数理解析研究所講究録 1226 巻 2001 年 239-250

Non-Hermitian dynamics of vortices in a shear flow

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1 Introduction

Non-orthogonality of eigenfunctions (modes) is the determining characteristic of non-Hermitian systems, which brings about interactions among different modes. This resembles the mode couplings in nonlinear systems, and hence, the diversity of transient behavior in non-Hermitian system is rather rich.

Let us consider an abstract autonomous evolution equation of the Schrödinger type

$$\begin{cases} i\partial_t u = \mathcal{H}u \\ u(0) = u_0 \end{cases}, \tag{1}$$

where $\mathcal{H}$ is a certain linear operator. When we can generate an exponential function (propagator) $e^{-it\mathcal{H}}$, we can write the solution of (1) as

$$u(t) = e^{-it\mathcal{H}}u_0.$$

When $u \in \mathbf{C}$ and $\mathcal{H} \in \mathbf{C}$, then $e^{-it\mathcal{H}}$ is nothing but the exponential function of elementary mathematics. For vectors $u \in \mathbf{C}^N$ and a linear map $\mathcal{H} : \mathbf{C}^N \to \mathbf{C}^N$, we can define $e^{-it\mathcal{H}}$ by the standard power series

$$e^{-it\mathcal{H}} = \sum_{n=1}^{\infty} \frac{(-it\mathcal{H})^n}{n!}, \tag{2}$$

or the Cauchy integral (inverse Laplace transform)

$$e^{-it\mathcal{H}} = \frac{1}{2\pi i}\oint e^{-it\lambda}(\lambda I - \mathcal{H})^{-1}\, d\lambda. \tag{3}$$

For u in a Hilbert space V, $\mathcal{H}$ is an operator in V. For some different classes of operators, we have theories to generate $e^{-it\mathcal{H}}$. For bounded operators, we can invoke the Dunford integral that is similar to (3). A most general theory of generating an exponential function of the type $e^{t\mathcal{A}}$ for positive t (so-called semigroup theory) is due to Hille and Yosida[1]. Although this theory warrants the solvability of initial value problems for a wide class of generators, understanding of the behavior of the solution is not simple. Indeed, the exponential functions of matrices or operators are not necessarily "exponential" in the conventional sense.

The von Neumann theory for Hermitian (self-adjoint) operators provides a deep insight into the structure of $e^{-it\mathcal{H}}$, which invokes the spectral resolution of the generator $\mathcal{H}$ in terms of a complete set of orthogonal modes. The basic idea is that the $e^{-it\mathcal{H}}$ may be represented as a sum of independent harmonic oscillators, each of which is an eigenfunction of $\mathcal{H}$ and the corresponding eigenvalue (real number) gives the frequency of the oscillation. Unlike the case of finite dimension vector space, however, the conventional eigenfunctions may not be complete to span the Hilbert space. The most essential generalization needed to study an infinite dimension space was the introduction of continuous spectra that correspond to singular eigenfunctions. The spectral resolution of $\mathcal{H}$ is, in general, given by an integral over the spectra (an example will be given in Sec. 3.1). The contribution to the $e^{-it\mathcal{H}}$ from the continuous spectra brings about the "phase mixing" of oscillations with continuous frequencies, resulting in various types of damping. Hence, the reality of the spectra of an Hermitian operator does not necessarily imply stationary (non-dumping) oscillations.

For a linear map in a finite dimension vector space, the spectral resolution yields the Jordan canonical form, and the explicit representation of $e^{-it\mathcal{H}}$ can be constructed using the canonical form. It is well known that a nilpotent yields secular behavior of the corresponding generalized eigenvector. Therefore, even if every eigenvalue λ_j is real, the $e^{-it\mathcal{H}}$ can describe "instabilities" (growth of oscillations).

In a Hilbert space, however, such a general theory of spectral resolution is limited to either compact operators or Hermitian operators. This paper is an attempt to obtain a spectral resolution of a non-Hermitian operator that is not included in the above mentioned categories. This operator is related to an important physics problem (Sec. 2).

2 Non-Hermitian dynamics of vortices

The vortex dynamics equation in $\mathbf{R}^2$ [the coordinates are denoted by (x, y)] reads as a Liouville equation

$$\partial_t \Psi + \{H, \Psi\} = 0, \tag{4}$$

where Ψ is the vorticity, H is the Hamiltonian (stream function) of an incompressible flow $\boldsymbol{v} = (\partial_y H, -\partial_x H)^t$ that transports the vortices, and

$$\{a, b\} = (\partial_y a)(\partial_x b) - (\partial_x a)(\partial_y b) = -\nabla a \times \nabla b \cdot \nabla z$$

is the Poisson bracket.

