}\right)^{n/2} J_{-n}\left(2\sqrt{-xt}\right) \\
&= e^{x+t}\left[e^{-x-t}-\sum_{n=0}^{\infty}\left(\frac{-x}{t}\right)^{n/2} J_n\left(2\sqrt{-xt}\right)\right] \\
&= 1-e^{x+t}\sum_{n=0}^{\infty}\left(\frac{-x}{t}\right)^{n/2} J_n\left(2\sqrt{-xt}\right). \qquad (27)
\end{aligned}
$$

In conclusion, the solution to Eq. (19) satisfying the initial condition (23) is given by

$$
h = h_0(x,t) = \begin{cases} 1 & (x>0) \\ \bar{h}_0(-x,t) & (x<0) \end{cases} \qquad (28a)
$$

where

$$
\begin{aligned}
\bar{h}_0(\xi,t) &= 1-e^{-\xi+t}\sum_{n=0}^{\infty}\left(\frac{\xi}{t}\right)^{n/2} J_n\left(2\sqrt{\xi t}\right) \\
&= e^{-\xi+t}\sum_{n=1}^{\infty}\left(\frac{t}{\xi}\right)^{n/2}(-)^{n} J_n\left(2\sqrt{\xi t}\right). \qquad (28b)
\end{aligned}
$$

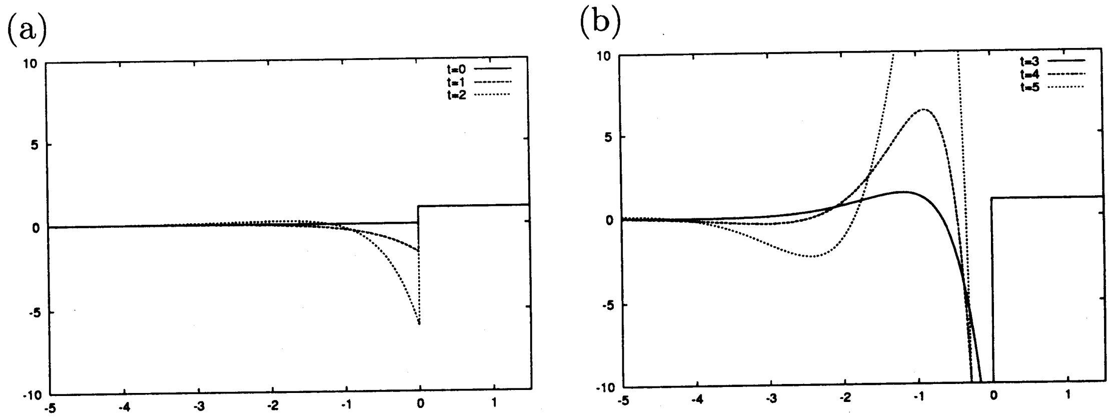

Figure 2: Numerical solution to Eq. (19)

Eq. (28) is understood also as a solution to the infinite-order parabolic equation Eq. (17). A solution for the general initial condition (18) could be obtained by convolution of f and $\partial_x h_0$.

To check the analytical solution (28), we solve Eq. (19) also numerically. A second-order finite difference scheme is very easily obtained by substituting into Eq. (19) the following central-difference formulae:

$$\partial_x h^{(*)} = \frac{1}{2}\left(\frac{h_{j+1}^{(0)} - h_j^{(0)}}{\Delta x} + \frac{h_{j+1}^{(+)} - h_j^{(+)}}{\Delta x}\right), \tag{29a}$$

$$\partial_t h^{(*)} = \frac{1}{2}\left(\frac{h_{j+1}^{(+)} - h_{j+1}^{(0)}}{\Delta t} + \frac{h_j^{(+)} - h_j^{(0)}}{\Delta t}\right), \tag{29b}$$

$$\partial_t \partial_x h^{(*)} = \frac{h_j^{(0)} + h_{j+1}^{(+)} - h_j^{(+)} - h_{j+1}^{(0)}}{\Delta t \Delta x}, \tag{29c}$$

where

$$x_j \stackrel{\text{def}}{=} x_0 + j\Delta x, \qquad h_j^{(0)} \stackrel{\text{def}}{=} h(t,\, x_j), \quad h_j^{(+)} \stackrel{\text{def}}{=} h(t+\Delta t,\, x_j),$$

and the superscript $(*)$ denotes evaluation at $(t + \Delta t/2,\, x_j + \Delta x/2)$.

