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Asymptotic expansions for the motion of a curved vortex filament and the localized induction hierarchy

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Summary (概要)

細長い渦管のことを‘渦糸’とよぶ．境界のない無限領域を満たす非圧縮・非粘性流体中の渦糸の 3 次元運動について考えよう．渦度が与えられたとすると，誘導速度は Biot-Savart の法則によって一意的に定まる．Helmholtz の渦定理によれば，渦線の運動速度は，これでもって原理的に計算できる．渦管を包む流れがポテンシャル流であるという事情から，渦管の各部がまわりに誘起する流れは遠方まで減衰しないので，曲線状の渦糸においては，単独でも，各切片同士が非局所的に長距離相互作用することになる．これによって，自己誘導運動が可能になるわけであるが，遠隔誘導をすべて取り込むための計算量は膨大である．しかも，渦線直上で Biot-Savart 積分を評価するのは容易ではない．体積積分を線積分で近似する従来のやり方では発散 (代数的，対数的) が避けられないからである．仮に，渦核内の渦度分布を取り入れて体積積分を巧妙に評価できたとしても，それは，渦線個々の運動速度を与えるに過ぎない．

渦糸を渦線群の束と見立てたとき，渦管内の渦線個々の動きと，渦管全体としての動きとは，同一とは限らないであろう.「これらの間にはどのような関係があるのだろうか？」「渦線の束としての渦糸の運動速度は，個々の速度からいかにして導出できるのであろうか？」

簡単のため，渦糸の波動運動や不安定性は考えない．時間スケールの長い‘準定常運動’と呼べるものだけに着目する．最も乱暴な近似は‘局所誘導近似 (LIA)’であろう；Biot-Savart の法則を線積分で置き換え，(i) 全長からの誘導のうち，考えている点の両側の有限長さ L の部分からの寄与のみ取り込み，さらに，(ii) 渦糸の太さの有限性 (断面半径 σ) の効果と称して，対数発散を正則化する．渦糸の循環を Γ とし，渦糸の中心線を，弧長パラメータ s と時間 t を用いて，$\boldsymbol{X} = \boldsymbol{X}(s,t)$ と表すと，この簡単化のもとでは，中心線の運動は局所誘導方程式

$$\frac{\partial \boldsymbol{X}}{\partial t} = \frac{\Gamma}{4\pi}\left[\log\left(\frac{L}{\sigma}\right)\right]\kappa \boldsymbol{b} \tag{1}$$

に従う (Da Rios 1906)．ここで，$\kappa = \kappa(s,t)$ と $\boldsymbol{b} = \boldsymbol{b}(s,t)$ は，それぞれ，中心線の曲率と陪法線ベクトルである．この扱いの範囲内では，定数 L と σ は決まらない．それでも，渦糸の運動に対する直感が得られた気になれる．渦輪の運動を経験的に知っているからである；進む方向は渦輪が囲む面に垂直 (= 陪法線 $\boldsymbol{b}$ 方向)，リング半径が小さい (= 曲率 κ が大きい) ほど速い．以上のことを，§1 で簡単に復習する．

定数 L を決める努力が連綿と続けられている．Crow (1970) は，Biot-Savart の法則のある種の正則化によって，(1) と全長にわたる非局所誘導を融合した運動速度に最初に到達した．‘カット・オフ積分法’として知られているものである．この積分法は，接合漸近展開法による系統的な導出によって支持されている (Widnall, Bliss & Zalay 1971; Callegari

& Ting 1978). 展開の微小パラメータ ϵ を，渦核半径 σ と代表的な曲率半径 $R_0 = 1/\kappa_0$ の比にとる．渦糸は，局所的にはまっすぐな円柱渦とみなすことができ，局所的な円形循環流を主要項 $O(\epsilon^0)$ と数える．運動速度の導出には，Euler 方程式の漸近展開を次のオーダ $O(\epsilon)$ まで行えばよい．確かに，この手続きにより，定数 L に対する表示は，渦核内の速度分布を反映させたものになる．が，運動のメカニズムに対する理解が深まったとは言いがたい.「その中身は何を意味するのであろうか？」

さて，現実の流れでは，極めて細い渦管に出会うことはまれである．太っていて，しかも曲がりくねったものが多い．現実的な渦管の運動を忠実に計算するためには，漸近展開の次数を上げる必要がある．本研究の主題は **高次漸近展開理論の建設** である．自身が誘導する局所ひずみ場が $O(\epsilon^2)$ で現れ，渦核断面が楕円形に変形される．この変形が運動速度におよぼす影響は，$O(\epsilon^3)$ で現れる．本研究では，近似精度をさらに 2 次上げて，$O(\epsilon^3)$ **まで有効な運動速度の導出に挑戦した** (§**3–5**).

軸対称渦輪においては，簡単な渦度分布の場合，Dyson (1983) によって 1 世紀以上前に解かれていた．

$$U = \frac{\Gamma}{4\pi R_0}\left\{\log\left(\frac{8}{\epsilon}\right) - \frac{1}{4} - \frac{3\epsilon^2}{8}\left[\log\left(\frac{8}{\epsilon}\right) - \frac{5}{4}\right] + O(\epsilon^4\log\epsilon)\right\}. \tag{2}$$

Fukumoto & Moffatt (2000) は，一般的な渦度分布や粘性効果の扱いも可能にする枠組みを作った．Navier-Stokes 方程式に対する接合漸近展開法の 3 次以上への拡張を行ったわけであるが，既存の理論の低次 $O(\epsilon^0), O(\epsilon)$ における不備を指摘し，修復しておいた．この綻びは，低次に留まっている限り問題にはならないが，高次へ進むときの重大な障害になるからである．しかし，軸対称という制約を外して，一般の 3 次元渦糸に拡張しようとすると，途方にくれてしまう．数式が膨れ上がって，道に迷ってしまうことは目に見えているからである．

目を転じて，LIA の歴史には独自の歩みがあることを思い起こそう．局所誘導方程式 (1) が非線形 Schrödinger 方程式と等価な完全可積分発展方程式であるということである (Hasimoto 1972)；曲率 $\kappa(s,t)$ と捩率 $\tau(s,t)$ を組み合わせた複素数値関数

$$\psi(s,t) = e^{i\int^s \tau ds} \tag{3}$$

を導入すると (‘橋本変換’ または ‘橋本写像’)，(1) は非線形 Schrödinger 方程式に帰着してしまう．

$$i\frac{\partial\psi}{\partial t} + C\left[\frac{\partial^2\psi}{\partial s^2} + \frac{1}{2}|\psi|^2\psi\right] + A(t)\psi = 0\,; \qquad C = \frac{\Gamma}{4\pi}\left[\log\left(\frac{L}{\sigma}\right)\right].$$

ここで，$A(t)$ は時間についての任意関数である．非線形 Schrödinger 方程式の 1 ソリトン解の渦糸版は，‘橋本ソリトン’ として広く知られている (§**1** 参照).

完全可積分発展方程式のより正確な言い方は，‘無限自由度完全可積分 Hamilton 力学系’ ということである；無限自由度 Hamilton 力学系で，互いに独立な第一積分を無限個もち，それらは包含系をなす．非線形 Schrödinger 方程式の場合について，Magri (1978) は，この性質の背後に biHamiltonian 構造が潜むことを見抜き，2 つの Hamiltonian 構造を利用して，無限個の積分 (= 保存量) および (しかるべき Lie 括弧に関して) 可換な

Hamilton ベクトル場を逐次生成する'再帰演算子'を具体的に構成した．Langer & Perline (1991) は，橋本写像が Poisson 写像であるという性質を利用して，局所誘導方程式での対応物を構成した．再帰演算子の直接的な導出は Tani (1995) まで待つ．得られた互いに可換な無限個のベクトル場は'局所誘導階層 (LIH)'と呼ばれる．LIH の n 番目のベクトル場を $\boldsymbol{V}^{(n)} = \boldsymbol{V}^{(n)}(s,t)$ とかくと，最初の 3 つは，

$$
\begin{aligned}
\boldsymbol{V}^{(1)} &= \kappa \boldsymbol{b}, & (4)\\
\boldsymbol{V}^{(2)} &= \frac{1}{2}\kappa^2 \boldsymbol{t} + \kappa_s \boldsymbol{n} + \kappa\tau \boldsymbol{b}, & (5)\\
\boldsymbol{V}^{(3)} &= \kappa^2\tau \boldsymbol{t} + (2\kappa_s\tau + \kappa\tau_s)\boldsymbol{n} + (\kappa\tau^2 - \kappa_{ss} - \frac{1}{2}\kappa^3)\boldsymbol{b} & (6)
\end{aligned}
$$

である．下付き添え字 s は，s についての偏微分を表す記号である．

§2 では，LIH に属する無限個のベクトル場をすべて足し上げることによって作られる新しい発展方程式 を提案する．進行波解に限定すれば，この方程式は Lund-Regge 方程式と等価である (Fukumoto & Miyajima 1996; Konno & Kakuhata 1999)．後者は，相対論的な'南部弦'と渦糸とを融合させる試みの中で生まれた (Lund & Regge 1976).

話を流体中の渦糸に戻そう．軸対称渦輪に対する Dyson の公式 (2) と，LIH に属する $\boldsymbol{V}^{(1)}$ (式 (4)) と $\boldsymbol{V}^{(3)}$ (式 (6)) を見比べると，完全に一致していることに気がつく．ただし，$\tau = 0, \kappa_s = \kappa_{ss} = \cdots = 0$ の場合に限った話に過ぎない．

少し欲を出して，Biot-Savart 積分の精密な評価を工夫してみる．このとき，軸対称渦輪の高次漸近展開理論の構築途上で獲得した，曲率をもつ渦管に対する次の描像が役に立つ；「3 次元の渦管は，自身が誘導する流れの中におかれた'双極子列'である．双極子は渦管の中心線上に並び，向きは運動方向，強さは曲率に比例する.」双極子の正体は，直線状の渦管の曲げに伴う渦線の伸び・縮みの差異によって生じる横断面内での渦度の増大・減少の非一様性である．実効的な'渦対'と考えてもよい．弾性棒を思い浮かべればこの状況が了解できよう．中心線上に並ぶ双極子列 $d^{(1)}\kappa(s)\boldsymbol{b}(s)$ が誘導する速度場は，

$$
\boldsymbol{v}_d(\boldsymbol{x}) = \frac{d^{(1)}}{2}\int\left\{\frac{\kappa(s)\boldsymbol{b}(s)}{|\boldsymbol{x}-\boldsymbol{X}(s)|^3} - \frac{3\kappa(s)\boldsymbol{b}(s)\cdot[\boldsymbol{x}-\boldsymbol{X}(s)]}{|\boldsymbol{x}-\boldsymbol{X}(s)|^5}[\boldsymbol{x}-\boldsymbol{X}(s)]\right\}\mathrm{d}s \qquad (7)
$$

の形をとることが想像できる．§3 で，このことを実際に確かめる．速度場 (7) は，§3 で，系統的に導かれる速度のベクトルポテンシャル (9) の第 2 項の回転になっているのである．係数 $d^{(1)}$ は (10) で定義されるが，さしあたっては，中味を問う必要はない．様子を探るために，局所誘導近似の精神による (7) の簡単化を試みよう．特異項と局所的な一様流のうち対数項だけを残すと，

$$
\begin{aligned}
\boldsymbol{v}_d = {} & d^{(1)}\left\{-\frac{\kappa}{r^2}[\sin\varphi \boldsymbol{e}_r - \cos\varphi \boldsymbol{e}_\theta] + \frac{\kappa^2}{2r}\cos 2\varphi \boldsymbol{e}_\theta\right\}\\
& -\frac{d^{(1)}}{2}\log\left(\frac{L}{r}\right)\left\{(2\kappa_s\tau + \kappa\tau_s)\boldsymbol{n} + \left[\kappa\tau^2 - \kappa_{ss} - \frac{1}{4}\kappa^3\right]\boldsymbol{b}\right\} + \cdots \qquad (8)
\end{aligned}
$$

となる．第 1 行目は文字通り双極子流の特異性に合致し，第 2 行目が，渦管の曲がりからくる誘導流である．$\tau \neq 0, \kappa_s \neq 0$ という一般的な場合においても，第 2 行目が $\boldsymbol{V}^{(3)}$ と驚くほど似ていることが見てとれる．$\kappa^3\boldsymbol{b}$ の係数がわずかにずれているだけである．

これら奇妙な符合は，渦糸の $O(\epsilon^3)$ ダイナミックスにまで手が届くのではないか，という希望を抱かせる．これが，本研究の出発点である．$O(\epsilon^3)$ では，log を含む項に対して，内部解からの寄与もある．$O(\epsilon)$ においては，内部解は，log 項に寄与せず，定数部分を調整するだけであったのとは対照的である．内部解を知りたい．

まず，§**3** で，外部解を求める．Biot-Savart の法則自身が外部解で，この積分の ϵ についての漸近展開を行う．曲率の 1 次の効果 $O(\epsilon)$ で生成される双極子列による誘導速度を，あいまいさなく計算することがその本質である．

渦度場 $\boldsymbol{\omega}(\boldsymbol{x})$ と速度場 $\boldsymbol{v}(\boldsymbol{x})$ を結びつける Biot-Savart の法則は，ベクトルポテンシャル $\boldsymbol{A}(\boldsymbol{x})$ を用いて，

$$\boldsymbol{v} = \nabla \times \boldsymbol{A}; \qquad \boldsymbol{A}(\boldsymbol{x}) = \frac{1}{4\pi}\iiint \frac{\boldsymbol{\omega}(\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}\mathrm{d}V'$$

とかける．有限太さの効果を取り込むべく，ϵ についての漸近展開を行う．少し長い計算の後，

$$\boldsymbol{A}(\boldsymbol{x}) \approx \frac{\Gamma}{4\pi}\int \frac{\boldsymbol{t}(s)}{|\boldsymbol{x}-\boldsymbol{X}(s)|}\mathrm{d}s - \frac{d^{(1)}}{2}\int \frac{\kappa(s)\boldsymbol{b}(s)\times(\boldsymbol{x}-\boldsymbol{X}(s))}{|\boldsymbol{x}-\boldsymbol{X}(s)|^3}\mathrm{d}s \tag{9}$$

が得られる．第 2 項が双極子列による流れである．「双極子の向きが $\boldsymbol{b}$ 方向で，強さが κ に比例する」ことは直感と一致している．係数 $d^{(1)}$ は渦度の $\boldsymbol{t}$ 方向成分 $\zeta(r,\varphi,t) = \zeta_0(r) + \kappa\hat{\zeta}_{11}^{(1)}(r)\cos\varphi + \cdots$ を用いて，

$$d^{(1)} = \frac{1}{2\pi}\left\{[\pi\int_0^\infty r^2\hat{\zeta}_{11}^{(1)}\mathrm{d}r] - \frac{1}{2}[\pi\int_0^\infty r^3\zeta^{(0)}\mathrm{d}r]\right\} \tag{10}$$

のように与えられる．渦糸中心 $\boldsymbol{X}(s)$ 上に原点をもつ局所円柱座標 (r,φ,s) を用いて表してある．強さ $O(\epsilon^1)$ の双極子列は，$O(\epsilon^3)$ で，渦核近傍での一様流を非局所的に誘導し，これが運動速度の補正項の一部をなす．

接合漸近展開法のプロセスの中では，(9) の内部極限を求め，内部解に対する接合条件として書き表すことになる (§3.4)．渦度分布はこの段階では $\zeta^{(0)}$ を除いて未定で，接合条件を課して，内部解を求めることによって，低次のものから逐次得られる．別の言い方をすれば，双極子の強さを計算するために，内部解が必要とされるわけである．

§**4,5** では，内部領域で Euler 方程式を解いて，内部解が外部解へ連続的につながるように渦糸の運動速度を定める．渦管の中心とともに動く局所運動座標系を導入して，Euler 方程式を書き直し，ϵ についての摂動展開の形で解を $O(\epsilon^2)$ まで求める (§4)．さらに，$O(\epsilon^3)$ での $\boldsymbol{n}$-$\boldsymbol{b}$ 断面流に対する一般解に接合条件を課すと，運動速度に対する $O(\epsilon^3)$ の補正項が導かれる (§5)．

局所誘導近似のもとでは，結果は簡潔な形で書き下せ，その数学的な構造がくっきりと浮かび上がる：

$$\frac{\partial\boldsymbol{X}}{\partial t} = C\left\{\kappa\boldsymbol{b} + \left[\frac{\pi}{\Gamma}\int_0^\infty \zeta^{(0)}r^3\mathrm{d}r\right]\left[(2\kappa_s\tau+\kappa\tau_s)\boldsymbol{n} + (\kappa\tau^2-\kappa_{ss})\boldsymbol{b} + \kappa^2\tau\boldsymbol{t}\right] + C_b\kappa^3\boldsymbol{b}\right\}. \tag{11}$$