When the Hamiltonian H depends on Ψ, the evolution equation (4) is nonlinear. The dynamics of Ψ can couple with other fields when they are included in H. The simplest example of nonlinear vortex dynamics is that of the Euler fluid (incompressible ideal flow), where

$$-\Delta H = \Psi, \tag{5}$$

or, denoting the Green operator of the Laplacian $-\Delta$ by $\mathcal{G}$

$$H = \mathcal{G}\Psi. \tag{6}$$

Let us linearize (4) with decomposing Ψ and H into their ambient (denoted by subscript 0) and fluctuation parts:

$$\begin{aligned} \Psi &= \Psi_0 + \psi, \\ H &= H_0 + h = \mathcal{G}\Psi_0 + \mathcal{G}\psi. \end{aligned}$$

Neglecting the second-order terms, (4) reads

$$\partial_t \psi + \{H_0, \psi\} + \{\Delta H_0, \mathcal{G}\psi\} = 0. \tag{7}$$

In this paper, we consider one-dimensional problem with

$$H_0 = H_0(x).$$

Since the ambient Hamiltonian H_0 is independent of y, the wavenumber in y is a good quantum number, and we can replace ∂_y by ik. In what follows, we assume $k \neq 0$, and normalize $k = 1$ [2]. We write

$$v(x) = -\partial_x H_0(x),$$

to obtain the standard Rayleigh equation

$$i\partial_t \psi = v(x)\psi + v''(x)\mathcal{G}\psi. \tag{8}$$

The Green operator $\mathcal{G}$ is represented by a convolution integral

$$(\mathcal{G}f)(x) = \int_{-\infty}^{+\infty} \frac{e^{-|x-\xi|}}{2} f(\xi)\, d\xi. \tag{9}$$

In what follows, we denote by $G(x, \xi)$ the Green function;

$$G(x, \xi) = \frac{e^{-|x-\xi|}}{2}. \tag{10}$$

3 Convection and oscillations

The generator of the vortex dynamics equation (8) consists of two terms, each of which describes different mechanism of vortex motion. The first term on the right-hand side of (8) [originating from $\{H_0, \psi\}$ in (7)] represents the transport of the vorticity by the ambient flow $v(x)$. An inhomogeneous (sheared) flow distorts vortices, and hence, no stationary structure can persist in a shear flow ($v(x) \neq$ constant). Such a dynamics is described by a continuous spectrum (Sec. 3.1). On the other hand, the second term [originating from $\{\Delta H_0, \mathcal{G}\psi\}$ in (7)] describes the interaction between the perturbation and the ambient field. When the ambient vorticity $\Psi_0 = -\Delta H_0$ has a spatial gradient, a flow induced by a perturbation yields a local change of the vorticity. This term, hence, can create perturbed vortices from the ambient field.

In this section, we study the role of both terms by formal calculations. In what follows, it is convenient to generalize (8) with replacing $v''(x)$ by a certain "real" function $w(x)$ that is independent of $v(x)$. With assuming $k \neq 0$, we consider

$$i\partial_t \psi = v(x)\psi + w(x)\mathcal{G}\psi. \tag{11}$$

The case when $w(x) = v''(x)$ recovers the physically relevant equation (8).

3.1 Convection – shear flow transport

Here, we assume $w(x) = 0$ in (11) and consider

$$i\partial_t \psi = v(x)\psi \tag{12}$$

with a "continuous" real function $v(x)$, which reads as a Schrödinger equation with a Hamiltonian $v(x)$.