Some of the profiles of the numerically calculated solution are depicted in Fig. 2. The agreement between the numerical solution and the analytical solution was quite satisfactory. Note that the waves propagate in the negative direction only, in spite of the positive phase velocity ($\omega/k > 0$ for $\forall k$) obtained from Eq. (20). The most dominant part of the waves, however, remain always near $x = 0$, i.e. evolves along the characteristic line defined by $x = \text{const.}\ (= 0)$.

5 Regularized equation for film flows

5.1 Padé approximation

The regularization procedure proposed in Sec. 4 is now applied to Eq. (15), which is a nonlinear partial differential equation describing the dynamics of film flows [6]. Since the essential idea is to regard Benney's long-wave expansion as Taylor expansion around $\partial_x = 0$ and then replace it by Padé approximation, it is appropriate to begin with a review of the systematic Padé approximation method.

Suppose that a function $\psi(k)$ is given in terms of a power series, as

$$\psi = c_0 + c_1 k + c_2 k^2 + \cdots . \tag{30}$$

This power series may be only poorly convergent or even not convergent at all; in such a case, a ratio of two polynomials may give a better approximation than a truncated power series. The procedure to determine these two polynomials involves converting the power-series (30) into another power-series

$$f = g\psi = a_0 + a_1 k + a_2 k^2 + \cdots \tag{31}$$

so that $\psi = f/g$, where $g = 1 + b_1 k + \cdots + b_n k^n$. What we expect is that a suitable choice of g can improve the convergence of the power series, as the singularity of $\psi(k)$ may be canceled (at least approximately) by the zero of $g(k)$. The problem is what is the best choice of g; we assume that the "best choice" should lead to termination of the power series (31) at k^m, so that $b_1, b_2, \ldots, b_n$ are determined by the conditions

$$a_{m+1} = a_{m+2} = \cdots = a_{m+n},$$

i.e. that the terms whose order is higher than k^m should vanish.

5.2 Regularization operator

Now let us remember that the long-wave expansion is essentially a power series expansion by a differential operator ∂_x, which suggests introducing an operator

$$\hat{L} = 1 + A^{(1)}\partial_x + A^{(2)}\partial_x^2 = 1 + \mu A^{(1)}\partial_{x_1} + \mu^2 A^{(2)}\partial_{x_1}^2 \tag{32}$$

corresponding to the polynomial g in the Padé approximation. The coefficients $A^{(j)}$ may depend on h. In analogy to Eq. (31) we define

$$S \stackrel{\text{def}}{=} \hat{L}Q = S_0 + \mu S_1 + \mu^2 S_2 + O(\mu^3), \tag{33}$$

where

$$S_0 = Q_0, \tag{34a}$$

$$S_1 = Q_1 + A^{(1)}\partial_{x_1} Q_0, \tag{34b}$$

$$S_2 = Q_2 + A^{(1)}\partial_{x_1} Q_1 + A^{(2)}\partial_{x_1}^2 Q_0. \tag{34c}$$

We can make $S_2 \simeq 0$ by defining the coefficients of the operator $\hat{L}$ appropriately. The operator $\hat{L}$, as well as g, plays the role of canceling the singularity; for this reason we call $\hat{L}$ "regularization operator".

5.3 Derivation of regularized equation

Let us proceed to determine $\hat{L}$, defined by Eq. (32). According to the philosophy of Padé approximation, $A^{(1)}$ and $A^{(2)}$ are determined so that "S_2 may vanish", which completes the regularization method and yields the regularized equation governing h.

Supposing μ to be small but finite, we decompose h as

$$h = \bar{h} + \phi, \tag{35}$$

where $\bar{h}$ represents the very-long-wave part whose wave number is significantly smaller than μ. The remaining part, ϕ, stands for the fluctuating wave components, and is characterized by the finite wave number $k \sim \mu$. Since the amplitude of ϕ is supposed to be small, we introduce the amplitude expansion parameter ϵ to write

$$\bar{h} \simeq h \sim 1, \tag{36a}$$
$$|\partial_{x_1}\phi| \sim |\phi| \sim \epsilon \ll 1, \tag{36b}$$
$$|\partial_{x_1}\bar{h}| \sim \epsilon^p |\partial_{x_1}\phi| \ll |\phi| \tag{36c}$$

with $p \geq 1$. By substituting the amplitude expansion (35) into the result of the long-wave expansion (14a) and taking the estimations (36) into account, we are led to