ここで，$\kappa^3\boldsymbol{b}$ 項の係数は

$$C_b = 2\pi d^{(1)}/\Gamma \tag{12}$$

である．橋本写像 (3) を用いると，

$$i\frac{\partial\psi}{\partial t}+C\left(\psi_{ss}+\frac{1}{2}|\psi|^2\psi\right)+A(t)\psi-Cc_1\left\{\psi_{ssss}+\frac{3}{2}\left(|\psi|^2\psi_{ss}+\psi_s^2\bar{\psi}\right)\right.$$
$$\left.+\left(\frac{3}{8}|\psi|^4+\frac{1}{2}\frac{\partial^2}{\partial s^2}|\psi|^2\right)\psi\right\}+C\left(C_b+\frac{c_1}{2}\right)\left\{\frac{\partial^2}{\partial s^2}(|\psi|^2\psi)+\frac{3}{4}|\psi|^4\psi\right\}=0 \qquad (13)$$

の形に変換される．ここで，

$$c_1=\frac{\pi}{\Gamma}\int_0^\infty \zeta^{(0)}r^3\mathrm{d}r$$

である．

発展方程式 (11) と LIH (4)–(6) とを見比べると，(11) は $\boldsymbol{V}^{(1)}$ と $\boldsymbol{V}^{(3)}$ の和に酷似していることに気づく．依然，$\kappa^3\boldsymbol{b}$ の係数だけが歩調を合わせない．渦輪の漸近展開で明らかになったように，双極子項の係数 $d^{(1)}$ は $O(\epsilon\sigma_0)$ の範囲内での局所運動座標系の原点の選びかたに敏感に依存する．ある横断面内で原点を $\boldsymbol{n}$ 方向に $-\epsilon\sigma_0 x_0$ だけずらすと，$d^{(1)}$ は　$x_0/2\pi$ だけ減り，(12) によれば，C_b は x_0 だけ減る：

$$C_b\to C_b-x_0\,.$$

このことはとりもなおさず，有限太さの渦核内部の曲線の選び方により，C_b が変わることを意味する．

かくして次の結論に到達した：**「局所誘導近似の範囲内で，渦核内部に完全可積分曲線が 1 本あり，その時間発展は LIH の $\boldsymbol{V}^{(1)}$ と $\boldsymbol{V}^{(3)}$ 和によって支配される.」** これに対応して，橋本写像 ψ も非線形 Schrödinger 階層の 1 番目と 3 番目の和 (13) によって支配されている．ただし，一般の C_b の場合 $(C_b+c_1/2\neq 0)$ には，これに付加項が加わる．完全可積分性の反映である LIH に導かれ，視界のきかない雑木林を手探りで進んでいるうちに，突然目の前に美しい景観が開けてきた，という感じがしている．

これまでの渦糸の 3 次元運動の直感的理解は，自身が誘導する流れに乗って受動的に流される，という通り一編のものであったと思う．渦管の運動には **双極子 (≈ 渦対)** が決定的な役割を果たす．渦対には自律的に運動できるという能力が備わっている．「渦管は単に自己誘導流によって流されるだけではなく，流れの中をさらに，内部にもつ‘渦対’を駆動力として能動的に動く.」これが，$O(\epsilon)$ の速度公式に対する 1 つの解釈である．加えて，$O(\epsilon)$ の双極子は，$O(\epsilon^3)$ で長距離におよぶ流れ ((8) の 2 行目) を引き起こし，この誘導流によって自身を運ぶという，2 重の働きがある．冒頭の問である「渦糸全体としての運動速度とそれを構成する渦線群の個別な速度との関係」にも，$O(\epsilon^0)$ の円柱渦流，$O(\epsilon)$ の双極子流，$O(\epsilon^2)$ の四重極子流という構造的観点から答えることができよう．

高次局所誘導方程式 (11) や，途中で出会った流れ場・圧力場の内容や意味を汲み取る作業は今後に待たれる．高次方程式の簡潔さは，少なくとも，渦糸の 3 次元動力学のより深い理解の仕方が可能であることを暗示している．しかし，現実の流れの中では，渦糸は不規則で激しい挙動を示す．高次近似を導くのにいくら努力してみたところで，より低次で現れる非局所誘導やそれが引き起こす渦線の伸長からの寄与が勝って，高次補正を目立たなくしてしまう恐れがある．全長にわたる Biot-Savart 積分を組み入れた渦糸の発展方程式の導出は今後の課題である．

以上の話の概略については，Fukumoto (2001) も参照されたい．

1 Localised induction approximation

Consider the three-dimensional motion of an isolated vortex filament embedded in an inviscid incompressible fluid of infinite expanse. According to Helmholtz' theorem, the filament is advected by the flow field induced by itself, which is dictated by the Biot-Savart law. In case the cross-section of the vortex tube is very small, the volume integral of the Biot-Savart law may be closely approximated by a line integral along the filament curve $\boldsymbol{X}(s)$ expressed as functions of arclength s, and the velocity $\boldsymbol{v}(\boldsymbol{x})$ at $\boldsymbol{x}$ is then reducible to

$$\boldsymbol{v}(\boldsymbol{x}) \approx -\frac{\Gamma}{4\pi}\int_{-\infty}^{\infty} \frac{(\boldsymbol{x}-\boldsymbol{X}(s)) \times \boldsymbol{t}(s)}{|\boldsymbol{x}-\boldsymbol{X}(s)|^3} \mathrm{d}s\,, \tag{1.1}$$

where $\boldsymbol{t}(s) = \mathrm{d}\boldsymbol{X}(s)/\mathrm{d}s$ is the unit tangent vector to the curve at $\boldsymbol{X}(s)$. Still, we are left with a line integral over the entire length.

The simplest approach to capture the leading-order behaviour of dynamics is the so called '*localised induction approximation (LIA)*' put forward by Da Rios (1906) under supervision of Levi-Civita (see also Batchelor (1967) and Ricca (1991)). The dominant contribution to the induced velocity at a point on the filament is considered to come from the neighboring segment, and thus the domain of integration is restricted to the interval, in arclength s, between $-L/2$ and $L/2$. The parameter L is indeterminate within the framework of this approximation. Thus, it is sufficient to deal exclusively with a curved segment of length L and to approximate it by a circular arc:

$$\boldsymbol{X}(s) \approx (s-s_0)\boldsymbol{t}(s_0) + \frac{1}{2}(s-s_0)^2\kappa(s_0)\boldsymbol{n}(s_0)\,, \tag{1.2}$$

where $\kappa(s)$ is the curvature of the filament at $\boldsymbol{X}(s)$, and $\{\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}\}$ is the Frenet-Serret frame of a curve. This is substituted into the integrand of (1.1). The integrand is expanded in powers of $(s-s_0)$, and then integration is performed with respect to s, from $s_0 - L/2$ to $s_0 + L/2$.

It is the logarithmic term that has a direct link with the self-induced motion. This term diverges logarithmically with distance r from the core centerline in the limit of $r \to 0$. A regularization is accomplished by setting $r = \sigma$. This procedure virtually amount to taking into account the effect of finite thickness of a circular core with radius σ.

The resulting expression is equated to the velocity of the centerline of the vortex filament. We are thus led to an evolution equation for a position vector $\boldsymbol{X}(s,t)$, represented as functions of s and time t, of the centerline, now being referred to as the *localized induction equation (LIA eq.)*:

$$\frac{\partial \boldsymbol{X}}{\partial t} = \frac{\Gamma}{4\pi}\left[\log\left(\frac{L}{\sigma}\right)\right]\kappa\boldsymbol{b}\,. \tag{1.3}$$

Da Rios (1906), and independently Betchov (1965), transformed (1.3) into a coupled system of intrinsic equations for curvature κ and torsion τ:

$$\frac{\partial \kappa}{\partial t} = -C\left(2\kappa_s\tau + \kappa\tau_s\right)\,, \tag{1.4}$$

$$\frac{\partial \tau}{\partial t} = C\frac{\partial}{\partial s}\left(\frac{\kappa_{ss}}{\kappa} - \tau^2 + \frac{\kappa^2}{2}\right)\,, \tag{1.5}$$

where

$$C = \frac{\Gamma}{4\pi}\left[\log\left(\frac{L}{\sigma}\right)\right], \tag{1.6}$$

and a subscript denotes partial differentiation with respect to the indicated variable. Hasimoto (1972) discovered that, by an introduction of a complex variable,

$$\psi(s,t) = e^{i\int^s \tau ds}, \tag{1.7}$$

(1.4) and (1.5) are combined to yield the *cubic nonlinear Schrödinger equation (NLS)*:

$$i\frac{\partial\psi}{\partial t} + C\left[\frac{\partial^2\psi}{\partial s^2} + \frac{1}{2}|\psi|^2\psi\right] + A(t)\psi = 0, \tag{1.8}$$

where $A(t)$ is an arbitrary function of t.

This remarkable finding implies that (1.3) is completely integrable, as a consequence of which a vortex filament is capable of supporting a soliton, a localised helical twist wave, now known as the *Hasimoto soliton.* The integrability remains intact even if the axial velocity is included in the core as far as we adhere to the LIA (Fukumoto & Miyazaki 1991). Moreover, this carries over to the *effect of finite thickness of the core* (§3–5), which is the central topic of this paper.

2 Summation of localised induction hierarchy

Originally, the concept of '*completely integrable*' makes sense within the framework of a system of ordinary differential equations with finite degrees of freedom. Here, by complete integrability, we mean that the evolution equation has an infinite sequence of independent integrals in involution. Magri (1978) uncovered the bi-Hamiltonian structure that underlies this integrability and thereby manipulated a recursion operator to generate an infinite sequence of integrals in involution and of commuting Hamiltonian vector fields. Langer & Perline (1991) made an effort at lifting the structure of the NLS to the LIA by taking the advantage of the Hasimoto map (1.7). They built a recursion operator to generate an infinite sequence of commuting vector fields associated with (1.3) (see also Tani 1995). We call this sequence the '*localised induction hierarchy (LIH)*'.

Let $\boldsymbol{X} = \boldsymbol{X}(s,t)$ be a point on the filament and $\boldsymbol{V}^{(n)} = \boldsymbol{V}^{(n)}(s,t)$ be the n-th term of the LIH. The first few terms are listed as follows:

$$\boldsymbol{V}^{(1)} = \kappa\boldsymbol{b}, \tag{2.1}$$

$$\boldsymbol{V}^{(2)} = \frac{1}{2}\kappa^2\boldsymbol{t} + \kappa_s\boldsymbol{n} + \kappa\tau\boldsymbol{b}, \tag{2.2}$$

$$\boldsymbol{V}^{(3)} = \kappa^2\tau\boldsymbol{t} + (2\kappa_s\tau + \kappa\tau_s)\boldsymbol{n} + (\kappa\tau^2 - \kappa_{ss} - \frac{1}{2}\kappa^3)\boldsymbol{b}, \tag{2.3}$$

$$\vdots$$

$$\boldsymbol{V}^{(n)} = -\boldsymbol{X}_s \times \boldsymbol{V}_s^{(n-1)} + \mathcal{T}^{(n)}\boldsymbol{X}_s, \tag{2.4}$$

Equation (2.4) is the recursion operator, in which $\mathcal{T}^{(n)}$ is a function to be determined by the condition of the arclength parameterization: $\boldsymbol{V}_s^{(n)} \cdot \boldsymbol{X}_s = 0$. Equating $\boldsymbol{V}^{(1)}$ with $\boldsymbol{X}_t$ gives (1.3) with an appropriately rescaled time. Next, if we take $\boldsymbol{X}_t = \boldsymbol{V}^{(1)} + \epsilon \boldsymbol{V}^{(2)}$, ϵ some parameter, we recover the localised induction equation of a vortex filament with axial flow in the core (Moore & Saffman 1972; Fukumoto & Miyazaki 1991).

With this observation, it is tempting to pursue the summation procedure of vector fields of the LIH. The objective of the present section is to establish an evolution equation of a curve by summing up all of the infinite vector fields of the LIH and to disclose its properties. See Fukumoto & Miyajima (1996) for the detail. Note that this is genuinely a mathematical argument, and a question arises whether the LIH has a bearing with practical flows. The answer is positive; we shall show in §3–5 that a superposition of (2.1) and (2.3) is indeed extracted from the Euler equations.

In §2.1, the summation procedure is implemented. We demonstrate that, if we restrict ourselves to traveling-wave solutions, the resulting equation is equivalent to the Lund-Regge equation. The latter was derived as a model for the motion of a relativistic string in a constant external field (Lund & Regge 1976). In §2.2, we rewrite our equation into an intrinsic form.

2.1 Summation of localised-induction hierarchy and the Lund-Regge equation

Consider the evolution equation of a curve obtained by summing up all of the terms of the LIH, namely,

$$\boldsymbol{X}_t = \boldsymbol{V}^{(1)} + \epsilon \boldsymbol{V}^{(2)} + \epsilon^2 \boldsymbol{V}^{(3)} + \cdots = \sum_{n=1}^{\infty} \epsilon^{n-1} \boldsymbol{V}^{(n)} . \tag{2.5}$$

Here the coefficient of each term is taken to be an integral power of some constant ϵ. This infinite summation is rather formal.

By virtue of the recursion relation (2.4), the resulting equation is expressed in a compact form:

$$\boldsymbol{X}_t = \boldsymbol{X}_s \times \boldsymbol{X}_{ss} - \epsilon \boldsymbol{X}_s \times \boldsymbol{X}_{ts} + \mathcal{T} \boldsymbol{X}_s , \tag{2.6}$$

where

$$\mathcal{T} = \frac{1}{2} \epsilon \boldsymbol{X}_t \cdot \boldsymbol{X}_t + c(t) , \tag{2.7}$$

with $c(t)$ being an arbitrary real function of t, and the condition $\boldsymbol{X}_s \cdot \boldsymbol{X}_s = 1$ is to be kept in view. The derivation of (2.7) is straightforward; we first differentiate the both sides of (2.6) with respect to s, and thereafter take the inner product with $\boldsymbol{X}_s$. Using (2.6) again, we have $\mathcal{T}_s = \epsilon \boldsymbol{X}_{st} \cdot \boldsymbol{X}_t$, from which (2.7) follows.

At first sight, the second term on the RHS of (2.6) appears to be a small perturbation to the LIA. However, it predominates in the time evolution in the sense that the first term is absorbed into the second one simply by the change of a variable $s \to s - t/\epsilon$. It deserves mention that this structure is accommodated in the equation derived by Moore & Saffman (1972) for the motion of a vortex filament with axial flow in the core.

It is illuminating to rewrite (2.6) into an alternative form. By taking the exterior product with $\boldsymbol{X}_s$, (2.6) is converted into

$$\boldsymbol{X}_s \times \boldsymbol{X}_t = -\boldsymbol{X}_{ss} + \epsilon \boldsymbol{X}_{st} . \tag{2.8}$$

Introduction of the new variables

$$\zeta = s, \quad \eta = 2t/\epsilon + s, \tag{2.9}$$

rewrites (2.8) into

$$\boldsymbol{X}_{\zeta\zeta} - \boldsymbol{X}_{\eta\eta} = -\frac{2}{\epsilon} \boldsymbol{X}_\zeta \times \boldsymbol{X}_\eta . \tag{2.10}$$

This equation is supplemented with two auxiliary conditions:

$$\boldsymbol{X}_\zeta^2 + \boldsymbol{X}_\eta^2 = \boldsymbol{X}_s^2 - \epsilon \boldsymbol{X}_s \cdot \boldsymbol{X}_t + \frac{\epsilon^2}{2} \boldsymbol{X}_t^2 = 1 - \epsilon c(t) , \tag{2.11}$$

$$\boldsymbol{X}_\zeta \cdot \boldsymbol{X}_\eta = \frac{\epsilon}{2} \boldsymbol{X}_t \cdot \left(\boldsymbol{X}_s - \frac{\epsilon}{2} \boldsymbol{X}_t \right) = \frac{\epsilon}{2} c(t) . \tag{2.12}$$

When $c(t) = 0$, (2.10)–(2.12) are no other than the Lund-Regge equation (Lund & Regge 1976). It was born as a byproduct of a unified theory of the Nambu string, a relativistic string, and the classical vortex filament. Fukumoto & Miyajima (1996) constructed a whole class of the traveling wave solution of (2.6), which will be touched on in §2.3. Konno & Kakuhata (1999) clarified that the equivalence between (2.6) and (2.10) is rather restrictive in that this is limited to this traveling wave solution.