The formal eigenvalue and the corresponding eigenfunction of the generator of (12), with setting

$$v(x)\psi = \omega\psi$$

(i.e., $\psi(t) = e^{-i\omega t}\psi$), is given by

$$\omega = v(\mu), \quad \psi = \delta(x - \mu), \tag{13}$$

where μ is an arbitrary real number and δ denotes the delta-measure. For the convenience, we write

$$(f(x), \delta(x - \mu)) = \int_{-\infty}^{+\infty} f(x)\delta(x - \mu)\, dx = f(\mu).$$

A formal spectral resolution of the generator is written as

$$\begin{aligned} v(x)f(x) &= \int_{-\infty}^{+\infty} v(\mu)(f, \delta(x-\mu))\delta(x - \mu)\, d\mu \\ &= \int_{-\infty}^{+\infty} v(\mu)f(\mu)\delta(x - \mu)\, d\mu. \end{aligned} \tag{14}$$

Rigorous mathematical representation of this "continuous spectrum" is given by the spectral resolution of the coordinate operator:

$$xf(x) = \int_{-\infty}^{+\infty} \mu dE(\mu)f(x), \tag{15}$$

where $\{E(\mu);\ \mu \in \mathbf{R}\}$ is a family of projectors (resolution of the identity) defined by

$$E(\mu)f(x) = \begin{cases} f(x) & \text{for } x \le \mu \\ 0 & \text{for } x > \mu \end{cases}. \tag{16}$$

The projector $E(\mu)$ gives a resolution of the identity:

$$I = \int_{-\infty}^{+\infty} dE(\mu). \tag{17}$$

Using this representation of the coordinate operator, we can write

$$v(x)f(x) = \int_{-\infty}^{+\infty} v(\mu)dE(\mu)f(x). \tag{18}$$

The solution of (12) with initial condition $\psi(x, 0) = \psi_0(x)$ is given by

$$\psi(x,t) = \int_{-\infty}^{+\infty} e^{-itv(\mu)}dE(\mu)\psi_0(x) = e^{-itv(x)}\psi_0(x). \tag{19}$$

3.2 Chandrasekhar model of surface-waves

Non-Hermitian property stems from the second term on the right-hand side of (11), because the multiplication of $w(x)$ and the integral operator $\mathcal{G}$ does not commute. As we have noted, this term represents the interaction between the perturbed flow and the ambient vorticity. Physically, the non-Hermitian property implies the non-conservation of the "energy" of the vorticity, i.e., the enstrophy $\int |\psi|^2 dx$. We also remark that the original nonlinear system (4) conserves the enstrophy, as well as all "Casimirs" $\int f(\Psi)\, dx$ (f is an arbitrary smooth function). The non-conservation of the enstrophy in the linearized system is due to the separation of the vorticity into the perturbed component and the ambient field. Because of the interaction between these two parts, which is enabled by the term $\{h, \Psi_0\}$, the perturbed component ψ does not describe a closed dynamical system.

The role of the non-Hermitian term $[w(x)\mathcal{G}\psi$ in (11)] is most simply highlighted by Chandrasekhar's model of a shear flow, which assumes a piece-wise linear flow $v(x)$ and the corresponding delta measure $v''(x)$ [3]. Before giving a mathematical justification, let us examine formal solutions of this model.

In this subsection, we assume $v(x) = 0$ and consider

$$i\partial_t \psi = w(x)\mathcal{G}\psi \tag{20}$$

with

$$w(x) = A\delta(x-a) \quad (A, a \in \mathbf{R}). \tag{21}$$

The formal eigenfunction of the generator, under the setting of $i\partial_t = \omega$ in (20), is determined by

$$A\delta(x-a)\int_{-\infty}^{+\infty} G(x,\xi)\psi(\xi)\, d\xi = \omega\psi(x), \tag{22}$$

where $G(x,\xi)$ is the Green function of $\mathcal{G}$ [see (10)]. Solving (22), we obtain

$$\omega = \frac{A}{2}, \quad \psi = \delta(x-a). \tag{23}$$

We thus have an oscillation of a "surface wave" that is localized at $x = a$ and has the wavenumber k in the y direction [4].

If we have multiple "sources" of the surface waves, these waves interact through spatial couplings induced by perturbed flows. Let us consider N (finite number) sources

$$w(x) = \sum_{j=1}^{N} A_j\delta(x-a_j) \quad (A_j,\ a_j \in \mathbf{R},\ j = 1, \ldots, N). \tag{24}$$

The frequencies of the coupled surface waves are given by solving

$$\sum_{j=1}^{N} A_j\delta(x-a_j)\int_{-\infty}^{+\infty} G(x,\xi)\psi(\xi)\, d\xi = \omega\psi(x). \tag{25}$$