$$Q_2 = \left[\frac{32}{63}R^2h^{10} - \frac{40}{63}(R\cot\alpha)h^7 + 2h^4\right]\partial_{x_1}^2\phi + \frac{40}{63}\tilde{W}Rh^7\partial_{x_1}^4\phi + O(\epsilon^2), \tag{37a}$$
$$\partial_{x_1}Q_1 = \left[\frac{8}{15}Rh^6 - \frac{2}{3}h^3\cot\alpha\right]\partial_{x_1}^2\phi + \frac{2}{3}\tilde{W}h^3\partial_{x_1}^4\phi + O(\epsilon^2), \tag{37b}$$
$$\partial_{x_1}^2 Q_0 = 2h^2\partial_{x_1}^2\phi + O(\epsilon^2); \tag{37c}$$

we substitute Eqs. (37) into S_2, defined by Eq. (34c), to obtain

$$S_2 = \left[\frac{32}{63}R^2h^{10} - \frac{40}{63}(R\cot\alpha)h^7 + 2h^4 + A^{(1)}\left(\frac{8}{15}Rh^6 - \frac{2}{3}h^3\cot\alpha\right)\right.$$
$$\left. + 2h^2A^{(2)}\right]\partial_{x_1}^2\phi + \tilde{W}\left[\frac{40}{63}Rh^7 + \frac{2}{3}h^3A^{(1)}\right]\partial_{x_1}^4\phi + O(\epsilon^2). \tag{38}$$

This expression for S_2 is made to vanish by a suitable choice of $A^{(1)}$ and $A^{(2)}$. We find that it vanishes by setting

$$A^{(1)} = -\frac{20}{21}Rh^4, \qquad A^{(2)} = -h^2; \tag{39}$$

note that the $A^{(j)}$'s are chosen to be independent of α. Thereby the regularization operator is determined to be

$$\hat{L} = 1 - \frac{20}{21}Rh^4\mu\partial_{x_1} - h^2\mu^2\partial_{x_1}^2 = 1 - \frac{20}{21}Rh^4\partial_x - h^2\partial_x^2, \tag{40}$$

and S_1, defined by Eq. (34b), becomes

$$S_1 = Q_1 - \frac{20}{21} R h^4 \partial_{x_1} Q_0$$
$$= \frac{2}{3}\left[-\frac{72}{35} R h^6 \partial_{x_1} h - (\cot\alpha) h^3 \partial_{x_1} h + \tilde{W} h^3 \partial_{x_1}^3 h\right]. \tag{41}$$

By substituting Eq. (41) into Eq. (33), where $S_0 = Q_0$ and $S_2 \simeq 0$ is already known, now S is completely determined. As a result, the power series representation of Q-h relation (14a) is replaced by its "regularized" representation,

$$\hat{L}Q = S = \frac{2}{3}\left[h^3 - \frac{72}{35} R h^6 \partial_x h - (\cot\alpha) h^3 \partial_x h + W h^3 \partial_x^3 h\right], \tag{42}$$

where μ is absorbed into $\partial_x = \mu \partial_{x_1}$ and $W = \mu^{-2}\tilde{W}$. Note that $Q = \hat{L}^{-1} S$ is determined uniquely as a solution of Eq. (42), as is shown in Appendix B.

The relation (42) must be combined with the mass conservation equation (5a),

$$\partial_t h + \partial_x Q = 0,$$

to close the equation. By eliminating Q from them, finally we are led to the regularized equation (1).

5.4 Numerical solutions

Numerical solutions of the regularized equation (1) are now compared with those of two long-wave equations. Fig. 3 shows a bifurcation diagram for permanent solitary wave solutions, where the wave velocity c is plotted against the Reynolds number R. The wall is vertical ($\alpha = \pi/2$) and the Weber number ($W = 90$) is fixed. The three lines stand for the three equations, namely Gjevik's long-wave equation (16), Nakaya's higher-order long-wave equation in Ref. [12], and the regularized equation (1). In the vicinity of the critical Reynolds number ($R_c = 0$ for this case), all the three equations yield almost the same result. Beyond $R = 1.5$, however, they behave quite differently. As for the long-wave equation of Gjevik, there occurs a saddle-node bifurcation (indicated by $\times$) at $R = 2.2099$, where the branch meets another branch to annihilate (not shown here). The equation of Nakaya is also subject to a similar bifurcation which occurs at $R = 1.5843$. Meanwhile, the regularized equation (1) does not exhibit such a bifurcation at all. This bifurcation is a false prediction of the long-wave equation; this is successfully avoided by the regularized equation, at least qualitatively. Further numerical tests and discussion on the validity of Eq. (1) are reported in Ref. [6].