2.2 Intrinsic Equations

We deduce the intrinsic form of (2.6) or (2.8) along the line of Hasimoto's procedure. Let us introduce a complex vector $\boldsymbol{N}$ defined by

$$\boldsymbol{N} = (\boldsymbol{n} + i\boldsymbol{b}) e^{i \int^s \tau ds} . \tag{2.13}$$

The Frenet-Serret formulae then read

$$\boldsymbol{t}_s = -\frac{1}{2} \left(\psi^* \boldsymbol{N} + \psi \boldsymbol{N}^* \right) , \quad \boldsymbol{N}_s = -\psi \boldsymbol{t} . \tag{2.14}$$

Here the asterisk indicates complex conjugate and ψ is the Hasimoto map (1.7). Using the identities $\boldsymbol{N} \cdot \boldsymbol{N}^* = 2$, $\boldsymbol{N} \cdot \boldsymbol{N} = \boldsymbol{N} \cdot \boldsymbol{t} = \boldsymbol{N}^* \cdot \boldsymbol{t} = 0$, the time derivatives of $\boldsymbol{t}$ and $\boldsymbol{N}$ can be generally expressed, by making use of some real function R and some complex function γ, as

$$\boldsymbol{t}_t = -\frac{1}{2} \left(\gamma^* \boldsymbol{N} + \gamma \boldsymbol{N}^* \right) , \quad \boldsymbol{N}_t = iR\boldsymbol{N} + \gamma \boldsymbol{t} . \tag{2.15}$$

Differentiating (2.6) with respect to s, we get, after some algebra,

$$\gamma = -i\psi_s + i\epsilon\psi_t - \left(\frac{\epsilon}{2} \boldsymbol{X}_t \cdot \boldsymbol{X}_t - \epsilon R \right) \psi . \tag{2.16}$$

The integrability condition $\boldsymbol{N}_{st} = \boldsymbol{N}_{ts}$ (or $\boldsymbol{t}_{st} = \boldsymbol{t}_{ts}$) requires

$$\psi_t = -\gamma_s + iR\psi\,, \tag{2.17}$$

$$R_s = \frac{i}{2}(\gamma\psi^* - \gamma^*\psi)\,. \tag{2.18}$$

Plugging (2.16) into (2.18), we have

$$R_s = \frac{1}{2}|\psi|_s^2 - \frac{\epsilon}{2}|\psi|_t^2\,. \tag{2.19}$$

On the other hand, using the identity $\gamma = -\boldsymbol{t}_t \cdot \boldsymbol{N}$, (2.18) leads to

$$R_s = \boldsymbol{t}_t \cdot \kappa\boldsymbol{b} = \boldsymbol{X}_{st} \cdot \boldsymbol{X}_t\,, \tag{2.20}$$

the last equality coming from from (2.6) and its spatial derivative. Equation (2.20) helps to simplify (2.16). It turns out that we may ignore the integration constant in R, being an arbitrary real function of t, because it can be absorbed into the phase factor of ψ without affecting the curve dynamics. Substitution of (2.16) and (2.19) into (2.17) yields, with the help of (2.20),

$$\psi_t = i\left(\psi_{ss} + \frac{1}{2}|\psi|^2\psi\right) - i\epsilon\left(\psi_{st} + \frac{1}{2}\psi\int^s |\psi|_t^2\,\mathrm{d}s\right)\,. \tag{2.21}$$

In keeping with the procedure of infinite summation (2.5), the same equation is reached via use of the recursion operator associated with the NLS hierarchy.

Splitting (2.21) into the real and imaginary parts, we are left with

$$\kappa_t = -(2\kappa_s\tau + \kappa\tau_s) + \epsilon\left(\kappa_t\tau + \kappa\tau_t + \kappa_s\int^s \tau_t\,\mathrm{d}s\right)\,, \tag{2.22}$$

$$\int^s \tau_t\,\mathrm{d}s = \frac{\kappa_{ss}}{\kappa} - \tau^2 + \frac{\kappa^2}{2} - \epsilon\left(\frac{\kappa_{st}}{\kappa} - \tau\int^s \tau_t\,ds + \int^s \kappa\kappa_t\,\mathrm{d}s\right)\,. \tag{2.23}$$

In a special case, (2.22) and (2.23) are collapsed into the sine-Gordon equation. In terms of the variables $\hat{t} = t$ and $\hat{s} = s + t/\epsilon$, they read

$$\kappa_{\hat{t}} + \frac{1}{\epsilon}\kappa_{\hat{s}} = \epsilon\left(\kappa_{\hat{t}}\tau + \kappa\tau_{\hat{t}} + \kappa_{\hat{s}}\int^{\hat{s}} \tau_{\hat{t}}\,\mathrm{d}\hat{s}\right)\,, \tag{2.24}$$

$$\int^{\hat{s}} \tau_{\hat{t}}\,\mathrm{d}\hat{s} + \frac{1}{\epsilon}\tau = -\epsilon\left(\frac{\kappa_{\hat{s}\hat{t}}}{\kappa} - \tau\int^{\hat{s}} \tau_{\hat{t}}\,\mathrm{d}\hat{s} + \int^{\hat{s}} \kappa\kappa_{\hat{t}}\,\mathrm{d}\hat{s}\right)\,. \tag{2.25}$$

The integral of torsion in the definition of (1.7) is an indefinite integral, and therefore a constant is at our disposal. If we set $\tau = 1/\epsilon$, the first equation is identically satisfied with a choice of the integration constant in such a way that $\int^{\hat{s}} \tau_{\hat{t}}\,d\hat{s} = 1/\epsilon^2$. For definiteness, we restrict our attention to balanced asymptotically linear curves, that is, curves approaching straight lines at infinity symmetrically in both directions. Their curvature vanishes at infinity. Under this restriction, (2.25) becomes

$$\frac{\kappa_{\hat{s}\hat{t}}}{\kappa} + \frac{1}{2}\left(\int_{-\infty}^{\hat{s}} \kappa\kappa_{\hat{t}}\,\mathrm{d}\hat{s} - \int_{\hat{s}}^{\infty} \kappa\kappa_{\hat{t}}\,\mathrm{d}\hat{s}\right) = -\frac{1}{\epsilon^3}\,. \tag{2.26}$$

Following Nakayama *et al.* (1992), we define

$$\theta = \int_{-\infty}^{\hat{s}} \kappa \, \mathrm{d}\hat{s} \,, \tag{2.27}$$

and prescribe the temporal evolution of κ as

$$\kappa_{\hat{t}} = -\frac{1}{\epsilon^3} \sin\theta \,. \tag{2.28}$$

Substituting from (2.27) and (2.28) and noting from (2.28) that $\sin\theta \to 0$ as $\hat{s} \to \pm\infty$, we find that (2.26) holds true. The consistency of (2.27) with (2.28) gives rise to the sine-Gordon equation:

$$\theta_{\hat{s}\hat{t}} = -\frac{1}{\epsilon^3} \sin\theta \,. \tag{2.29}$$

2.3 Remarks on an exact solution

In this section, we have highlighted some aspects of the localised induction hierarchy that show up when the summation is extended to the infinite order. The recursion operator of the LIH renders it feasible. In the restricted case of the invariant forms mentioned below, the resulting equation is shown to be reducible to the Lund-Regge equation.

Our model possesses exact solutions of the same type as derived by Kida (1981), namely, the invariant forms of a filament steadily rotating and translating in the three-dimensional space (Fukumoto & Miyajima 1996). The shape remains unaltered from Kida's solution, but a profound difference makes its appearance in the movement. Given the shape, the traveling and rotating speeds are not uniquely determined. Instead, there are two kinds, one of which is inherited from the solution of the LIA. The other is novel, because the speeds diverge in the limit that the model equation tends to the LIA. The symmetry of the Lund-Regge equation with respect to the interchange of the parameters accounts for the existence of the new mode.

When we make a mathematical model to mimic natural phenomena, a common tactic is to invoke a perturbation-expansions technique. Usually, on account of difficulty, we cannot help truncating the expansions at a finite order in powers of a small parameter. However, it is probable that there are modes that cannot be captured without completing the expansions to the infinite order. The analysis of the traveling wave solution reveals that our model provides us with an example to illustrate the insufficiency of finite truncation.

The LIH is an endproduct of a genuinely mathematical extension of the LIA, but the second term $\boldsymbol{V}^{(2)}$ happens to have some relevance to the effect of axial flow. The realizability of the third $\boldsymbol{V}^{(3)}$ is the question to be addressed in the rest of paper.

3 Asymptotic development of the Biot-Savart law

A potential flow is an exact solution of the Euler equations (and the Navier-Stokes equations as well). This is the case for the Biot-Savart law outside the vortex tube, if the vorticity field in the core is compatible with the Euler equations. In this section, we develop a systematic method to calculate an asymptotic expansion of the Biot-Savart law for a slender vortex tube, accommodating the effect of finite thickness successively.

We shall demonstrate that a dominant correction to the traditional formula stems from a line of dipoles arranged on the core centerline $\boldsymbol{X}(\xi,t)$, with their axes in the binormal direction $\boldsymbol{b}(\xi,t)$ and their strength proportional to the local curvature $\kappa(\xi,t)$. Here ξ is a parameter along the centerline to be defined by (3.2).

An expression valid near the core is then deduced in §3.4. This is the inner limit of the outer solution and serves as the matching condition on the inner solution worked out in §4 and 5.

3.1 Vorticity field in terms of local cylindrical coordinates

Once that the vorticity $\boldsymbol{\omega}(\boldsymbol{x})$ is specified at every point of the space, the velocity $\boldsymbol{v}(\boldsymbol{x})$ of the fluid at a position $\boldsymbol{x}$ is uniquely determined by the Biot-Savart law. The leading-order part was provided by (1.1). However this is not sufficient for our purpose of going into higher orders, and hence we must come back to the full form:

$$\boldsymbol{v} = \nabla \times \boldsymbol{A}; \quad \boldsymbol{A}(\boldsymbol{x}) = \frac{1}{4\pi}\iiint \frac{\boldsymbol{\omega}(\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}\,\mathrm{d}V'. \tag{3.1}$$

The assumption that vorticity is localised in a slender tube-like region ensures the existence of the volume integral (3.1).

In order to evaluate (3.1) at positions near the core, it is expedient to introduce local coordinates $(\tilde{x},\tilde{y},\xi)$, or local cylindrical coordinates (r,φ,ξ), moving with the filament. Here ξ is a parameter along the central curve $\boldsymbol{X}$ of the vortex tube, defined so as to satisfy

$$\frac{\partial \boldsymbol{X}}{\partial t}(\xi,t)\cdot \boldsymbol{t}(\xi,t) = 0\,, \tag{3.2}$$

for the sake of simplicity. Given a point $\boldsymbol{x}$ sufficiently close to the core, there corresponds uniquely the nearest point $\boldsymbol{X}(\xi,t)$ on the centerline of filament. Then $\boldsymbol{x}$ is expressed, in terms of the spatial parameters and time t, as

$$\begin{aligned} \boldsymbol{x} &= \boldsymbol{X}(\xi,t) + \tilde{x}\boldsymbol{n}(\xi,t) + \tilde{y}\boldsymbol{b}(\xi,t) && (3.3)\\ &= \boldsymbol{X} + r\cos\varphi\boldsymbol{n} + r\sin\varphi\boldsymbol{b}\,, && (3.4)\end{aligned}$$

where (r,φ) are cylindrical coordinates in the plane perpendicular to $\boldsymbol{t}(\xi,t)$, with the angle φ measured from the $\boldsymbol{n}$-axis. Inconveniently, (r,φ,ξ) do not constitute orthogonal coordinates. They are converted into orthogonal coordinates (r,θ,ξ) by adjusting the origin of angle, depending on torsion, as

$$\theta(\varphi,\xi,t) = \varphi - \int_{s_0}^{s(\xi,t)} \tau(s',t)\,\mathrm{d}s'\,, \tag{3.5}$$

where $s = s(\xi,t)$ is the arclength along the centerline.

We define the relative velocity $\boldsymbol{V} = (u(r,\theta,\xi,t), v(r,\theta,\xi,t), w(r,\theta,\xi,t))$ by

$$\boldsymbol{v} = \dot{\boldsymbol{X}}(\xi,t) + u\boldsymbol{e}_r + v\boldsymbol{e}_\theta + w\boldsymbol{t}\,, \tag{3.6}$$

where a dot stands for a derivative in t with fixing ξ, and $\boldsymbol{e}_r$ and $\boldsymbol{e}_\theta$ are the unit vectors in the radial and azimuthal directions respectively. The vorticity $\boldsymbol{\omega} = \nabla \times \boldsymbol{v}$ is calculated through

$$\begin{aligned}
\boldsymbol{\omega} &= \omega_r \boldsymbol{e}_r + \omega_\theta \boldsymbol{e}_\theta + \zeta \boldsymbol{t} \qquad (3.7)\\
&= \left\{ \frac{1}{r}\frac{\partial w}{\partial \theta} - \frac{1}{h_3}\frac{\partial v}{\partial \xi} + \frac{\eta}{h_3}\kappa w \sin\varphi - \frac{1}{h_3}\frac{\partial \dot{\boldsymbol{X}}}{\partial \xi}\cdot \boldsymbol{e}_\theta \right\} \boldsymbol{e}_r \\
&+ \left\{ -\frac{\partial w}{\partial r} + \frac{1}{h_3}\frac{\partial u}{\partial \xi} + \frac{\eta}{h_3}\kappa w \cos\varphi + \frac{1}{h_3}\frac{\partial \dot{\boldsymbol{X}}}{\partial \xi}\cdot \boldsymbol{e}_r \right\} \boldsymbol{e}_\theta + \left\{ \frac{1}{r}\frac{\partial}{\partial r}(rv) - \frac{1}{r}\frac{\partial u}{\partial \theta} \right\} \boldsymbol{t}\,, \qquad (3.8)
\end{aligned}$$

where

$$\eta = \left| \frac{\partial \boldsymbol{X}}{\partial \xi} \right|, \tag{3.9}$$

$$h_3 = \eta(1 - \kappa r \cos\varphi)\,. \tag{3.10}$$

We are concerned with a *'quasi-steady'* motion of a vortex filament. In our setting, the leading-order flow field consists only of circulatory motion with circular symmetry, and accordingly we pose the following form for the perturbation solution in a power series in $\epsilon = \sigma_0/R_0$, the ratio of a typical core radius σ_0 to a typical curvature radius R_0:

$$u = \epsilon u^{(1)} + \epsilon^2 u^{(2)} + \epsilon^3 u^{(3)} + \cdots\,, \tag{3.11}$$

$$v = v^{(0)}(r) + \epsilon v^{(1)} + \epsilon^2 v^{(2)} + \epsilon^3 v^{(3)} + \cdots\,, \tag{3.12}$$

$$w = \epsilon w^{(1)} + \epsilon^2 w^{(2)} + \cdots\,, \tag{3.13}$$

$$\dot{\boldsymbol{X}} = \dot{\boldsymbol{X}}^{(0)} + \epsilon \dot{\boldsymbol{X}}^{(1)} + \epsilon^2 \dot{\boldsymbol{X}}^{(2)} + \cdots\,. \tag{3.14}$$

As will be stated in the beginning of §4, $\dot{\boldsymbol{X}}^{(0)}$ is looked upon as the first order. Further, an analysis of the inner expansion will tell us that $w^{(1)} = w^{(1)}(\xi,t)$, being independent of r and θ, is compatible with the Euler equations. By inspection from (3.8) and the form (3.11)–(3.14), we readily find that

$$\omega_r = \epsilon^2 \omega_r^{(2)} + \cdots\,, \tag{3.15}$$

$$\omega_\theta = \epsilon^2 \omega_\theta^{(2)} + \cdots\,, \tag{3.16}$$

$$\zeta = \zeta^{(0)}(r) + \epsilon \zeta^{(1)} + \epsilon^2 \zeta^{(2)} + \epsilon^3 \zeta^{(3)} + \cdots\,, \tag{3.17}$$

with

$$\zeta^{(0)} = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(rv^{(0)}\right)\,. \tag{3.18}$$

In accord with our intention, the vorticity is dominated by the tangential component.

First, in the following subsection (§3.2), we evaluate contribution to the Biot-Savart law from tangential vorticity $\zeta\boldsymbol{t}$ and after that in §3.3, an evaluation of contribution from transversal vorticity $\omega_r \boldsymbol{e}_r + \omega_\theta \boldsymbol{e}_\theta$ follows.

3.2 Contribution of tangential vorticity

We denote the vector potential induced by the tangential vorticity component

$$\boldsymbol{\omega}_{\|} = \zeta(\tilde{x}, \tilde{y}, \xi, t)\boldsymbol{t}(\xi, t)\,, \tag{3.19}$$

by $\boldsymbol{A}_{\|}$. We stipulate that $|\zeta|$ decays sufficiently rapidly to zero with distance r from the vortex centerline.