Substituting

$$\psi = \sum_{j=1}^{N} \alpha_j\delta(x-a_j),$$

into (25), we obtain the "dispersion relation"

$$M\begin{pmatrix}\alpha_1\\ \vdots\\ \alpha_N\end{pmatrix}=\omega\begin{pmatrix}\alpha_1\\ \vdots\\ \alpha_N\end{pmatrix} \tag{26}$$

with

$$M_{i,j}=A_iG(a_i,a_j)=A_i\frac{e^{-|a_i-a_j|}}{2}. \tag{27}$$

The eigenvalue problem (26) determines the frequencies of the coupled oscillations. Obviously, the matrix M is non-symmetric (except for the case of $A_j = C$ for all j), representing the non-Hermitian property of the generator. For some sets of coefficients A_j ($j = 1, \cdots, N$), the frequency ω can be complex. The imaginary part of ω gives the growth rate of the unstable mode of oscillation which corresponds to the "Kelvin-Helmholtz instability".

3.3 Coupling of the two generators

We have seen the dynamics of vortices induced by each of the two different generators in (11), separately. Now, we study the coupling of these two generators.

Let us first consider the case of single source; see (21). The eigenvalue problem associated with the generalized Rayleigh equation (11) reads

$$v(x)\psi+A\delta(x-a)\int_{-\infty}^{+\infty}G(x,\xi)\psi(\xi)\,d\xi=\omega\psi, \tag{28}$$

where $G(x,\xi) = e^{-|x-\xi|}/2$ is the Green function [see (10)]. Let us try to find a formal solution with assuming

$$\psi=\alpha\delta(x-a)+\beta\delta(x-\mu), \tag{29}$$

where μ is an arbitrary "fixed" real number [see (13) and (23)]. Substituting (29) into (28), we obtain an eigenvalue problem

$$L\begin{pmatrix}\alpha\\ \beta\end{pmatrix}=\omega\begin{pmatrix}\alpha\\ \beta\end{pmatrix} \tag{30}$$

where

$$L=\begin{pmatrix}v(a)+AG(a,a) & AG(a,\mu)\\ 0 & v(\mu)\end{pmatrix}. \tag{31}$$

We can solve (30) to find a set of eigenvalues and eigenfunctions:

$$\omega=\Omega_1(a):=v(a)+\frac{A}{2},\quad \begin{pmatrix}\alpha\\ \beta\end{pmatrix}=\boldsymbol{U}_1:=\begin{pmatrix}1\\ 0\end{pmatrix}, \tag{32}$$

and

$$\omega=\Omega_c(\mu):=v(\mu),\quad \begin{pmatrix}\alpha\\ \beta\end{pmatrix}=\boldsymbol{U}_c:=m(\mu)\begin{pmatrix}\frac{AG(a,\mu)}{\Omega_c(\mu)-\Omega_1(a)}\\ 1\end{pmatrix}, \tag{33}$$

where the normalization factor is

$$m(\mu) = \left[1 + \left(\frac{AG(a,\mu)}{\Omega_c(\mu) - \Omega_1(a)}\right)^2\right]^{-1/2} . \tag{34}$$

The first eigenvalue $\Omega_1 = v(a) + (A/2)$ gives the "Doppler-shifted" frequency of the surface wave [see (23)]. The corresponding formal eigenfunction is exactly $\psi = \delta(x-a)$. The second eigenvalue $\Omega_c = v(\mu)$ represents the local flow velocity [see (13)], while the corresponding formal eigenfunction describes a combination of the surface wave and a local vortex.

By the transforms

$$T = (\boldsymbol{U}_1\ \boldsymbol{U}_c) = \begin{pmatrix} 1 & \frac{m(\mu)AG(a,\mu)}{\Omega_c(\mu)-\Omega_1(a)} \\ 0 & m(\mu) \end{pmatrix}, \quad T^{-1} = \begin{pmatrix} 1 & -\frac{AG(a,\mu)}{\Omega_c(\mu)-\Omega_1(a)} \\ 0 & m(\mu)^{-1} \end{pmatrix}, \tag{35}$$

the matrix L is diagonalized;

$$T^{-1}LT = \begin{pmatrix} \Omega_1 & 0 \\ 0 & \Omega_c \end{pmatrix}.$$

We note that T is not a unitary transform, reflecting the fact that the generator is not a Hermitian operator.