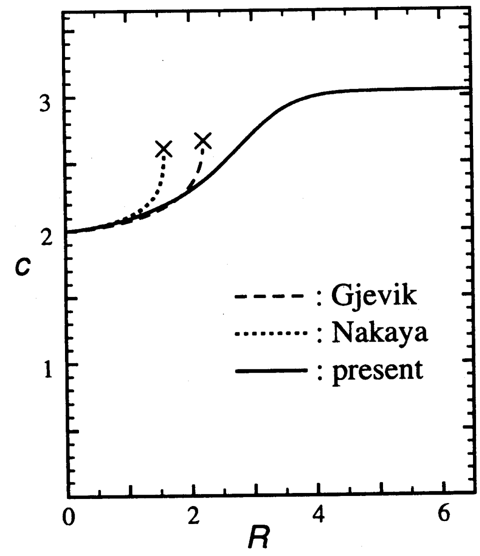

Figure 3: Bifurcation of permanent solitary waves

6 Discussion

6.1 Regularized equation as infinite-order long-wave equation

The inverse of the regularization operator (40) is formally expressed as an infinite summation,

$$\hat{L}^{-1} = 1 + \left(\frac{20}{21}Rh^4\partial_x + h^2\partial_x^2\right) + \left(\frac{20}{21}Rh^4\partial_x + h^2\partial_x^2\right)^2 + \cdots, \tag{43}$$

if we do not care about its convergence. We may formally operate $\hat{L}^{-1}$ either upon the regularized Q-h relation (42) or upon the regularized equation (1), to recover a long-wave equation which consists of infinite number of terms. Several leading terms in this infinite-order equation are exactly identical with those contained in the traditional long-wave equations, such as Gjevik's long-wave equation (16), so that the regularized equation (1) asymptotically agrees with Eq. (16). The remaining higher-order terms does not strictly coincide with the straightforward higher-order results of the long-wave expansion, but they do agree with the major part of the long-wave result. For example, it is easy to show that every Q_n includes a term proportional to $R^n h^{2+4n}\partial_x^n h \sim R^n\mu^n$ according to the long-wave expansion; it is therefore inappropriate to truncate the power-series, unless $R\mu$ is sufficiently small. In this respect the regularized equation (1) is equivalent to an infinite-order version of long-wave equation. We would like to emphasize, however, that such an infinite-order equation is meaningful only with the aid of summation method, such as the regularization method adopted here.

It is convenient to introduce the rescaled Reynolds number

$$\delta_* = \frac{R}{W^{1/3}} \tag{44}$$

so that $R\mu \sim R^{3/2}W^{-1/2} = \delta_*^{2/3}$ according to the estimation of μ in Eq. (10). This parameter δ_* is analogous to the parameter δ used by Chang *et al* [18]. Then we can reformulate one of the assumptions in the long-wave expansion, i.e. Eq. (12), as

$$\delta_* \ll 1, \tag{45}$$

because Eq. (12) actually states that R is small as compared to μ^{-1}. The condition (45), which is equivalent to $R\mu \ll 1$, is also what validates the formal expansion (43) and its truncation, i.e. Benney's long-wave expansion. This condition (45) is relaxed by the regularization method, so that the regularized equation (1) is valid for $\delta_* \lesssim 1$.

6.2 Effect of zero-modes due to the cross-differential terms

The remarkable features of the regularized equations, such as Eqs. (19) and (1), is that they include cross-differential terms like $\partial_x\partial_t h$. Such terms incorporates the influences of the (nearly) zero-wavenumber modes, i.e. the mode with arbitrarily long wave length, upon the linear growth rate and the linear dispersion relation of other modes. The influences of the zero-modes are referred to as the "baseline effect" by Ooshida and Kawahara [19], who developed a model equation for one-dimensional fluidized beds:

$$\left(1 - \gamma\partial_X - \partial_X^2\right)\partial_T\Psi + \left(1 + \Psi - \mu'\Psi^2\right)\partial_X\Psi - \gamma\partial_X^2\Psi = 0, \tag{46}$$

where Ψ is related to the so-called void fraction, with μ' and γ being positive constants. Eq. (46) includes two cross-differentiational terms, namely $\partial_X\partial_T\Psi$ and $\partial_X^2\partial_T\Psi$.