In the kinematical treatment, we may make dependence on t implicit, and, as to the parameter s or ξ along the filament, we shall use whichever seems more convenient. Noting that the Jacobian for the coordinate transformation to $(\tilde{x}, \tilde{y}, s)$ is $1 - \kappa\tilde{x}$, we have

$$\boldsymbol{A}_{\|}(\boldsymbol{x}) = \frac{1}{4\pi}\iiint \zeta(\tilde{x}, \tilde{y})\frac{\boldsymbol{t}(s)}{|\boldsymbol{x} - \boldsymbol{X} - \tilde{x}\boldsymbol{n} - \tilde{y}\boldsymbol{b}|}(1 - \kappa\tilde{x})\,\mathrm{d}\tilde{x}\mathrm{d}\tilde{y}\mathrm{d}s\,. \tag{3.20}$$

This expression is legitimate only when

$$1 - \kappa\tilde{x} > 0\,. \tag{3.21}$$

This condition is met when ζ is negligibly small outside a slender tube-like region with thickness much shorter than the curvature radius $1/\kappa_0$.

Use of a shift-operator, being adapted from Dyson's technique (Dyson 1893), facilitates to rewrites (3.20) in a form amenable to a multi-pole expansion:

$$\boldsymbol{A}_{\|}(\boldsymbol{x}) = \frac{1}{4\pi}\int \mathrm{d}s\left\{\iint \mathrm{d}\tilde{x}\mathrm{d}\tilde{y}\zeta(\tilde{x}, \tilde{y})(1 - \kappa\tilde{x})\,\exp\left[-\tilde{x}(\boldsymbol{n}\cdot\nabla) - \tilde{y}(\boldsymbol{b}\cdot\nabla)\right]\right\}\frac{\boldsymbol{t}(s)}{|\boldsymbol{x} - \boldsymbol{X}(s)|}\,. \tag{3.22}$$

The exponential function is formally expanded in powers of $\tilde{x}$ and $\tilde{y}$ as

$$\begin{aligned}\boldsymbol{A}_{\|}(\boldsymbol{x}) \;=\; & \frac{1}{4\pi}\int \mathrm{d}s\Big\{\iint \mathrm{d}\tilde{x}\mathrm{d}\tilde{y}\zeta(\tilde{x}, \tilde{y})\Big(1 - \kappa\tilde{x} - \tilde{x}(\boldsymbol{n}\cdot\nabla) - \tilde{y}(\boldsymbol{b}\cdot\nabla)\\ & + \frac{1}{2}\left[\tilde{x}^2(\boldsymbol{n}\cdot\nabla)^2 + 2\tilde{x}\tilde{y}(\boldsymbol{n}\cdot\nabla)(\boldsymbol{b}\cdot\nabla) + \tilde{y}^2(\boldsymbol{b}\cdot\nabla)^2\right] + \kappa\tilde{x}^2(\boldsymbol{n}\cdot\nabla)\\ & + \kappa\tilde{x}\tilde{y}(\boldsymbol{b}\cdot\nabla) + \cdots\Big)\Big\}\frac{\boldsymbol{t}(s)}{|\boldsymbol{x} - \boldsymbol{X}(s)|}\,.\end{aligned} \tag{3.23}$$

We shall know from the inner expansion in §4 and 5 that, in accordance with the solution of a vortex ring, the axial component ζ of vorticity has the following dependence on the local azimuthal coordinate φ:

$$\zeta(\tilde{x}, \tilde{y}) = \zeta_0(r) + \zeta_{11}(r, \xi, t)\cos\varphi + \zeta_{12}(r, \xi, t)\sin\varphi + \zeta_{21}(r, \xi, t)\cos 2\varphi + \cdots\,, \tag{3.24}$$

where

$$\zeta_0 = \zeta^{(0)}(r) + \kappa^2\hat{\zeta}_0^{(2)}(r, \xi, t) + \cdots\,, \tag{3.25}$$

$$\zeta_{11} = \kappa\hat{\zeta}_{11}^{(1)}(r) + \kappa^3\hat{\zeta}_{11}^{(3)}(r, \xi, t) + \cdots\,, \tag{3.26}$$

$$\zeta_{12} = \kappa\hat{\zeta}_{12}^{(1)}(r) + \cdots\,, \tag{3.27}$$

$$\zeta_{21} = \kappa^2\hat{\zeta}_{21}^{(2)}(r, \xi, t) + \cdots\,. \tag{3.28}$$

In $\hat{\zeta}_{ij}^{(k)}$, the superscript k stands for order of perturbation, and i labels the Fourier mode with $j = 1$ and 2 being corresponding to $\cos i\theta$ and $\sin i\theta$ respectively. It will be shown that our assumptions of *a slender tube* and *quasi-steady motion* permit $\zeta^{(0)}$, $\hat{\zeta}_{11}^{(1)}$ and $\hat{\zeta}_{12}^{(1)}$ to be uniform along the filament.

Substituting (3.24)–(3.28) into (3.23), we get the first two terms, $\boldsymbol{A}_m$ and $\boldsymbol{A}_{\parallel d}$, of a multi-pole expansion of $\boldsymbol{A}_{\parallel}$, as

$$\boldsymbol{A}_{\parallel}(\boldsymbol{x}) = \boldsymbol{A}_m(\boldsymbol{x}) + \boldsymbol{A}_{\parallel d}(\boldsymbol{x}) + \cdots , \tag{3.29}$$

where

$$\boldsymbol{A}_m(\boldsymbol{x}) = \frac{\Gamma}{4\pi}\int \frac{\boldsymbol{t}(s)}{|\boldsymbol{x} - \boldsymbol{X}(s)|}\mathrm{d}s , \tag{3.30}$$

with

$$\Gamma = 2\pi \int_0^\infty r\zeta^{(0)}(r)\mathrm{d}r , \tag{3.31}$$

and

$$\boldsymbol{A}_{\parallel d}(\boldsymbol{x}) = \frac{1}{4}\int \mathrm{d}s \int_0^\infty \mathrm{d}r\left\{-r^3\zeta^{(0)}\left[\frac{1}{2}(\boldsymbol{t}\cdot\nabla)^2 - \kappa(\boldsymbol{n}\cdot\nabla)\right] - r^2\hat{\zeta}_{11}^{(1)}\left[(\boldsymbol{n}\cdot\nabla) + \kappa\right]\right\}\frac{\boldsymbol{t}(s)}{|\boldsymbol{x} - \boldsymbol{X}(s)|} . \tag{3.32}$$

In deriving (3.32), we have invoked

$$\nabla^2 = (\boldsymbol{t}\cdot\nabla)^2 + (\boldsymbol{n}\cdot\nabla)^2 + (\boldsymbol{b}\cdot\nabla)^2 , \tag{3.33}$$

and

$$\nabla^2\frac{1}{|\boldsymbol{x} - \boldsymbol{X}(s)|} = -4\pi\delta\left(\boldsymbol{x} - \boldsymbol{X}(s)\right) , \tag{3.34}$$

where δ is Dirac's delta function and vanishes outside the core. Using

$$(\boldsymbol{t}\cdot\nabla)\frac{1}{|\boldsymbol{x} - \boldsymbol{X}(s)|} = -\frac{\partial}{\partial s}\frac{1}{|\boldsymbol{x} - \boldsymbol{X}(s)|} , \tag{3.35}$$

(3.32) is further simplified, by a repetition of partial integration, to

$$\begin{aligned}\boldsymbol{A}_{\parallel d}(\boldsymbol{x}) &= -\frac{1}{16\pi}[2\pi\int_0^\infty r^3\zeta^{(0)}\mathrm{d}r]\int\frac{\kappa_s\boldsymbol{n} + \kappa\tau\boldsymbol{b}}{|\boldsymbol{x} - \boldsymbol{X}(s)|}\mathrm{d}s \\ &\quad -\frac{d^{(1)}}{2}\int \mathrm{d}s[\kappa(\boldsymbol{n}\cdot\nabla) + \kappa^2]\frac{\boldsymbol{t}}{|\boldsymbol{x} - \boldsymbol{X}(s)|} ,\end{aligned} \tag{3.36}$$

where

$$d^{(1)} = \frac{1}{4\pi}\left\{[2\pi\int_0^\infty r^2\hat{\zeta}_{11}^{(1)}\mathrm{d}r] - \frac{1}{2}[2\pi\int_0^\infty r^3\zeta^{(0)}\mathrm{d}r]\right\} , \tag{3.37}$$

is the strength of dipole. This is constant in ξ in accord with $\zeta^{(0)}$ and $\hat{\zeta}_{11}^{(1)}$.

The first term $\boldsymbol{A}_m$ in (3.29) pertains to a flow field induced by a curved vortex line of infinitesimal thickness, and is called the '*monopole field*'. The correction term $\boldsymbol{A}_{\parallel d}$ corresponds to a part of the flow field induced by *a line of dipoles*, based at the vortex centerline, with their axes oriented in the binormal direction. The origin of dipole field is attributable to the curvature effect; by bending the vortex tube, the vortex lines on the convex side are stretched, while those on the concave side are contracted, producing effectively a vortex pair (Fukumoto & Moffatt 2000). The flow field associated with this pair is equivalent to the above dipole field augmented by the contribution from the vorticity lying in the cross-section. The latter is elaborated in the following subsection.

3.3 Contribution of transversal vorticity

The components of vorticity perpendicular to $\boldsymbol{t}$

$$\boldsymbol{\omega}_\perp = \omega_r \boldsymbol{e}_r + \omega_\theta \boldsymbol{e}_\theta \,, \tag{3.38}$$

makes its appearance at $O(\epsilon^2)$. In view of (3.8), the second-order terms $\omega_r^{(2)}$ and $\omega_\theta^{(2)}$ are expressible, in terms of the streamfunction at $O(\epsilon)$, as

$$\omega_r^{(2)} = \hat{\omega}_r^{(2)}(r)(\kappa_s \cos\varphi + \kappa\tau \sin\varphi) + \hat{\omega}_{r0}^{(2)}(r) \,, \tag{3.39}$$

$$\omega_\theta^{(2)} = \hat{\omega}_\theta^{(2)}(r)(\kappa\tau \cos\varphi - \kappa_s \sin\varphi) + \hat{\omega}_{\theta 0}^{(2)}(r) \,, \tag{3.40}$$

where

$$\hat{\omega}_r^{(2)} = \frac{\zeta^{(0)}}{v^{(0)}} \hat{\psi}_{11}^{(1)} \,, \tag{3.41}$$

$$\hat{\omega}_\theta^{(2)} = \frac{r\zeta^{(0)}}{v^{(0)}} \left[\left(\frac{2}{r} - \frac{\zeta^{(0)}}{v^{(0)}} \right) \hat{\psi}_{11}^{(1)} + \frac{\partial \hat{\psi}_{11}^{(1)}}{\partial r} - r v^{(0)} \right] , \tag{3.42}$$

and $\hat{\psi}_{11}^{(1)}$ will be defined as a solution of (4.5) in §4. The axisymmetric parts $\hat{\omega}_{r0}^{(2)}$ and $\hat{\omega}_{\theta 0}^{(2)}$ do not affect the flow field at $O(\epsilon^2)$, and thus their detail is left untouched.

Since $\boldsymbol{\omega}_\perp = O(\epsilon^2)$, the vector potential $\boldsymbol{A}_\perp$ associated with it is, to $O(\epsilon^2)$,

$$\boldsymbol{A}_\perp(\boldsymbol{x}) = \frac{1}{4\pi} \int \frac{\mathrm{d}s}{|\boldsymbol{x} - \boldsymbol{X}(s)|} \left[\iint \boldsymbol{\omega}_\perp(\tilde{x}, \tilde{y}, s) \mathrm{d}\tilde{x} \mathrm{d}\tilde{y} \right] . \tag{3.43}$$

Substituting from (3.39)–(3.40),

$$\iint \boldsymbol{\omega}_\perp(\tilde{x}, \tilde{y}, s) \mathrm{d}\tilde{x} \mathrm{d}\tilde{y} = \left[\pi \int_0^\infty r \left(\hat{\omega}_r^{(2)} + \hat{\omega}_\theta^{(2)} \right) \mathrm{d}r \right] (\kappa_s \boldsymbol{n} + \kappa\tau \boldsymbol{b}) \,. \tag{3.44}$$

Equation (4.5) helps to simplify the coefficient to

$$r \left(\hat{\omega}_r^{(2)} + \hat{\omega}_\theta^{(2)} \right) = -r^2 \left(a \hat{\psi}_{11}^{(1)} + r\zeta^{(0)} \right) + \frac{\partial}{\partial r} \left(\frac{r^3 \zeta^{(0)}}{r v^{(0)}} \hat{\psi}_{11}^{(1)} \right) , \tag{3.45}$$

and, upon integration, we are left only with

$$\int_0^\infty r \left(\hat{\omega}_r^{(2)} + \hat{\omega}_\theta^{(2)} \right) \mathrm{d}r = \int_0^\infty r^2 \hat{\zeta}_{11}^{(1)} \mathrm{d}r \,. \tag{3.46}$$

where we have taken advantage of the expression (4.15) for vorticity $\hat{\zeta}_{11}^{(1)}$ at $O(\epsilon)$. Eventually, (3.43) is reduced to

$$\boldsymbol{A}_{\perp}(\boldsymbol{x}) = \frac{1}{4}\left[\int_0^\infty r^2\hat{\zeta}_{11}^{(1)}\mathrm{d}r\right]\int \frac{\kappa_s(s)\boldsymbol{n}(s) + \kappa(s)\tau(s)\boldsymbol{b}(s)}{|\boldsymbol{x} - \boldsymbol{X}(s)|}\mathrm{d}s\,, \tag{3.47}$$

a counterpart of the *dipole field* originating from the transversal vorticity.

Collecting (3.30), (3.36) and (3.47) gives rise to the first two components of a multipole expansion of the Biot-Savart law:

$$\boldsymbol{A}(\boldsymbol{x}) \approx \boldsymbol{A}_{\parallel}(\boldsymbol{x}) + \boldsymbol{A}_{\perp}(\boldsymbol{x}) \tag{3.48}$$

$$= \frac{\Gamma}{4\pi}\int \frac{\boldsymbol{t}(s)}{|\boldsymbol{x} - \boldsymbol{X}(s)|}\mathrm{d}s - \frac{d^{(1)}}{2}\int \frac{\kappa(s)\boldsymbol{b}(s) \times (\boldsymbol{x} - \boldsymbol{X}(s))}{|\boldsymbol{x} - \boldsymbol{X}(s)|^3}\mathrm{d}s\,. \tag{3.49}$$

It is informative to provide the form of expansion for velocity field $\boldsymbol{v}(\boldsymbol{x})$ by taking curl of (3.49):

$$\begin{aligned}\boldsymbol{v}(\boldsymbol{x}) \approx\; & -\frac{\Gamma}{4\pi}\int \frac{(\boldsymbol{x} - \boldsymbol{X}(s)) \times \boldsymbol{t}(s)}{|\boldsymbol{x} - \boldsymbol{X}(s)|^3}\mathrm{d}s \\ & +\frac{d^{(1)}}{2}\int\left\{\frac{\kappa(s)\boldsymbol{b}(s)}{|\boldsymbol{x} - \boldsymbol{X}(s)|^3} - \frac{3\kappa(s)\boldsymbol{b}(s)\cdot[\boldsymbol{x} - \boldsymbol{X}(s)]}{|\boldsymbol{x} - \boldsymbol{X}(s)|^5}[\boldsymbol{x} - \boldsymbol{X}(s)]\right\}\mathrm{d}s\,. \end{aligned} \tag{3.50}$$

The structure of dipole field manifests itself in the second integral.

3.4 Inner limit of the Biot-Savart law

We shall manipulate the limiting form of (3.49) as the vortical core is approached. We deal exclusively with vortex tubes whose centerlines $\boldsymbol{X} = \boldsymbol{X}(\xi, t)$ are *closed curves of finite length L*. A similar treatment can be available for filaments extending to infinity. We rely on an asymptotic method contrived by Margerit (1998). We describe its outline in Appendix A, and are contented with the resulting expressions for the asymptotic expansions.

For the sake of clarity, we choose the arcwise parameter at the point under consideration to be $s = \xi = 0$ and attach suffix 0 to the quantities at this point. We write

$$\boldsymbol{x}_0 = \boldsymbol{X}_0 + r\cos\varphi\boldsymbol{n}_0 + r\sin\varphi\boldsymbol{b}_0\,, \tag{3.51}$$

with $\boldsymbol{X}_0 = \boldsymbol{X}(0)$ and similarly for $\boldsymbol{n}_0$ and $\boldsymbol{b}_0$.