If the "resonance" $\Omega_1 = \Omega_c$ $[v(a) + A/2 = v(\mu)]$ occurs, the second solution (33) degenerates into the first one (32). This is the case when the matrix L of (30) transforms into a Jordan block. We introduce a generalized eigenfunction belonging to the degenerate eigenvalue Ω_1;

$$\boldsymbol{U}'_c = \begin{pmatrix} 1 \\ (AG(a,\mu))^{-1} \end{pmatrix}, \tag{36}$$

which satisfies $(L - \Omega_1 I)^2 \boldsymbol{U}'_c = 0$. By transforms

$$T' = (\boldsymbol{U}_1\ \boldsymbol{U}'_c) = \begin{pmatrix} 1 & 1 \\ 0 & (AG(a,\mu))^{-1} \end{pmatrix}, \quad T'^{-1} = \begin{pmatrix} 1 & -AG(a,\mu) \\ 0 & AG(a,\mu) \end{pmatrix},$$

we can transform L into a Jordan canonical form

$$T'^{-1}LT' = \begin{pmatrix} \Omega_1 & 1 \\ 0 & \Omega_1 \end{pmatrix}.$$

To unify both the non-resonant and resonant (nilpotent) cases, we define

$$\tilde{m}(\mu) = \begin{cases} m(\mu) & \text{if } \Omega_c(\mu) \neq \Omega_1(a) \\ (AG(a,\mu))^{-1} & \text{if } \Omega_c(\mu) = \Omega_1(a) \text{ (i.e. } m(\mu) = 0), \end{cases} \tag{37}$$

and combine $\boldsymbol{U}_c$ and $\boldsymbol{U}'_c$ as

$$\tilde{\boldsymbol{U}}_c(\mu) = \begin{pmatrix} \frac{m(\mu)AG(a,\mu)}{\Omega_c(\mu)-\Omega_1(a)} \\ \tilde{m}(\mu) \end{pmatrix}. \tag{38}$$

The transform

$$\tilde{T} = \left(\boldsymbol{U}_1\ \tilde{\boldsymbol{U}}_c(\mu)\right) = \begin{pmatrix} 1 & \frac{m(\mu)AG(a,\mu)}{\Omega_c(\mu)-\Omega_1(a)} \\ 0 & \tilde{m}(\mu) \end{pmatrix} \tag{39}$$

is regular for all μ.

Next, we study the case of multiple sources; see (24). We solve

$$v(x)\psi + \sum_{j=1}^{N} A_j\delta(x-a_j)\int_{-\infty}^{+\infty} G(x,\xi)\psi(\xi)\,d\xi = \omega\psi \tag{40}$$

with assuming

$$\psi = \sum_{j=1}^{N} \alpha_j\delta(x-a_j) + \beta\delta(x-\mu).$$

To generalize the above calculations, we prepare notation [see (32)]

$$\Omega_j(a_j) = v(a_j) + \frac{A_j}{2} \quad (j = 1, \ldots, N). \tag{41}$$

The dispersion relation is

$$L\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_N \\ \beta \end{pmatrix} = \omega \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_N \\ \beta \end{pmatrix}. \tag{42}$$

where the matrix L generalizes (31) as

$$L = \begin{pmatrix} \Omega_1(a_1) & \cdots & A_1G(a_1,a_N) & A_1G(a_1,\mu) \\ \vdots & \ddots & \vdots & \vdots \\ A_NG(a_N,a_1) & \cdots & \Omega_N(a_N) & A_NG(a_N,\mu) \\ 0 & \cdots & 0 & \Omega_c(\mu) \end{pmatrix}. \tag{43}$$

We have two different classes of solutions. The first group, corresponding to (32), is obtained with setting $\beta = 0$. Then, the eigenvalue problem (42) reduces into

$$\begin{pmatrix} \Omega_1(a_1) & \cdots & A_1G(a_1,a_N) \\ \vdots & \ddots & \vdots \\ A_NG(a_N,a_1) & \cdots & \Omega_N(a_N) \end{pmatrix}\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_N \end{pmatrix} = \omega\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_N \end{pmatrix}, \tag{44}$$

which reads as the dispersion relation that is Doppler shifted from (26). The second class of eigenvectors, corresponding to (33), is given by setting $\beta \neq 0$. The eigenvalue is

$$\Omega_c(\mu) = v(\mu),$$

and the corresponding eigenfunction is determined by

$$\begin{pmatrix} \Omega_1(a_1)-\Omega_c(\mu) & \cdots & A_1G(a_1,a_N) \\ \vdots & \ddots & \vdots \\ A_NG(a_N,a_1) & \cdots & \Omega_N(a_N)-\Omega_c(\mu) \end{pmatrix}\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_N \end{pmatrix} = -\beta\begin{pmatrix} A_1G(a_1,\mu) \\ \vdots \\ A_NG(a_N,\mu) \end{pmatrix}. \tag{45}$$

As discussed above, the resonances $\Omega_j(a_j) = \Omega_c(\mu)$ $(j = 1, \cdots, N)$ yield singularities in the matrix of (45), and then, we must consider nilpotents.