A KdV-like equation with a cross-differential term $\partial_x^2\partial_t h$ is known as Benjamin-Bona-Mahony (BBM) equation [20], historically also known as the "Regularized Long-Wave (RLW) equation." Unfortunately, the studies of BBM equation seem to have concentrated on its linear dispersion relation, rather than the influences of the zero-modes. For falling film flows, Indireshkumar and Frenkel [21] pointed out that an equation including a cross-differential term $\partial_x\partial_t\left(h^5\right)$ might provide a model better than Eq. (16), but further investigations were not made.

As an example to observe the influence of the zero-modes, let us linearize Eq. (46) around Ψ and examine the complex linear dispersion relation. Substituting $\Psi = \Psi_b + \hat{\Psi}\exp(\sigma T + ikX)$ with $|\hat{\Psi}| \ll 1$ into Eq. (46), we obtain

$$\begin{aligned}\sigma &= \frac{-i(1 + \Psi_b - \mu'\Psi_b^2)k - \gamma k^2}{1 - i\gamma k + k^2} \\ &= \begin{cases} -i(1 + \Psi_b - \mu'\Psi_b^2)k + \gamma(\Psi_b - \mu'\Psi_b^2)k^2 + \cdots & \text{(for long waves)} \\ -\gamma + O(k^{-1}) & \text{(for short waves)} \end{cases}\end{aligned} \tag{47}$$

to find that $\mathrm{Re}\,\sigma$ depends on Ψ_b. We may say that the zero-wavenumber mode Ψ_b is influential through the implicit nonlinearity introduced by the cross-differential terms.

It is worth while to note here that permanent solutions to Eq. (46) can be obtained explicitly. Substituting $\Psi = \Psi(Z)$ with $Z = X - cT$ into Eq. (46) leads to an ordinary differential equation,

$$(1-c)\partial_Z\Psi + c\partial_Z^3\Psi + \left(\Psi - \mu'\Psi^2\right)\partial_Z\Psi - (1-c)\gamma\partial_Z^2\Psi = 0, \tag{48}$$

which poses a nonlinear eigenvalue problem under a suitable boundary condition. The "eigenvalue" c is easily determined as follows: let us multiply (48) by Ψ and integrate with respect to Z, to find

$$(1-c)\gamma \int dZ\ (\partial_Z\Psi)^2 = 0 \tag{49}$$

by partial integration. Obviously $c = 1$ if Ψ is to be non-trivial. Then the terms with γ in Eq. (48) completely cancels out each other, so that we obtain permanent solutions traveling with $c = 1$. Especially, when $\mu' = 0$, a family of cnoidal wave solutions are obtained:

$$\Psi = \frac{12}{\ell^2}\left[m^2\,\mathrm{cn}^2\left(\frac{x-ct}{\ell}, m\right) + \frac{1}{3}\left(1-2m^2\right)\right], \qquad c = 1. \tag{50}$$

Note that ℓ can take any positive value if Ψ_b is given in accord, i.e. arbitrary long waves are possible.

Eq. (49), prescribing a selection rule for the permanent waves of Eq. (46), is interpreted as the balance between $\partial_X\partial_T\Psi$ and $\partial_X\partial_T\Psi$; their balance leads to the selection of the *velocity*, i.e. $c = 1$. This is in contrast with the case for Kuramoto-Sivashinsky (KS) equation,

$$\partial_T\eta + \partial_X(\eta^2) + \partial_X^2\eta + \partial_X^4\eta = 0, \tag{51}$$

where the rule to select permanent solutions is the balance between $\partial_X^2\eta$ and $\partial_X^4\eta$, so that solutions with appropriate *wave length* are selected.

6.3 Whitham's wave hierarchy equation

Whitham studied coupled equations of the general form

$$\eta\left(\partial_t + c_1\partial_x\right)\left(\partial_t + c_2\partial_x\right)\varphi + \left(\partial_t + a\partial_x\right)\varphi = 0, \tag{52}$$

which appears in his book [22] as Eq. (10.5). In Eq. (52), η is a positive constant, and a, c_1, c_2 are positive or negative constants which would have the dimension of velocity if written in dimensional form; without loss of generality we may assume that $c_1 > c_2$. The relevance of this type of equations to film flows was pointed out