Putting together (A.8) and (A.14), we obtain the inner limit of the monopole component $\boldsymbol{A}_m(\boldsymbol{x}_0)$ defined by (3.30):

$$\begin{aligned}\frac{4\pi}{\Gamma}\boldsymbol{A}^m(\boldsymbol{x}_0) =\; & 2\log\left(\frac{L}{r}\right)\boldsymbol{t}_0 + \kappa_0\boldsymbol{t}_0\left[\log\left(\frac{L}{r}\right) - 1\right] r\cos\varphi \\ & +\left\{\kappa_0^2\boldsymbol{t}_0\left(\frac{2}{\kappa_0^2L^2} + \frac{3}{4}\left[\log\left(\frac{L}{r}\right) - \frac{31}{36}\right] + \frac{3}{8}\left[\log\left(\frac{L}{r}\right) - \frac{4}{3}\right]\cos 2\varphi\right)\right.\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{2}(\kappa_{0s}\boldsymbol{n}_0+\kappa_0\tau_0\boldsymbol{b}_0)\left[\log\left(\frac{L}{r}\right)-\frac{1}{2}\right]\Bigg\}r^2\\
&+\Bigg\{\kappa_0^3\boldsymbol{t}_0\left(\frac{3}{\kappa_0^2L^2}\cos\varphi+\frac{33}{32}\left[\log\left(\frac{L}{r}\right)-\frac{12}{11}\right]\cos\varphi+\frac{5}{32}\left[\log\left(\frac{L}{r}\right)-\frac{23}{15}\right]\cos 3\varphi\right)\\
&-\left(\frac{1}{8}\left[(\kappa_{0ss}-\kappa_0\tau_0^2)\cos\varphi+(2\kappa_{0s}\tau_0+\kappa_0\tau_{0s})\sin\varphi\right]\boldsymbol{t}_0\right.\\
&\left.+\left[\frac{5\kappa_0\kappa_{0s}}{4}\cos\varphi+\frac{\kappa_0^2\tau_0}{2}\sin\varphi\right]\boldsymbol{n}_0+\frac{3\kappa_0^2\tau_0}{4}\cos\varphi\boldsymbol{b}_0\right)\left[\log\left(\frac{L}{r}\right)-\frac{5}{6}\right]\Bigg\}r^3\\
&+\boldsymbol{Q}_m+O\left(\frac{r^5}{R_0^3\alpha^2},\frac{r^4}{\alpha^4}\right)\,. \qquad (3.52)
\end{aligned}
$$

Here $\boldsymbol{Q}_m$ designates regularized integrals expressed in the form of a series in ϵ. To (ϵ^2), we have

$$
\boldsymbol{Q}_m=\boldsymbol{Q}^{(0)}+r\left(\boldsymbol{Q}_{11}^{(1)}\cos\varphi+\boldsymbol{Q}_{12}^{(1)}\sin\varphi\right)+r^2\left(\boldsymbol{Q}_0^{(2)}+\boldsymbol{Q}_{21}^{(2)}\cos 2\varphi+\boldsymbol{Q}_{22}^{(2)}\sin 2\varphi\right)+\cdots\,. \qquad (3.53)
$$

Each term is an integral over the entire length of the filament and is represented, by use of an abbreviated notation,

$$
R_n=\boldsymbol{R}\cdot\boldsymbol{n}_0\,,\qquad R_b=\boldsymbol{R}\cdot\boldsymbol{b}_0\,, \qquad (3.54)
$$

as

$$
\begin{aligned}
\boldsymbol{Q}^{(0)}&=\frac{\Gamma}{4\pi}\oint\left(\frac{\boldsymbol{t}(s)}{R}-\frac{\boldsymbol{t}_0}{|s|}\right)\mathrm{d}s\,, && (3.55)\\
\boldsymbol{Q}_{11}^{(1)}&=\frac{\Gamma}{4\pi}\oint\left(\frac{R_n}{R^3}\boldsymbol{t}(s)-\frac{\kappa_0}{2|s|}\boldsymbol{t}_0\right)\mathrm{d}s\,, && (3.56)\\
\boldsymbol{Q}_{12}^{(1)}&=\frac{\Gamma}{4\pi}\oint\frac{R_b}{R^3}\boldsymbol{t}(s)\mathrm{d}s\,, && (3.57)\\
\boldsymbol{Q}_0^{(2)}&=\frac{3\Gamma}{8\pi}\oint\left(\frac{R_n^2+R_b^2}{2R^5}\boldsymbol{t}(s)-\frac{\kappa_0^2}{8|s|}\boldsymbol{t}_0\right)\mathrm{d}s-\frac{\Gamma}{8\pi}\oint\Bigg\{\frac{\boldsymbol{t}(s)}{R}-\frac{1}{|s|s^2}\Big[\boldsymbol{t}_0+\kappa_0\boldsymbol{n}_0s\\
&\quad+\frac{1}{2}\Big(-\frac{3}{4}\kappa_0^2\boldsymbol{t}_0+\kappa_{0s}\boldsymbol{n}_0+\kappa_0\tau_0\boldsymbol{b}_0\Big)s^2\Big]\Bigg\}\mathrm{d}s\,, && (3.58)\\
\boldsymbol{Q}_{21}^{(2)}&=\frac{3\Gamma}{8\pi}\oint\left(\frac{R_n^2+R_b^2}{2R^5}\boldsymbol{t}(s)-\frac{\kappa_0^2}{8|s|}\boldsymbol{t}_0\right)\mathrm{d}s\,, && (3.59)\\
\boldsymbol{Q}_{22}^{(2)}&=\frac{3\Gamma}{8\pi}\oint\frac{R_nR_b}{R^5}\boldsymbol{t}(s)\mathrm{d}s\,. && (3.60)
\end{aligned}
$$

In the same way, the second term of (3.49), the dipole field $\boldsymbol{A}_d$, is evaluated in the neighbourhood of the core. After some algebra, we get

$$
\frac{1}{d^{(1)}}\boldsymbol{A}_d(\boldsymbol{x}_0)\equiv\frac{1}{2}\int\frac{\kappa(s)\boldsymbol{b}(s)\times(\boldsymbol{x}-\boldsymbol{X}(s))}{|\boldsymbol{x}-\boldsymbol{X}(s)|^3}\mathrm{d}s \qquad (3.61)
$$

$$
\begin{aligned}
&= \kappa_0 \boldsymbol{t}_0 \frac{\cos\varphi}{r} + \kappa_0^2 \boldsymbol{t}_0 \left\{ -\frac{1}{2}\left[\log\left(\frac{L}{r}\right) + \frac{1}{2}\right] + \frac{1}{4}\cos 2\varphi \right\} + (\kappa_{0s}\boldsymbol{n}_0 + \kappa_0\tau_0\boldsymbol{b}_0)\log\left(\frac{L}{r}\right) \\
&\quad + \left\{ \kappa_0^3 \boldsymbol{t}_0 \left(-\frac{2}{\kappa_0^2 L^2}\cos\varphi - \frac{5}{8}\left[\log\left(\frac{L}{r}\right) - \frac{59}{60}\right]\cos\varphi + \frac{5}{32}\cos 3\varphi \right) \right. \\
&\qquad + \left(\frac{1}{2}\left[(\kappa_{0ss} - \kappa_0\tau_0^2)\cos\varphi + (2\kappa_{0s}\tau_0 + \kappa_0\tau_{0s})\sin\varphi \right] \boldsymbol{t}_0 \right. \\
&\qquad \left. \left. + \left[2\kappa_0\kappa_{0s}\cos\varphi + \kappa_0^2\tau_0\sin\varphi \right]\boldsymbol{n}_0 + \kappa_0^2\tau_0\cos\varphi\boldsymbol{b}_0 \right) \left[\log\left(\frac{L}{r}\right) - 1\right] \right\} r \\
&\quad + \boldsymbol{D}_0^{(2)}|_{s=0} + O\left(\frac{r^2}{R_0^2\alpha^2}, \frac{r^3}{R_0^3\alpha^2} \right) .
\end{aligned} \tag{3.62}
$$

The last integral $\boldsymbol{D}_0^{(2)}$ is the second-order part among nonlocal contributions and is written as

$$
\begin{aligned}
\boldsymbol{D}_0^{(2)} &= -\frac{d^{(1)}}{2} \oint \left(\kappa(s)\frac{\boldsymbol{n}(s)\cdot\boldsymbol{R}}{R^3}\boldsymbol{t}(s) + \frac{\kappa_0^2}{2|s|}\boldsymbol{t}_0 \right) \mathrm{d}s \\
&\quad - \frac{d^{(1)}}{2} \oint \left\{ \frac{\kappa(s)^2\boldsymbol{t}(s) - \kappa_s(s)\boldsymbol{n}(s) - \kappa(s)\tau(s)\boldsymbol{b}(s)}{R} - \left(\kappa_0^2\boldsymbol{t}_0 - \kappa_{0s}\boldsymbol{n}_0 - \kappa_0\tau_0\boldsymbol{b}_0\right)\frac{1}{|s|} \right\} \mathrm{d}s .
\end{aligned} \tag{3.63}
$$

In §4, we seek an asymptotic solution of the Euler equation in the inner region, subjected to the matching conditions derived above. To this end, it is advantageous to eliminate the pressure beforehand by introducing

$$
\psi(\boldsymbol{x}) = (1 - \kappa r\cos\varphi)\boldsymbol{A}(\boldsymbol{x})\cdot\boldsymbol{t}(\xi) , \tag{3.64}
$$

a three-dimensional extension of the Stokes streamfunction for an axisymmetric problem. The asymptotic expansions (3.52) and (3.62) are cast into the inner limit of ψ, valid for $\sigma_0 \ll r \ll R_0$, as

$$
\begin{aligned}
\psi(\boldsymbol{x}) &= \frac{\Gamma}{2\pi}\log\left(\frac{L}{r}\right) - \frac{\Gamma}{4\pi}\kappa\left[\log\left(\frac{L}{r}\right) + 1\right] r\cos\varphi + d^{(1)}\kappa\frac{\cos\varphi}{r} \\
&\quad + \frac{\Gamma}{4\pi}\kappa^2 \left\{ \frac{2}{\kappa^2 L^2} + \frac{1}{4}\left[\log\left(\frac{L}{r}\right) - \frac{7}{12}\right] - \frac{1}{8}\log\left(\frac{L}{r}\right)\cos 2\varphi \right\} r^2 \\
&\quad - \frac{d^{(1)}}{2}\kappa^2 \left\{ \log\left(\frac{L}{r}\right) + \frac{3}{2} + \frac{1}{2}\cos 2\varphi \right\} \\
&\quad + \frac{\Gamma}{4\pi} \left\{ \kappa^3 \left(\frac{1}{\kappa^2 L^2}\cos\varphi + \frac{3}{32}\left[\log\left(\frac{L}{r}\right) - \frac{22}{9}\right]\cos\varphi - \frac{1}{32}\left[\log\left(\frac{L}{r}\right) - \frac{1}{3}\right]\cos 3\varphi \right) \right. \\
&\qquad \left. - \frac{1}{8}\left[(\kappa_{ss} - \kappa\tau^2)\cos\varphi + (2\kappa_s\tau + \kappa\tau)\sin\varphi \right] \left[\log\left(\frac{L}{r}\right) - \frac{5}{6} \right] \right\} r^3 \\
&\quad + d^{(1)} \left\{ -\kappa^3 \left(\frac{2}{\kappa^2 L^2}\cos\varphi + \frac{1}{8}\left[\log\left(\frac{L}{r}\right) - \frac{71}{12}\right]\cos\varphi + \frac{1}{4}\cos 3\varphi \right) \right.
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2}\left[(\kappa_{ss}-\kappa\tau^2)\cos\varphi+(2\kappa_s\tau+\kappa\tau_s)\sin\varphi\right]\left[\log\left(\frac{L}{r}\right)-1\right]\Bigg\}r \\
&+Q_T^{(0)}+\left[\left(Q_{11T}^{(1)}-\kappa Q_T^{(0)}\right)\cos\varphi+Q_{12T}^{(1)}\sin\varphi\right]r \\
&+\left[Q_{0T}^{(2)}-\frac{\kappa}{2}Q_{11T}^{(1)}+\left(Q_{21T}^{(2)}-\frac{\kappa}{2}Q_{11T}^{(1)}\right)\cos 2\varphi+\left(Q_{22T}^{(2)}-\frac{\kappa}{2}Q_{12T}^{(1)}\right)\sin 2\varphi\right]r^2 \\
&+D_{0T}^{(2)}+\cdots, \qquad (3.65)
\end{aligned}
$$

where, to avoid confusion with a derivative in time t, the subscript T is used for representing the tangential component:

$$
Q_{ijT}^{(k)}(\xi)=\boldsymbol{Q}_{ij}^{(k)}(\xi)\cdot\boldsymbol{t}(\xi). \qquad (3.66)
$$

The nonlocal contributions $\boldsymbol{Q}_{ij}^{k}$ are not independent from each other. In the outer region, $\boldsymbol{A}_m$ is constrained by the conditions not only of the *Coulomb gauge* but also of *null vorticity*, namely

$$
\nabla\cdot\boldsymbol{A}_m=0, \qquad \nabla^2\boldsymbol{A}_m=\boldsymbol{0}. \qquad (3.67)
$$

Imposition of the first of (3.67) on the limiting form (3.52) of $\boldsymbol{A}_m$ brings in

$$
\begin{aligned}
\frac{1}{\eta}\frac{\partial Q_T^{(0)}}{\partial\xi} &= \kappa Q_n^{(0)}-\left(Q_{11n}^{(1)}+Q_{12b}^{(1)}\right), && (3.68)\\
\frac{1}{\eta}\frac{\partial Q_{11T}^{(1)}}{\partial\xi} &= \kappa\left(2Q_{11n}^{(1)}+Q_{12b}^{(1)}\right)+\tau Q_{12T}^{(1)}-2\left(Q_{0n}^{(2)}+Q_{21n}^{(2)}+Q_{22b}^{(2)}\right), && (3.69)\\
\frac{1}{\eta}\frac{\partial Q_{12T}^{(1)}}{\partial\xi} &= \kappa Q_{12n}^{(1)}-\tau Q_{11T}^{(1)}-2\left(Q_{0b}^{(2)}-Q_{21b}^{(2)}+Q_{22n}^{(2)}\right), && (3.70)
\end{aligned}
$$

where, as (3.66),

$$
Q_{ijn}^{(k)}=\boldsymbol{Q}_{ij}^{(k)}(\xi)\cdot\boldsymbol{n}(\xi), \qquad Q_{ijb}^{(k)}=\boldsymbol{Q}_{ij}^{(k)}(\xi)\cdot\boldsymbol{b}(\xi). \qquad (3.71)
$$

The second of (3.67) imposes

$$
\begin{aligned}
\left(\frac{1}{\eta}\frac{\partial}{\partial\xi}\right)^2 Q_n^{(0)} &= -\frac{3\Gamma}{4\pi}\kappa_s-4Q_{0n}^{(2)}+\kappa Q_{11n}^{(1)}+\frac{2\tau}{\eta}\frac{\partial Q_b^{(0)}}{\partial\xi}-\kappa_s Q_T^{(0)}-(\kappa^2-\tau^2)Q_n^{(0)} \\
&\quad +\tau_s Q_b^{(0)}+2\kappa\left(2Q_{11n}^{(1)}+Q_{12b}^{(1)}\right), \qquad (3.72)\\
\left(\frac{1}{\eta}\frac{\partial}{\partial\xi}\right)^2 Q_b^{(0)} &= -\frac{3\Gamma}{4\pi}\kappa\tau-4Q_{0b}^{(2)}+\kappa Q_{11b}^{(1)}-\frac{2\tau}{\eta}\frac{\partial Q_n^{(0)}}{\partial\xi}-\kappa\tau Q_T^{(0)}+\tau^2 Q_b^{(0)}-\tau_s Q_n^{(0)}. \\
&\qquad (3.73)
\end{aligned}
$$

The matching condition on the axial velocity w is obtainable by taking curl of (3.52) and (3.62), giving to $O(\epsilon^2)$,

$$
\begin{aligned}
w(\boldsymbol{x}) &= Q_{11T}^{(1)}-Q_{12T}^{(1)}-\frac{\Gamma}{4\pi}\left[\kappa\tau\cos\varphi-\kappa_s\sin\varphi\right]\left[\log\left(\frac{L}{r}\right)-1\right]r \\
&\quad +2\left[\left(Q_{0b}^{(2)}+Q_{21b}^{(2)}-Q_{22n}^{(2)}\right)\cos\varphi-\left(Q_{0n}^{(2)}-Q_{21n}^{(2)}-Q_{22b}^{(2)}\right)\sin\varphi\right]r \\
&\quad +\cdots. \qquad (3.74)
\end{aligned}
$$

4 Inner solution to second order

The inner solution is addressed by solving the Euler equations in the moving coordinates. We introduce the following dimensionless variables endowed with star:

$$\left.\begin{array}{l} r = \sigma_0 r^* , \quad \xi = R_0 \xi^* , \quad \boldsymbol{X} = R_0 \boldsymbol{X}^* , \quad \kappa = \frac{\kappa^*}{R_0} , \quad \tau = \frac{\tau^*}{R_0} , \quad t = \frac{R_0^2}{\Gamma} t^* , \\ (u, v, w) = \frac{\Gamma}{\sigma_0}(u^*, v^*, w^*) , \quad \psi = \Gamma R_0 \psi^* , \quad \dot{\boldsymbol{X}} = \frac{\Gamma}{R_0} \dot{\boldsymbol{X}}^* , \quad \frac{p}{\rho_f} = \left(\frac{\Gamma}{\sigma_0}\right)^2 \frac{p^*}{\rho_f^*} , \end{array}\right\} \tag{4.1}$$

where ρ_f is used for fluid density, and it should be remembered that σ_0 and R_0 signify measures of the core radius and the curvature radius, respectively. The symbol overdot implies partial differentiation in t with fixing r, θ and ξ. In Appendix B, we write down dimensionless form of the Euler equations and their curl, viewed from the moving coordinates (r, θ, ξ), along with the subsidiary relation that holds between ζ and ψ.