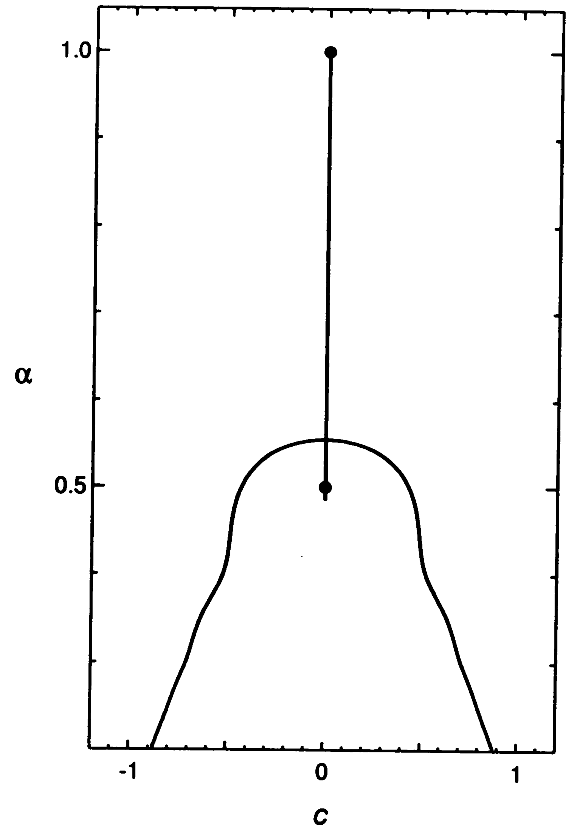

Figure 4: Bifurcation of periodic solutions to KS equation

by Alekseenko *et al.* [23], who referred to them as having "two-wave structure." The criterion for stability of Eq. (52) is given by

$$c_2 < a < c_1, \tag{53}$$

which is known as Whitham's wave hierarchy condition.

To see what occurs when Whitham's wave hierarchy condition (53) is not satisfied, let us consider a case with

$$a > 0, \qquad c_1 = 0, \quad c_2 = -\epsilon^{-1}a < 0. \tag{54}$$

If we take the limit

$$\epsilon \to +0, \qquad \eta \to +0$$

with the ratio $\tau_0 = \eta/\epsilon$ kept finite, then Whitham's equation (52) is formally reduced to the regularized model equation (19). This suggests that the regularization method is related to the treatment of the characteristics with negative infinite phase velocity, $c_2 \to -\infty$.

6.4 Kuramoto-Sivashinsky equation

Now let us return to the regularized equation of films flows, i.e. Eq. (1). Considering the fact that a major contribution of the higher-order terms are *implicitly* incorporated through the regularization operator $\hat{L}$, we expect that a "weakly nonlinear"

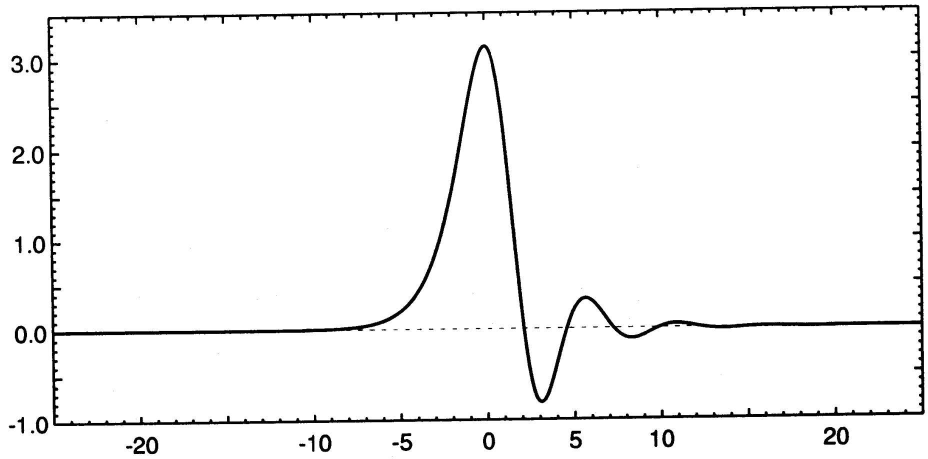

Figure 5: Permanent solitary wave solution to KS equation

version of the regularized equation may as well be a nice model equation for waves on a film flow at a certain stage of its evolution. While the traditional long-wave equation (16) was simplified to KS equation by setting $h = 1 + \eta$ and postulating that $|\eta| \ll 1$, the regularized equation (1) becomes

$$\left(1 - \frac{20}{21}R\partial_x - \partial_x^2\right)\partial_t\eta + \frac{2}{3}\partial_x\left[3\eta + 3\eta^2 - \frac{72}{35}R\partial_x\eta - (\cot\alpha)\partial_x\eta + W\partial_x^3\eta\right] = 0. \tag{55}$$

Though the explicit nonlinearity is only seen at η^2, higher-order nonlinearity is implicitly incorporated through the regularization operator. In particular, Eq. (55) incorporates the "baseline effect" which is lacking for KS equation.