Recall our assumption that the leading-order flow consists only of the azimuthal component $v^{(0)}$ possessing both *rotational and translational symmetry* about the local central axis. We can confirm that this assumption is compatible with the Euler equations. In passing we remark that the axial symmetry of $v^{(0)}$ may not be an assumption but a necessary restriction on the solution, as proved, in the context of elliptic partial differential equations, by Caffarelli & Friedman (1980). The local stretching of vortex lines may enter only through dependence on t. The Euler equations are immediately integrated for the pressure at $O(\epsilon^0)$ as

$$p^{(0)} = \int_0^r \frac{[v^{(0)}(r')]^2}{r'} \mathrm{d}r' . \tag{4.2}$$

Going into higher orders, we are led to the form (3.11)–(3.14) of the inner expansions. The vorticity is expanded as (3.15)–(3.17). In harmony with these, the streamfunction ψ is expanded as

$$\psi = \psi^{(0)}(r,t) + \epsilon\psi^{(1)} + \epsilon^2\psi^{(2)} + \epsilon^3\psi^{(3)} + \cdots . \tag{4.3}$$

The solution at $O(\epsilon)$ is constructed in the following way. Let us set

$$\begin{aligned} \psi^{(1)} \; &= \; \left[\kappa\hat{\psi}_{11}^{(1)} + r\left(\frac{1}{\eta}\frac{\partial Q_n^{(0)}}{\partial \xi} - \tau Q_b^{(0)} - \dot{\boldsymbol{X}}^{(0)} \cdot \boldsymbol{b}\right)\right] \cos\varphi \\ &\quad + \left[\kappa\hat{\psi}_{12}^{(1)} + r\left(\frac{1}{\eta}\frac{\partial Q_b^{(0)}}{\partial \xi} + \tau Q_n^{(0)} + \dot{\boldsymbol{X}}^{(0)} \cdot \boldsymbol{n}\right)\right] \sin\varphi + \psi_0^{(1)} . \end{aligned} \tag{4.4}$$

The axisymmetric part $\psi_0^{(1)}$ plays little role on the movement at low orders. The functions $\hat{\psi}_{11}^{(1)}$ and $\hat{\psi}_{12}^{(1)}$ are determined by integrating the first-order part of the coupled system of (B.5) and (B.8) supplemented with (B.6) and (B.7):

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \left(\frac{1}{r^2} + a\right)\right] \hat{\psi}_{11}^{(1)} \; = \; v^{(0)} + 2r\zeta^{(0)} , \tag{4.5}$$

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \left(\frac{1}{r^2} + a\right)\right] \hat{\psi}_{12}^{(1)} \; = \; 0 , \tag{4.6}$$

where

$$a = \frac{1}{v^{(0)}}\frac{\partial \zeta^{(0)}}{\partial r}. \tag{4.7}$$

The solution, meeting the condition that the relative velocity $u^{(1)}$ and $v^{(1)}$ are finite at $r = 0$, is

$$\hat{\psi}_{11}^{(1)} = \Psi_{11}^{(1)} + c_{11}^{(1)} v^{(0)}, \tag{4.8}$$

$$\hat{\psi}_{12}^{(1)} = c_{12}^{(1)} v^{(0)}, \tag{4.9}$$

with

$$\Psi_{11}^{(1)} = v^{(0)} \left\{ \frac{r^2}{2} + \int_0^r \frac{\mathrm{d}r'}{r'[v^{(0)}(r')]^2} \int_0^{r'} r'' \left[v^{(0)}(r'')\right]^2 \mathrm{d}r'' \right\}, \tag{4.10}$$

and $c_{11}^{(1)}$ and $c_{12}^{(1)}$ are constants bearing with the freedoms of shifting the local origin $r = 0$ of the moving frame, in the ρ- and z-directions respectively, within an accuracy of $O(\epsilon)$ (Fukumoto & Moffatt 2000). The matching condition (3.65) then demands

$$\dot{\boldsymbol{X}}^{(0)} \cdot \boldsymbol{b} = \frac{\Gamma\kappa}{4\pi}\left[\log\left(\frac{L}{\epsilon}\right) + A + \frac{3}{2}\right] - Q_{11T}^{(1)} + \kappa Q_T^{(0)} + \frac{1}{\eta}\frac{\partial Q_n^{(0)}}{\partial \xi} - \tau Q_b^{(0)}, \tag{4.11}$$

$$\dot{\boldsymbol{X}}^{(0)} \cdot \boldsymbol{n} = Q_{12T}^{(1)} - \frac{1}{\eta}\frac{\partial Q_b^{(0)}}{\partial \xi} - \tau Q_n^{(0)}, \tag{4.12}$$

where the dimensional variables are recovered and

$$A = \lim_{r\to\infty}\left\{\frac{4\pi^2}{\Gamma^2}\int_0^r r'[v^{(0)}(r')]^2 \mathrm{d}r' - \log r\right\}. \tag{4.13}$$

This is an alternative form, being refashioned for our convenience, of the well-known result obtained by Widnall, Bliss & Zalay (1971) and Callegari & Ting (1978).

Integration of the $O(\epsilon)$-part of (B.5) couples the vorticity $\zeta^{(1)}$ at $O(\epsilon)$ with $\psi^{(1)}$ through

$$\zeta^{(1)} = \kappa\left(\hat{\zeta}_{11}^{(1)}\cos\varphi + \hat{\zeta}_{12}^{(1)}\sin\varphi\right) + \zeta_0^{(1)}(r), \tag{4.14}$$

where

$$\hat{\zeta}_{11}^{(1)} = -\left(a\hat{\psi}_{11}^{(1)} + r\zeta^{(0)}\right), \qquad \hat{\zeta}_{12}^{(1)} = -a\hat{\psi}_{12}^{(1)}, \tag{4.15}$$

and again we may forget $\zeta_0^{(1)}$. The axial velocity at $O(\epsilon)$

$$w^{(1)} = Q_{11b}^{(1)} - Q_{12n}^{(1)}, \tag{4.16}$$

a locally uniform flow, complies with both the matching condition (3.74) and the tangential component (B.3) of the Euler equations.

Subsequently we proceed to the second order. Fortunately an explicit form of $p^{(1)}$ is available by integrating the transversal components (B.1) and (B.2) of the Euler equations:

$$p^{(1)} = \kappa\left\{\left[v^{(0)}\frac{\partial\hat{\psi}_{11}^{(1)}}{\partial r} - \zeta^{(0)}\hat{\psi}_{11}^{(1)} - r(v^{(0)})^2\right]\cos\varphi + \left[v^{(0)}\frac{\partial\hat{\psi}_{12}^{(1)}}{\partial r} - \zeta^{(0)}\hat{\psi}_{12}^{(1)}\right]\sin\varphi\right\} + p_0^{(1)}. \tag{4.17}$$

The gradient of $p^{(1)}$, in turn, drives axial flow at $O(\epsilon^2)$. Discarding the irrelevant terms from the Euler equation (B.3), we are left with

$$-v^{(0)}(\boldsymbol{e}_\theta \cdot \dot{\boldsymbol{t}}^{(0)}) + \kappa w^{(1)} v^{(0)} \sin\varphi + \frac{v^{(0)}}{r}\frac{\partial w^{(2)}}{\partial \theta} = -\frac{1}{\eta}\frac{\partial p^{(1)}}{\partial \xi}. \qquad (4.18)$$

With the aid of (3.68)–(3.73), a derivative in t of (4.11) and (4.12) becomes

$$\begin{aligned}\boldsymbol{e}_\theta \cdot \dot{\boldsymbol{t}}^{(0)} &= \frac{\Gamma\kappa}{4\pi}\left[\log\left(\frac{L}{\epsilon}\right) + A + \frac{3}{2}\right](\kappa_s \cos\varphi + \kappa\tau\sin\varphi) - 2\left(Q_{0n}^{(2)} - Q_{21n}^{(2)} - Q_{22b}^{(2)}\right)\cos\varphi \\ &\quad - \left[\kappa\left(Q_{12n}^{(1)} - Q_{11b}^{(1)}\right) + 2\left(Q_{0b}^{(2)} + Q_{21b}^{(2)} - Q_{22n}^{(2)}\right)\right]\sin\varphi. \qquad (4.19)\end{aligned}$$

Equation (4.18), the subjected to the matching condition that $w^{(2)}$ approaches the $O(\epsilon^2)$-part of (3.74), admits a compact form of the solution for $w^{(2)}$ as

$$\begin{aligned} w^{(2)} &= \left\{-\frac{\Gamma\kappa}{4\pi}\left[\log\left(\frac{L}{\epsilon}\right) + A + \frac{3}{2}\right] + \left(\frac{\partial\hat{\psi}_{11}^{(1)}}{\partial r} - \frac{\zeta^{(0)}}{v^{(0)}}\hat{\psi}_{11}^{(1)} - r v^{(0)}\right)\right\} r(\kappa\tau\cos\varphi - \kappa_s\sin\varphi) \\ &\quad + \left(\frac{\partial\hat{\psi}_{12}^{(1)}}{\partial r} - \frac{\zeta^{(0)}}{v^{(0)}}\hat{\psi}_{12}^{(1)}\right) r(\kappa_s\cos\varphi + \kappa\tau\sin\varphi) + 2r\Big[\left(Q_{0b}^{(2)} + Q_{21b}^{(2)} - Q_{22n}^{(2)}\right)\cos\varphi \\ &\quad - \left(Q_{0n}^{(2)} - Q_{21n}^{(2)} - Q_{22b}^{(2)}\right)\sin\varphi\Big] + w_0^{(2)}(r,\xi,t), \qquad (4.20)\end{aligned}$$

where the last term $w_0^{(2)}(r,\xi,t)$ is the axisymmetric part yet undetermined at this level.

It follows from (4.20) that, for a curved vortex filament, torsion and arcwise variation of curvature are vital for the presence of pressure gradient and thus of nontrivial axial velocity at $O(\epsilon^2)$. Otherwise stated, a circular vortex ring alone is capable of being free from swirl.

The streamfunction $\psi^{(2)}$ at $O(\epsilon^2)$ for flow in the transversal plane is built in parallel with the case of a circular vortex ring (Fukumoto & Moffatt 2000). The speed at $O(\epsilon^2)$ that is compatible with the matching condition (3.65) is

$$\dot{\boldsymbol{X}}^{(1)} = \boldsymbol{0}. \qquad (4.21)$$

The detail is postponed to a full paper.

5 Higher-order localised induction approximation

We are now in a position to make headway to deduce the third-order velocity. At $O(\epsilon^3)$, the vorticity equation (B.5) in the axial direction gives

$$\begin{aligned}&\dot{\zeta}^{(1)} + \frac{1}{\sigma}\frac{\partial\zeta^{(1)}}{\partial\xi}w^{(1)} - \left(\dot{\boldsymbol{e}}_r^{(0)}\cdot\boldsymbol{e}_\theta\right)\frac{\partial\zeta^{(1)}}{\partial\theta} + \dot{T}\frac{\partial\zeta^{(1)}}{\partial\theta} \\ &\quad + \frac{v^{(1)}}{r}\frac{\partial\zeta^{(2)}}{\partial\theta} + u^{(2)}\frac{\partial\zeta^{(1)}}{\partial r} + \frac{v^{(2)}}{r}\frac{\partial\zeta^{(1)}}{\partial\theta} + u^{(1)}\frac{\partial\zeta^{(2)}}{\partial r} + + \frac{v^{(0)}}{r}\frac{\partial\zeta^{(3)}}{\partial\theta} + u^{(3)}\frac{\partial\zeta^{(0)}}{\partial r}\end{aligned}$$

$$
\begin{aligned}
= \quad & \kappa v^{(0)}\zeta^{(2)}\sin\varphi + \kappa\zeta^{(1)}(-u^{(1)}\cos\varphi + v^{(1)}\sin\varphi) \\
& +\kappa\zeta^{(0)}(-u^{(2)}\cos\varphi + v^{(2)}\sin\varphi) + \frac{\kappa^2}{2} r v^{(0)}\zeta^{(1)}\sin 2\varphi \\
& +\frac{\kappa^2}{2} r\zeta^{(0)}\left[-u^{(1)}(1+\cos 2\varphi) + v^{(1)}\sin 2\varphi\right] + \frac{\kappa^3}{4} r^2 v^{(0)}\zeta^{(0)}(\sin\varphi + \sin 3\varphi) \\
& +\frac{\zeta^{(0)}}{\eta}\frac{\partial w^{(2)}}{\partial \xi} + \frac{\zeta^{(1)}}{\eta}\frac{\partial w^{(1)}}{\partial \xi} + \left(\zeta^{(1)} + \kappa r\cos\varphi\zeta^{(0)}\right)\frac{1}{\eta}\frac{\partial \dot{\boldsymbol{X}}^{(0)}}{\partial \xi}\cdot \boldsymbol{t}\,, \qquad (5.1)
\end{aligned}
$$

where

$$
T(\xi, t) = \int_0^{s(\xi,t)} \tau(s', t)\mathrm{d}s'\,. \qquad (5.2)
$$

The third-order velocity $\dot{\boldsymbol{X}}^{(2)}$ under question is included in $\zeta^{(3)}$ and $u^{(3)}$ on the right hand side, and is determined by imposing the matching condition, the $O(\epsilon^3)$ terms of (3.65), on the solution of of (5.1). Relevant to the traveling speed is the terms proportional to $\cos\varphi$ and $\sin\varphi$, the dipole components.

Equation (5.1) has much in common with that for a circular vortex ring. The three-dimensional effect featured by τ, κ_s, κ_{ss}, *etc.*, which is missing in the axisymmetric case, makes its appearance only in the first few terms $\dot{\zeta}^{(1)}$, $(\partial\zeta^{(1)}/\partial\xi)w^{(1)}/\sigma$, $(\dot{\boldsymbol{e}}_r^{(0)}\cdot\boldsymbol{e}_\theta)\partial\zeta^{(1)}/\partial\theta$, $\dot{T}\partial\zeta^{(1)}/\partial\theta$, and in the last few terms including $w^{(1)}, w^{(2)}$ and $\partial\dot{\boldsymbol{X}}^{(0)}/\partial\xi$. The first term $\dot{\zeta}^{(1)}$ is , from (4.14),

$$
\dot{\zeta}^{(1)} = -\left(a\hat{\psi}_{11}^{(1)} + r\zeta^{(0)}\right)\left(\dot{\kappa}\cos\varphi + \kappa\dot{T}\sin\varphi\right)\,. \qquad (5.3)
$$

In order to gain an insight into the higher-order effect of local curvature and torsion, we appeal to the *localised induction approximation*, a kind of skeleton analysis. We dismiss the nonlocal-induction terms $\boldsymbol{Q}_m$ and $\boldsymbol{D}_0$, the integrals over the entire length. In addition, we throw away other terms except for those including $\log(L/\sigma)$. Following this procedure, the first-order translation speed $\dot{\boldsymbol{X}}^{(0)}$ becomes (1.3). Use of the Betchov-Da Rios equations (1.4) and (1.5) achieves a great simplification of the terms in (5.1) associated with the three-dimensional effect. For instance,

$$
\begin{aligned}
\dot{\boldsymbol{e}}_r^{(0)}\cdot\boldsymbol{e}_\theta &= -C\left(\tau^2 - \frac{\kappa_{ss}}{\kappa}\right)\,, \qquad (5.4) \\
w^{(1)} &= \frac{\partial\dot{\boldsymbol{X}}^{(0)}}{\partial\xi}\cdot\boldsymbol{t} = 0\,. \qquad (5.5)
\end{aligned}
$$

The terms with κ^3 are exactly the same as for an axisymmetric vortex ring, whose analysis was exhausted by Fukumoto & Moffatt (2000). Hence it suffices to concentrate our attention on the terms tied with torsion and non-constancy of curvature. The matching condition (3.65) at $O(\epsilon^3)$ reads, when only those terms in the $\cos\varphi$ and $\sin\varphi$ components are retained,

$$
\begin{aligned}
\psi^{(3)} \sim \; & \left(-\frac{3\Gamma}{32\pi}r^3 + \frac{d^{(1)}}{2}r\right)\log\left(\frac{L}{\epsilon r}\right)\left[(\kappa_{ss} - \kappa\tau^2)\cos\varphi + (2\kappa_s\tau + \kappa\tau_s)\sin\varphi\right] \\
& + \cdots \quad \text{as } r\to\infty\,. \qquad (5.6)
\end{aligned}
$$