First, we consider special cases, where both $\mu \sim \sqrt{R/W} \ll$ and $R \sim O(1)$ are satisfied; there are the conditions which validate Benney's long-wave expansion. Then the operator before $\partial_t\eta$ in Eq. (55) can be inverted as

$$\left(1 - \frac{20}{21}R\partial_x - \partial_x^2\right)^{-1} = 1 + \frac{20}{21}R\partial_x + O(\mu^2), \tag{56}$$

so that Eq. (55) reduces itself, after a suitable rescaling and a change of frame, to Kuramoto-Sivashinsky equation (51); in fact, several higher-order terms will appear, but they are found to be negligible. This is an expected result, because the regularized long-wave expansion method is reduced to Benney's method in the case under consideration. Permanent solutions to KS equation (51) are obtained as a eigenfunction of a nonlinear ordinary differential equation,

$$-c\partial_Z\eta + \partial_Z(\eta^2) + \partial_Z^2\eta + \partial_Z^4\eta = 0. \tag{57}$$

When the second condition $R \sim O(1)$ (or equivalently $\delta_* \ll 1$) is not satisfied, it is not possible to utilize Eq. (56) to simplify Eq. (55). However, in the limiting case of $W/R \to +\infty$, the term $\partial_x^2\partial_t\eta$ becomes negligibly small as compared to other terms. Considering such a case and assuming a permanent wave solution, we have

$$\partial_x\left[(-c+2)\eta + 2\eta^2\right] + \left(\frac{20}{21}Rc - \frac{48}{35}R - \frac{2}{3}\cot\alpha\right)\partial_x^2\eta + \frac{2}{3}W\partial_x^4\eta = 0; \tag{58}$$

this is essentially the same equation as Eq. (57), except that c appears not only as a coefficient of $\partial_x\eta$ but also as a coefficient of $\partial_x^2\eta$. The selection of permanent solution consists in the balance of *three* terms in Eq. (55), namely $\partial_x\partial_t\eta$, $\partial_x^2\eta$, and $\partial_x^4\eta$.

Although the applicability of KS equation is much more limited as compared to the regularized equation (1), still we emphasize the significance of KS equation as a prototype of all the simplified equations of falling film flows. Eq. (51) admits many permanent solutions, i.e. solutions describing a wave traveling in a constant velocity c. Under the periodic boundary condition with the periodicity $L = 2\pi/\alpha$, such permanent solutions are possible only for $\alpha < 1$. The most important branches of the bifurcation diagram are shown in Fig. 4, where the wave velocity c is plotted against α; for more detail, see Fig. 3 in Ref. [4, p.116] and Fig. 2 in Ref. [3, p.2210].

In the limit of an infinite domain, i.e. $\alpha \to +0$, KS equation (51) admits a solitary wave solution. In Fig. 5 we show the wave profile 2η against $z = x - ct$, where $c = 1.21615012396$; see also the calculation by Toh [24, p.956]. It is interesting that the regularized equation (1) also admits a solitary wave solution, but the shape of the solitary wave is different from that of KS equation; see the discussion on the tail length of solitary waves in Ref. [6].

Appendix

A Klein-Gordon equation

Klein-Gordon equation in $1+1$-dimensional space-time may be written either in the first standard form

$$\partial_\xi\partial_\eta\phi \pm \phi = 0 \tag{59}$$

or in the second standard form

$$\left(\partial_t^2 - c^2\partial_x^2\right)\phi + \alpha^2\phi = 0. \tag{60}$$

The second form, Eq. (60), is more familiar to physicists; note that c and α could be set to unity but are left present to help the physicist's intuition. The first form (59) clarifies that the characteristics are given by $\xi =$ const. and $\eta =$ const.; usually we assume that these characteristics does not coincide with the initial data. The peculiarity of Eq. (22) discussed in Sec. 4 consists in that the initial data (18) is given exactly on the characteristic line, i.e. $t = 0$.

Solutions to Eq. (59) or (60) may be given by separation of variables, unless we mind the initial and boundary conditions. Let us begin with the first standard form, Eq. (59). We introduce new independent variables (r, s), instead of (ξ, η), by the relation

$$\xi = \frac{r}{2} e^{+s}, \qquad \eta = \frac{r}{2} e^{-s};$$

note that r is real for $\xi\eta \geq 0$ and purely imaginary for $\xi\eta < 0$. After some calculation, we have

$$\begin{bmatrix} \partial_\xi \\ \partial_\eta \end{bmatrix} = \begin{bmatrix} \frac{\partial r}{\partial \xi} & \frac{\partial s}{\partial \xi} \\ \frac{\partial r}{\partial \eta} & \frac{\partial s}{\partial \eta} \end{bmatrix} \begin{bmatrix} \partial_r \\ \partial_s \end{bmatrix} = \begin{bmatrix} e^{-s} \left(\partial_r + r^{-1} \partial_s \right) \\ e^{+s} \left(\partial_r - r^{-1} \partial_s \right) \end{bmatrix}, \tag{61}$$

which enables to rewrite Eq. (59) as

$$\left(\partial_r^2 + \frac{1}{r} \partial_r \right) \phi - \frac{1}{r^2} \partial_s^2 \phi \pm \phi = 0. \tag{62}$$

Now we can separate the variables by assuming $\phi(r, s) = R(r)\, S(s)$, so that Klein-Gordon equation (59), or Eq. (62), is reduced either to Bessel equation or modified Bessel equation.

As for the second standard form (60), we assume $t > 0$ and $-ct < x < ct$, i.e. we consider only the domain within the light cone. In terms of new variables (r, θ) introduced by

$$t = \frac{r}{\alpha} \cosh \theta, \qquad x = \frac{cr}{\alpha} \sinh \theta,$$

the $1 + 1$-dimensional d'Alembertian is found to be given by

$$\partial_t^2 - c^2 \partial_x^2 = \alpha^2 \left(\partial_r^2 + \frac{1}{r} \partial_r - \frac{1}{r^2} \partial_\theta^2 \right). \tag{63}$$

Note that the right-hand side of Eq. (63) is reminiscent of the Laplacian in cylindrical coordinate; this is no wonder if we remark that the Euclideanization of the $1 + 1$-dimensional d'Alembertian would yield the 2-dimensional Laplacian, and that r stands for the Minkowski distance from the origin of the space-time. This suggests that the Klein-Gordon equation (60) can be solved quite similarly to the 2-dimensional Helmholtz equation. In fact, the general solution is given in terms of Bessel functions:

$$\phi = \sum_{n=-\infty}^{+\infty} c_n\, J_n(r)\, e^{n\theta}, \tag{64}$$

where c_n's are arbitrary constants. Note that (r, θ) can be expressed in terms of (x, t):

$$r = \alpha \sqrt{t^2 - \left(\frac{x}{c} \right)^2}, \qquad e^\theta = \sqrt{\frac{ct + x}{ct - x}}.$$

B One-to-one mapping property of regularization operator

Let us prove that Eq. (42) gives a one-to-one correspondence between Q and S.

Suppose that a linear operator $\hat{L}$ is given, such that

$$\hat{L}: \ \psi \mapsto \hat{L}[\psi] = \psi + A_1 \frac{d\psi}{dx} + A_2 \frac{d^2\psi}{dx^2}, \tag{65}$$

where A_1 and A_2 need not to be constant but satisfy

$$|A_1| < +\infty, \quad |A_2| < +\infty \qquad (\text{for } \forall x); \tag{66a}$$

$$\sup A_2 < 0. \tag{66b}$$

Then the boundary-value problem for given f,

$$\hat{L}\phi = f \quad (a < x < b), \tag{67}$$

$$\phi(a) = \phi(b) = 0, \tag{68}$$

is shown to have at most one solution.

To prove this, we assume that the homogeneous equation

$$\hat{L}\phi = \phi + A_1\phi' + A_2\phi'' = 0 \tag{69}$$

had a non-trivial solution and show that it would lead to absurdity. Since the solution ϕ satisfies the boundary condition (68), it should have either a positive maximum or a negative minimum; without loss of generality we may consider the former. Then we have

$$\phi(\exists x_0) = \max \phi > 0, \quad a < x_0 < b, \quad \phi'(x_0) = 0, \tag{70}$$

which implies, due to Eq. (69) and the condition (66b), that

$$\phi''(x_0) = -\frac{\phi(x_0)}{A_2(x_0)} > 0; \tag{71}$$

this is against the assumption that ϕ has a maximum at $x = x_0$.

Thus we have proved that the correspondence from ϕ to f is one-to-one, which obviously applies to Eq. (42) as well.

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