Imposition of the matching condition (5.6) on (5.1) gives rise to, after some manipulation, the third-order correction $\dot{\boldsymbol{X}}^{(2)}$ to the traveling speed. Combining with (1.3), we eventually arrive at the evolution equation of a vortex filament in the LIA, which is expressed, in terms of dimensional variables, as

$$\frac{\partial \boldsymbol{X}}{\partial t} = C\Big\{\kappa \boldsymbol{b} + \Big[\frac{\pi}{\Gamma}\int_0^\infty \zeta^{(0)} r^3 \mathrm{d}r\Big]\Big[(2\kappa_s\tau + \kappa\tau_s)\boldsymbol{n} + (\kappa\tau^2 - \kappa_{ss})\boldsymbol{b} + \kappa^2\tau\boldsymbol{t}\Big] + C_b\kappa^3\boldsymbol{b}\Big\}, \quad (5.7)$$

$$C_b = 2\pi d^{(1)}/\Gamma\,. \quad (5.8)$$

Hasimoto map (1.7) transforms (5.7) into

$$i\frac{\partial\psi}{\partial t} + C\left(\psi_{ss} + \frac{1}{2}|\psi|^2\psi\right) + A(t)\psi - Cc_1\Big\{\psi_{ssss} + \frac{3}{2}\left(|\psi|^2\psi_{ss} + \psi_s^2\bar{\psi}\right)$$
$$+\left(\frac{3}{8}|\psi|^4 + \frac{1}{2}\frac{\partial^2}{\partial s^2}|\psi|^2\right)\psi\Big\} + C\left(C_b + \frac{c_1}{2}\right)\left\{\frac{\partial^2}{\partial s^2}(|\psi|^2\psi) + \frac{3}{4}|\psi|^4\psi\right\} = 0\,. \quad (5.9)$$

We see that the third-order correction closely resembles $\boldsymbol{V}^{(3)}$ given by (2.3), or its correspondent in the nonlinear Schröinger hierarchy. Still a discrepancy survives in the coefficient of $\kappa^3\boldsymbol{b}$. As discussed at length by Fukumoto & Moffatt (2000), the strength $d^{(1)}$ of dipole is sensitive to the location of the origin $r = 0$ of the moving coordinates; by a displacement of origin in the $\boldsymbol{n}$-direction by ϵx_0, measured in the inner variable,

$$d^{(1)} \to d^{(1)} + x_0/2\pi\,, \quad (5.10)$$

and thence

$$C_b \to C_b + x_0\,. \quad (5.11)$$

It is confirmed that C_b is adjustable to be coincident with (2.3), and that the local origin for this coincident case is indeed contained inside the core. We conclude that there is an *integrable line* that obeys a summation of the first and the third terms of the LIH.

This fact is illustrated by selecting a specific vorticity distribution at $O(\epsilon^0)$ of constant vorticity in the circular domain $r \le \sigma$ of radius σ surrounded by an irrotational flow. This is known as the *Rankine vortex*. The azimuthal velocity at $O(\epsilon^0)$ takes, in dimensionless variables,

$$v^{(0)} = \begin{cases} \dfrac{r}{2\pi}\,, & (r \le 1) \\ \dfrac{1}{2\pi r}\,. & (r > 1) \end{cases} \quad (5.12)$$

This vortex is one of a few examples that are tractable in explicit form in both inner and outer regions to a high order. The strength $d^{(1)}$ of dipole, defined by (3.37), is evaluated as

$$d^{(1)} = -\frac{3\Gamma\sigma^2}{32\pi}\,, \quad (5.13)$$

when the local origin $r = 0$ is sitting at the stagnation point relative to the moving frame. For a general value of x_0, (5.7) reduces to

$$\frac{\partial \boldsymbol{X}}{\partial t} = C\Big\{\kappa\boldsymbol{b} + \frac{\sigma^2}{4}\Big[(2\kappa_s\tau + \kappa\tau_s)\boldsymbol{n} + \left(\kappa\tau^2 - \kappa_{ss} + [1 + 4x_0]\kappa^3\right)\boldsymbol{b} + \kappa^2\tau\boldsymbol{t}\Big]\Big\}\,. \quad (5.14)$$

Recall that the choice $x_0 = -5/8$ corresponds to placing the origin $r = 0$ at the form center of circular core. Choice of $x_0 = -3/8$, a value between $x_0 = -5/8$ and 0, renders (5.14) completely integrable.

At present, we do not figure out the origin of this resemblance, but it indicates at least the structural stability of the Hasimoto soliton. A brief summary of §3–5 is given by Fukumoto (2001).

Appendix

A Asymptotic expansions of the Biot-Savart integral

In this appendix, the procedure for asymptotic evaluation is exemplified for the monopole field $\boldsymbol{A}_m$ defined by (3.30). Choose a constant α such that

$$\sigma_0 \ll \alpha \ll R_0 . \tag{A.1}$$

With this parameter, the integration range along one circuit of a closed filament is divided into two parts, the neighbouring and the far portions. Thus (3.30) is decomposed into two integrals as

$$\boldsymbol{A}_m(\boldsymbol{x}_0) = \boldsymbol{A}_m^N(\boldsymbol{x}_0) + \boldsymbol{A}_m^F(\boldsymbol{x}_0) , \tag{A.2}$$

each being defined by

$$\boldsymbol{A}_m^N(\boldsymbol{x}_0) = \frac{\Gamma}{4\pi}\int_{-\alpha}^{\alpha} \frac{\boldsymbol{t}(s)}{|\boldsymbol{x}_0 - \boldsymbol{X}(s)|} \mathrm{d}s , \tag{A.3}$$

$$\boldsymbol{A}_m^F(\boldsymbol{x}_0) = \frac{\Gamma}{4\pi}\int_{[2\alpha]} \frac{\boldsymbol{t}(s)}{|\boldsymbol{x}_0 - \boldsymbol{X}(s)|} \mathrm{d}s \equiv \frac{\Gamma}{4\pi}\left\{\int_{-L/2}^{-\alpha} \mathrm{d}s + \int_{\alpha}^{L/2} \mathrm{d}s\right\} \frac{\boldsymbol{t}(s)}{|\boldsymbol{x}_0 - \boldsymbol{X}(s)|} . \tag{A.4}$$

Here we have used, with some abuse of notation, L for total arclength of the closed centerline.

We are requested to inquire into an asymptotic evaluation of $\boldsymbol{A}_m(\boldsymbol{x}_0)$, valid for $\sigma_0 \ll r \ll \alpha$, at a point $\boldsymbol{x}_0$ represented by (3.51). We begin with (A.3). The integrand is expanded in powers of κ_0, τ_0 and their derivatives in s. It is to be noted that each derivative in s increases order of expansion by one. The numerator is, to $O(\epsilon^3)$,

$$\begin{aligned} \boldsymbol{t}(s) &= \boldsymbol{t}_0 + \kappa_0 \boldsymbol{n}_0 r s + \frac{1}{2}(-\kappa_0^2 \boldsymbol{t}_0 + \kappa_{0s}\boldsymbol{n}_0 + \kappa_0\tau_0\boldsymbol{b}_0) r^2 s^2 \\ &\quad + \frac{1}{6}\left[-3\kappa_0\kappa_{0s}\boldsymbol{t}_0 + (\kappa_{0ss} - \kappa_0\tau_0^2 - \kappa_0^3)\boldsymbol{n}_0 + (2\kappa_{0s}\tau_0 + \kappa_0\tau_{0s})\boldsymbol{b}_0\right] r^3 s^3 + \cdots , \end{aligned} \tag{A.5}$$

where, for example,

$$\kappa_{0s} = \left.\frac{\partial \kappa}{\partial s}\right|_{s=0} . \tag{A.6}$$

The denominator is, to the same order of accuracy,

$$
\begin{aligned}
|\boldsymbol{x}_0-\boldsymbol{X}_s|^{-1} = {}& \frac{1}{r(1+s^2)^{1/2}}\Bigg\{1+\frac{\kappa_0}{2}r\cos\varphi\frac{s^2}{1+s^2} \\
&+\frac{r^2}{6(1+s^2)}\left[(\kappa_{0s}\cos\varphi+\kappa_0\tau_0\sin\varphi)s^3+\frac{\kappa_0^2}{4}s^2\right]+\frac{3\kappa_0^2}{16}r^2(1+\cos 2\varphi)\frac{s^4}{(1+s^2)^2} \\
&+\frac{r^3}{24(1+s^2)}\left(\kappa_0\kappa_{0s}s^5+[(\kappa_{0ss}-\kappa_0\tau_0^2-\kappa_0^3)\cos\varphi+(2\kappa_{0s}\tau_0+\kappa_0\tau_{0s})\sin\varphi]s^4\right) \\
&+\frac{\kappa_0}{4}r^3\cos\varphi\frac{s^2}{(1+s^2)^2}\left[\frac{\kappa_0^2}{4}s^4+(\kappa_{0s}\cos\varphi+\kappa_0\tau_0\sin\varphi)s^3\right] \\
&+\frac{5\kappa_0^3}{64}r^3(3\cos\varphi+\cos 3\varphi)\frac{s^6}{(1+s^2)^3}+\cdots\Bigg\}. \qquad \text{(A.7)}
\end{aligned}
$$

With this form, the integration (A.3) is carried through, giving,

$$
\begin{aligned}
\frac{4\pi}{\Gamma}\boldsymbol{A}_m^N(\boldsymbol{x}_0) = {}& 2\log\left(\frac{2\alpha}{r}\right)\boldsymbol{t}_0+\kappa_0\boldsymbol{t}_0\left[\log\left(\frac{2\alpha}{r}\right)-1\right]r\cos\varphi \\
&+\Bigg\{\kappa_0^2\boldsymbol{t}_0\left(\frac{3}{4}\left[\log\left(\frac{2\alpha}{r}\right)-\frac{31}{36}\right]+\frac{3}{8}\left[\log\left(\frac{2\alpha}{r}\right)-\frac{4}{3}\right]\cos 2\varphi\right) \\
&\quad -\frac{1}{2}(\kappa_{0s}\boldsymbol{n}_0+\kappa_0\tau_0\boldsymbol{b}_0)\left[\log\left(\frac{2\alpha}{r}\right)-\frac{1}{2}\right]\Bigg\}r^2 \\
&+\frac{1}{2\alpha^2}\boldsymbol{t}_0r^2+\frac{1}{2}\left(-\frac{11}{12}\kappa_0^2\boldsymbol{t}_0+\kappa_{0s}\boldsymbol{n}_0+\kappa_0\tau_0\boldsymbol{b}_0\right)\alpha^2+\frac{3\kappa_0}{4\alpha^2}\boldsymbol{t}_0r^3\cos\varphi \\
&+\Bigg\{\kappa_0^3\boldsymbol{t}_0\left(\frac{33}{32}\left[\log\left(\frac{2\alpha}{r}\right)-\frac{12}{11}\right]\cos\varphi+\frac{5}{32}\left[\log\left(\frac{2\alpha}{r}\right)-\frac{23}{15}\right]\cos 3\varphi\right) \\
&\quad -\left(\frac{1}{8}\left[(\kappa_{0ss}-\kappa_0\tau_0^2)\cos\varphi+(2\kappa_{0s}\tau_0+\kappa_0\tau_{0s})\sin\varphi\right]\boldsymbol{t}_0\right. \\
&\quad \left.+\left[\frac{5\kappa_0\kappa_{0s}}{4}\cos\varphi+\frac{\kappa_0^2\tau_0}{2}\sin\varphi\right]\boldsymbol{n}_0+\frac{3\kappa_0^2\tau_0}{4}\cos\varphi\boldsymbol{b}_0\right)\left[\log\left(\frac{2\alpha}{r}\right)-\frac{5}{6}\right]\Bigg\}r^3 \\
&+\Bigg\{\frac{1}{24}\left[\left(\kappa_{0ss}-\kappa_0\tau_0^2-\frac{11}{2}\kappa_0^3\right)\cos\varphi+(2\kappa_{0s}\tau_0+\kappa_0\tau_{0s})\sin\varphi\right]\boldsymbol{t}_0 \\
&\quad +\frac{1}{6}\left(\frac{5\kappa_0\kappa_{0s}}{2}\cos\varphi+\kappa_0^2\tau_0\sin\varphi\right)\boldsymbol{n}_0+\frac{3\kappa_0^2\tau_0}{4}\cos\varphi\boldsymbol{b}_0\Bigg\}r\alpha^2+O\left(\frac{r^5}{R_0^3\alpha^2},\frac{r^4}{\alpha^4}\right). \qquad \text{(A.8)}
\end{aligned}
$$

The terms including α, an artificial parameter, will be compensated for by the corresponding terms in (A.4), and α will be excluded from the final asymptotic form of $\boldsymbol{A}_m$.

Next, the asymptotic evaluation of $\boldsymbol{A}_m^F(\boldsymbol{x}_0)$ is performed in the following manner. The denominator is expanded in powers of r, yielding

$$
\begin{aligned}
\boldsymbol{A}_m^F(\boldsymbol{x}_0) = {}& \frac{\Gamma}{4\pi}\int_{[2\alpha]}\boldsymbol{t}(s)\Bigg\{\frac{1}{R}+\frac{\boldsymbol{e}_{0r}\cdot\boldsymbol{R}}{R^3}r+\frac{1}{2R^3}\left[3\frac{(\boldsymbol{e}_{0r}\cdot\boldsymbol{R})^2}{R^2}-1\right]r^2 \\
&+\frac{1}{2R^4}\left[5\frac{(\boldsymbol{e}_{0r}\cdot\boldsymbol{R})^3}{R^3}-3\frac{\boldsymbol{e}_{0r}\cdot\boldsymbol{R}}{R}\right]r^3+\cdots\Bigg\}\mathrm{d}s, \qquad \text{(A.9)}
\end{aligned}
$$

$$\boldsymbol{e}_{0r} = \cos\varphi \boldsymbol{n}_0 + \sin\varphi \boldsymbol{b}_0 , \tag{A.10}$$
$$\boldsymbol{R} = \boldsymbol{X}(s) - \boldsymbol{X}_0 . \tag{A.11}$$

The cut-off integrals in (A.9) get rid of singularity which otherwise emerges at $s = 0$. These integrals are converted, term by term, into a form of one-circuit integral over the entire length by subtracting terms with the same singular behaviour at $s = 0$, being supplemented with the contribution from the subtracted terms and the contribution from the regularized integral over the range $(-\alpha, \alpha)$. For instance, the first term is

$$\begin{aligned}\int_{[2\alpha]} \frac{\boldsymbol{t}(s)}{R} \mathrm{d}s &= \oint \left[\frac{\boldsymbol{t}(s)}{R} - \frac{\boldsymbol{t}_0}{|s|}\right] \mathrm{d}s + 2\log\left(\frac{2\alpha}{r}\right) \boldsymbol{t}_0 \\ &\quad + \frac{1}{2}\left[\frac{11}{2}\kappa_0^2 \boldsymbol{t}_0 - (\kappa_{0s}\boldsymbol{n}_0 + \kappa_0\tau_0\boldsymbol{b}_0)\right]\alpha^2 , \end{aligned} \tag{A.12}$$

because

$$\int_{-\alpha}^{\alpha} \left[\frac{\boldsymbol{t}(s)}{R} - \frac{\boldsymbol{t}_0}{|s|}\right] \mathrm{d}s = \frac{1}{2}\left[-\frac{11}{2}\kappa_0^2 \boldsymbol{t}_0 + \kappa_{0s}\boldsymbol{n}_0 + \kappa_0\tau_0\boldsymbol{b}_0\right]\alpha^2 + O(\alpha^4/R^4) . \tag{A.13}$$

Repeating this procedure, we eventually reach

$$\begin{aligned}
\frac{4\pi}{\Gamma}\boldsymbol{A}_m^F(\boldsymbol{x}_0) &= 2\log\left(\frac{L}{2\alpha}\right)\boldsymbol{t}_0 + \kappa_0 \boldsymbol{t}_0 r \log\left(\frac{L}{2\alpha}\right)\cos\varphi \\
&+ \left[\kappa_0^2 \boldsymbol{t}_0 \left(\frac{3}{4} + \frac{3}{8}\cos 2\varphi\right) - \frac{1}{2}(\kappa_{0s}\boldsymbol{n}_0 + \kappa_0\tau_0\boldsymbol{b}_0)\right] r^2 \log\left(\frac{L}{2\alpha}\right) + \frac{2}{L^2} r^2 \boldsymbol{t}_0 \\
&\quad + \frac{1}{2}\left(\frac{11}{12}\kappa_0^2 \boldsymbol{t}_0 - \kappa_{0s}\boldsymbol{n}_0 - \kappa_0\tau_0\boldsymbol{b}_0\right)\alpha^2 - \frac{1}{2\alpha^2} r^2 \boldsymbol{t}_0 \\
&- \Bigg\{ \bigg(\Big[-\frac{33}{32}\kappa_0^3 + \frac{1}{8}(\kappa_{0ss} - \kappa_0\tau_0^2) \Big] \cos\varphi + \frac{1}{8}(2\kappa_{0s}\tau_0 + \kappa_0\tau_{0s})\sin\varphi - \frac{5}{32}\kappa_0^3 \cos 3\varphi \bigg) \boldsymbol{t}_0 \\
&\quad + \Big(\frac{5\kappa_0\kappa_{0s}}{4}\cos\varphi + \frac{\kappa_0^2\tau_0}{2}\sin\varphi \Big) \boldsymbol{n}_0 + \frac{3\kappa_0^2\tau_0}{4}\cos\varphi \boldsymbol{b}_0 \Bigg\} r^3 \log\left(\frac{L}{2\alpha}\right) \\
&- \Bigg\{ \frac{1}{24}\left[\left(\kappa_{0ss} - \kappa_0\tau_0^2 - \frac{11}{2}\kappa_0^3\right)\cos\varphi + (2\kappa_{0s}\tau_0 + \kappa_0\tau_{0s})\sin\varphi \right] \boldsymbol{t}_0 \\
&\quad + \frac{1}{6}\Big(\frac{5\kappa_0\kappa_{0s}}{2}\cos\varphi + \kappa_0^2\tau_0 \sin\varphi \Big) \boldsymbol{n}_0 + \frac{3\kappa_0^2\tau_0}{4}\cos\varphi \boldsymbol{b}_0 \Bigg\} r\alpha^2 - \frac{3\kappa_0}{4\alpha^2}\boldsymbol{t}_0 r^3 \cos\varphi \\
&+ \frac{4\pi}{\Gamma}\boldsymbol{Q}_m|_{s=0} + \cdots .
\end{aligned} \tag{A.14}$$

Here $\boldsymbol{Q}_m$ designates regularized integrals expressed in the form of a series in ϵ whose detailed form is provided, to (ϵ^2), by (3.53)–(3.60) in the text. Collecting (A.8) and (A.14) yields the inner limit (3.52) of $\boldsymbol{A}_m(\boldsymbol{x}_0)$.

The second term of (3.49) is handled in parallel with the first, and the result is (3.62).

B Equations of motion

This appendix collects the equations motion to be solved in §4 and 5. Dropping the stars, the Euler equations for dimensionless variables, expressed in terms of the moving coordinates (r, θ, ξ), take the following form:

$$\begin{aligned}
&\epsilon^3 \ddot{\boldsymbol{X}} \cdot \boldsymbol{e}_r + \epsilon^2 \left[\dot{u} + w\left(\dot{\boldsymbol{t}} \cdot \boldsymbol{e}_r\right) - (\dot{\boldsymbol{e}}_r \cdot \boldsymbol{e}_\theta) \frac{\partial u}{\partial \theta}\right] \\
&+ \frac{\epsilon}{h_3}\left[w + \epsilon^2 r(\dot{\boldsymbol{t}} \cdot \boldsymbol{e}_r)\right]\left(\epsilon \frac{\partial \dot{\boldsymbol{X}}}{\partial \xi} \cdot \boldsymbol{e}_r + \frac{\partial u}{\partial \xi} + \eta\kappa \cos\varphi w\right) \\
&+ u\frac{\partial u}{\partial r} + \frac{v}{r}\left(\frac{\partial u}{\partial \theta} - v\right) = -\frac{\partial p}{\partial r},
\end{aligned} \tag{B.1}$$

$$\begin{aligned}
&\epsilon^3 \ddot{\boldsymbol{X}} \cdot \boldsymbol{e}_\theta + \epsilon^2 \left[\dot{v} + w\left(\dot{\boldsymbol{t}} \cdot \boldsymbol{e}_\theta\right) - (\dot{\boldsymbol{e}}_r \cdot \boldsymbol{e}_\theta) \frac{\partial v}{\partial \theta}\right] \\
&+ \frac{\epsilon}{h_3}\left[w + \epsilon^2 r(\dot{\boldsymbol{t}} \cdot \boldsymbol{e}_r)\right]\left(\epsilon \frac{\partial \dot{\boldsymbol{X}}}{\partial \xi} \cdot \boldsymbol{e}_\theta + \frac{\partial v}{\partial \xi} - \eta\kappa \sin\varphi w\right) \\
&+ u\frac{\partial v}{\partial r} + \frac{v}{r}\left(\frac{\partial v}{\partial \theta} + u\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta},
\end{aligned} \tag{B.2}$$

$$\begin{aligned}
&\epsilon^3 \ddot{\boldsymbol{X}} \cdot \boldsymbol{t} + \epsilon^2 \left[\dot{w} - u\left(\dot{\boldsymbol{t}} \cdot \boldsymbol{e}_r\right) - v\left(\dot{\boldsymbol{t}} \cdot \boldsymbol{e}_\theta\right) - (\dot{\boldsymbol{e}}_r \cdot \boldsymbol{e}_\theta) \frac{\partial w}{\partial \theta}\right] \\
&+ \frac{\epsilon}{h_3}\left[w + \epsilon^2 r(\dot{\boldsymbol{t}} \cdot \boldsymbol{e}_r)\right]\left[\epsilon \frac{\partial \dot{\boldsymbol{X}}}{\partial \xi} \cdot \boldsymbol{t} + \frac{\partial w}{\partial \xi} - \eta\kappa(u\cos\varphi - v\sin\varphi)\right] \\
&+ \left(u\frac{\partial}{\partial r} + \frac{v}{r}\frac{\partial}{\partial \theta}\right) w = -\frac{\epsilon}{h_3}\frac{\partial p}{\partial \xi}.
\end{aligned} \tag{B.3}$$

Here an overdot implies partial differentiation in t with fixing r, θ and ξ. The equation of continuity is

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\epsilon}{h_3}\frac{\partial w}{\partial \xi} + \epsilon\frac{\eta\kappa}{h_3}(-u\cos\varphi + v\sin\varphi) + \frac{\epsilon^2}{h_3}\dot{\boldsymbol{X}}_\xi \cdot \boldsymbol{t} = 0. \tag{B.4}$$

Elimination of pressure from the Euler equations by trial and error is not easy, and we can sidestep this procedure by turning to the tangential component of vorticity equation, the curl of the Euler equations, at the outset. The vorticity equation in the axial direction becomes

$$\begin{aligned}
&\epsilon^2\left[\dot{\zeta} - \left(\dot{\boldsymbol{t}} \cdot \boldsymbol{e}_r\right)\omega_r - \left(\dot{\boldsymbol{t}} \cdot \boldsymbol{e}_\theta\right)\omega_\theta\right] + \frac{\epsilon}{h_3}\left[w + \epsilon^2 r\left(\dot{\boldsymbol{t}} \cdot \boldsymbol{e}_r\right)\right]\left[\frac{\partial \zeta}{\partial \xi} - \eta\kappa(\omega_r\cos\varphi - \omega_\theta\sin\varphi)\right] \\
&-\epsilon^2(\dot{\boldsymbol{e}}_r \cdot \boldsymbol{e}_\theta)\frac{\partial \zeta}{\partial \theta} + u\frac{\partial \zeta}{\partial r} + \frac{v}{r}\frac{\partial \zeta}{\partial \theta} \\
&= \omega_r\frac{\partial w}{\partial r} + \frac{\omega_\theta}{r}\frac{\partial w}{\partial \theta} + \epsilon\frac{\zeta}{h_3}\frac{\partial w}{\partial \xi} - \frac{\epsilon\eta}{h_3}\kappa(u\cos\varphi - v\sin\varphi)\zeta + \frac{\epsilon^2}{h_3}\left(\frac{\partial \dot{\boldsymbol{X}}}{\partial \xi} \cdot \boldsymbol{t}\right)\zeta.
\end{aligned} \tag{B.5}$$

Use of the vector potential $\boldsymbol{A}$ enables us to skip (B.4). The vector potential, or the extended Stokes streamfunction (3.64), is tied with the relative velocity (u, v) in the transversal plane via

$$u = \frac{\eta}{h_3 r}\frac{\partial\psi}{\partial\theta} - \frac{1}{h_3}\frac{\partial A_\theta}{\partial\xi} - \left[\left(\dot{\boldsymbol{X}}\cdot\boldsymbol{n}\right)\cos\varphi + \left(\dot{\boldsymbol{X}}\cdot\boldsymbol{b}\right)\sin\varphi\right], \tag{B.6}$$

$$v = -\frac{\eta}{h_3}\frac{\partial\psi}{\partial r} + \frac{1}{h_3}\frac{\partial A_r}{\partial\xi} + \left[\left(\dot{\boldsymbol{X}}\cdot\boldsymbol{n}\right)\sin\varphi - \left(\dot{\boldsymbol{X}}\cdot\boldsymbol{b}\right)\cos\varphi\right]. \tag{B.7}$$

Introducing these into (3.8) produces the subsidiary condition relating ψ to ζ:

$$\begin{aligned}
&\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} + \epsilon^2\frac{1}{h_3}\frac{\partial}{\partial\xi}\left(\frac{1}{h_3}\frac{\partial\psi}{\partial\xi}\right) + \epsilon\kappa\frac{\eta}{h_3}\left(\cos\varphi\frac{\partial\psi}{\partial r} - \frac{\sin\varphi}{r}\frac{\partial\psi}{\partial\theta}\right)\\
&+\epsilon^3 r\left[\frac{(\kappa_{ss} - \kappa\tau^2)\cos\varphi + (2\kappa_s\tau + \kappa\tau)\sin\varphi}{(1-\epsilon\kappa r\cos\varphi)^3} + 3\epsilon\frac{(\kappa_s\cos\varphi + \kappa\tau\sin\varphi)^2}{(1-\epsilon\kappa r\cos\varphi)^4}\right]\psi\\
&+2\epsilon^3 r\frac{\kappa_s\cos\varphi + \kappa\tau\sin\varphi}{(1-\epsilon\kappa r\cos\varphi)^2 h_3}\frac{\partial\psi}{\partial\xi} - 2\epsilon^3\frac{\kappa}{h_3}\left(\frac{\partial A_r}{\partial\xi}\cos\varphi - \frac{\partial A_\theta}{\partial\xi}\sin\varphi\right)\\
&-\epsilon^2\left(\frac{\eta}{h_3}\right)^2\left[(\kappa_s\cos\varphi + \kappa\tau\sin\varphi)A_r + (\kappa\tau\cos\varphi - \kappa_s\sin\varphi - \epsilon\kappa^2\tau r)A_\theta\right] = -\frac{h_3}{\eta}\zeta\,.
\end{aligned} \tag{B.8}$$

For the sake of convenience, we present the form of the relative velocity (u, v) expanded in powers of ϵ. This is obtained by substituting (3.14), (4.3), (3.52) and (3.62) into (B.6) and (B.7).
At $O(\epsilon^0)$,

$$u^{(0)} = 0, \qquad v^{(0)} = -\frac{\partial\psi^{(0)}}{\partial r}. \tag{B.9}$$

At $O(\epsilon^1)$,

$$\begin{aligned}
u^{(1)} &= \frac{1}{r}\frac{\partial\psi^{(1)}}{\partial\theta} - \left(\frac{1}{\eta}\frac{\partial Q_b^{(0)}}{\partial\xi} + \tau Q_n^{(0)}\right)\cos\varphi + \left(\frac{1}{\eta}\frac{\partial Q_n^{(0)}}{\partial\xi} - \tau Q_b^{(0)}\right)\sin\varphi\\
&\quad - \left[(\dot{\boldsymbol{X}}^{(0)}\cdot\boldsymbol{n})\cos\varphi + (\dot{\boldsymbol{X}}^{(0)}\cdot\boldsymbol{b})\sin\varphi\right],
\end{aligned} \tag{B.10}$$

$$\begin{aligned}
v^{(1)} &= -\left(\frac{\partial\psi^{(1)}}{\partial r} + \kappa r\cos\varphi\frac{\partial\psi^{(0)}}{\partial r}\right) + \left(\frac{1}{\eta}\frac{\partial Q_n^{(0)}}{\partial\xi} - \tau Q_b^{(0)}\right)\cos\varphi + \left(\frac{1}{\eta}\frac{\partial Q_b^{(0)}}{\partial\xi} + \tau Q_n^{(0)}\right)\sin\varphi\\
&\quad + \left[(\dot{\boldsymbol{X}}^{(0)}\cdot\boldsymbol{n})\sin\varphi - (\dot{\boldsymbol{X}}^{(0)}\cdot\boldsymbol{b})\cos\varphi\right].
\end{aligned} \tag{B.11}$$

At $O(\epsilon^2)$,

$$\begin{aligned}
u^{(2)} &= \frac{1}{r}\frac{\partial\psi^{(2)}}{\partial\theta} + \kappa\cos\varphi\frac{\partial\psi^{(1)}}{\partial\theta} - \frac{r}{2}\left[\kappa\left(\frac{1}{\eta}\frac{\partial Q_b^{(0)}}{\partial\xi} + \tau Q_n^{(0)}\right) + \frac{1}{\eta}\left(\frac{\partial Q_{11b}^{(1)}}{\partial\xi} - \frac{\partial Q_{12n}^{(1)}}{\partial\xi}\right)\right]\\
&\quad - \left[\frac{\epsilon\kappa}{2}\left(\frac{1}{\eta}\frac{\partial Q_b^{(0)}}{\partial\xi} + \tau Q_n^{(0)}\right) + \frac{1}{2\eta}\left(\frac{\partial Q_{11b}^{(1)}}{\partial\xi} + \frac{\partial Q_{12n}^{(1)}}{\partial\xi}\right) + \tau\left(Q_{11n}^{(1)} - Q_{12b}^{(1)}\right)\right] r\cos 2\varphi
\end{aligned}$$

$$
\begin{aligned}
&+\left[\frac{\epsilon\kappa}{2}\left(\frac{1}{\eta}\frac{\partial Q_n^{(0)}}{\partial\xi}-\tau Q_b^{(0)}\right)+\frac{1}{2\eta}\left(\frac{\partial Q_{11n}^{(1)}}{\partial\xi}-\frac{\partial Q_{12b}^{(1)}}{\partial\xi}\right)-\tau\left(Q_{11b}^{(1)}+Q_{12n}^{(1)}\right)\right] r\sin 2\varphi \\
&-\left[(\dot{\boldsymbol{X}}^{(1)}\cdot\boldsymbol{n})\cos\varphi+(\dot{\boldsymbol{X}}^{(1)}\cdot\boldsymbol{b})\sin\varphi\right], \qquad \text{(B.12)}
\end{aligned}
$$

$$
\begin{aligned}
v^{(2)} &= -\left[\frac{\partial\psi^{(2)}}{\partial r}+\kappa r\cos\varphi\frac{\partial\psi^{(1)}}{\partial r}+\frac{\kappa^2}{2}r^2(1+\cos 2\varphi)\frac{\partial\psi^{(0)}}{\partial r}\right] \\
&+\frac{r}{2}\left[\kappa\left(\frac{1}{\eta}\frac{\partial Q_n^{(0)}}{\partial\xi}-\tau Q_b^{(0)}\right)+\frac{1}{\eta}\left(\frac{\partial Q_{11n}^{(1)}}{\partial\xi}+\frac{\partial Q_{12b}^{(1)}}{\partial\xi}\right)\right] \\
&+\left[\frac{\epsilon\kappa}{2}\left(\frac{1}{\eta}\frac{\partial Q_n^{(0)}}{\partial\xi}-\tau Q_b^{(0)}\right)+\frac{1}{2\eta}\left(\frac{\partial Q_{11n}^{(1)}}{\partial\xi}-\frac{\partial Q_{12b}^{(1)}}{\partial\xi}\right)-\tau\left(Q_{11b}^{(1)}+Q_{12n}^{(1)}\right)\right] r\cos 2\varphi \\
&+\left[\frac{\epsilon\kappa}{2}\left(\frac{1}{\eta}\frac{\partial Q_b^{(0)}}{\partial\xi}+\tau Q_n^{(0)}\right)+\frac{1}{2\eta}\left(\frac{\partial Q_{11b}^{(1)}}{\partial\xi}+\frac{\partial Q_{12n}^{(1)}}{\partial\xi}\right)+\tau\left(Q_{11n}^{(1)}-Q_{12b}^{(1)}\right)\right] r\sin 2\varphi \\
&+\left[(\dot{\boldsymbol{X}}^{(1)}\cdot\boldsymbol{n})\sin\varphi-(\dot{\boldsymbol{X}}^{(1)}\cdot\boldsymbol{b})\cos\varphi\right]. \qquad \text{(B.13)}
\end{aligned}
$$

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