4 Spectral resolution of coupled non-Hermitian generator

In this section, we formulate the vortex dynamics equation (11) with the delta-measure field (24) as an evolution equation in an appropriate Hilbert space, and give a spectral resolution of the generator. The generator reads

$$\mathcal{L}\psi = v(x)\psi + \sum_{j=1}^{N} A_j\delta(x-a_j)\int_{-\infty}^{+\infty} G(x,\xi)\psi(\xi)\,d\xi, \tag{46}$$

where $v(x) \in C(\mathbf{R})$, $A_j \in \mathbf{R}$, $a_j \in \mathbf{R}$ $(j = 1,\ldots,N)$, and $G(x,\xi) = e^{-|x-\xi|}/2$ is the Green function [see (10)]. In what follows, we assume $|v(x)| < c$ $(\forall x)$ with some finite number c.

Since the delta measure $\delta(x-a_j)$ is not a member of the Lebesgue space, we encounter a difficulty in formulating the problem in the conventional L^2 Hilbert space.

4.1 Mathematical formulation of the generator

Let us consider a Hilbert space

$$V = \mathbf{C}^N \oplus L^2(\mathbf{R}), \tag{47}$$

where $\mathbf{C}^N$ is the unitary space of dimension N, and $L^2(\mathbf{R})$ is the complex Lebesgue space on $\mathbf{R}$ endowed with the standard inner product. The member of V is written as

$$\psi = \begin{pmatrix} \boldsymbol{\alpha} \\ \varphi(x) \end{pmatrix} \quad [\boldsymbol{\alpha} \in \mathbf{C}^N,\ \varphi(x) \in L^2(\mathbf{R})]. \tag{48}$$

The inner product of V is, thus, defined as

$$\langle \psi, \psi' \rangle = (\boldsymbol{\alpha}, \boldsymbol{\alpha}') + (\varphi, \varphi') = \sum_{j=1}^{N} \alpha_j \bar{\alpha}'_j + \int_{-\infty}^{+\infty} \varphi(x)\bar{\varphi}'(x)\,dx \tag{49}$$

We identify

$$\psi = \begin{pmatrix} \boldsymbol{\alpha} \\ \varphi(x) \end{pmatrix} \Leftrightarrow \psi(x) = \sum_{j=1}^{N} \alpha_j\delta(x-a_j) + \varphi(x). \tag{50}$$

It is essential to decompose the delta-measure part (representing the surface waves) from the total vorticity ψ. Although the supports (in the sense of distributions) of both components $\delta(x-a_j)$ and $\varphi(x)$ may overlap, we separate them into different degrees of freedom.

Because $\mathcal{G}\psi \in C(\mathbf{R})$ for all $\psi \in V$, the generator $\mathcal{L}$ is a bounded operator on V.

Following (50), the generator $\mathcal{L}$ of (46) is now written in a matrix form [see (43)]

$$\mathcal{L}\psi = \begin{pmatrix} \Omega_1(a_1) & \cdots & A_1G(a_1,a_N) & \int A_1G(a_1,x)\cdot\,dx \\ \vdots & \ddots & \vdots & \vdots \\ A_NG(a_N,a_1) & \cdots & \Omega_N(a_N) & \int A_NG(a_N,x)\cdot\,dx \\ 0 & \cdots & 0 & \Omega_c(x) \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_N \\ \varphi(x) \end{pmatrix} \tag{51}$$

In the previous section, we dealt delta functions in a formal way and did calculations using $\delta(x-\mu)$ with an arbitrary $\mu \in \mathbf{R}$ [see (13) and (29)]. We note that such formal functions are not the member of the Hilbert space V. In this section, they are justified as generalized eigenfunctions corresponding to "continuous spectra".

4.2 Spectral resolution of the generator

First, we consider the simple case of single "source", i.e., $w(x) = A\delta(x-a)$ [see (21)]. The surface wave mode has only one degree of freedom ($N = 1$). Here, the generator $\mathcal{L}$ of (51) simplifies as

$$\mathcal{L} = \begin{pmatrix} \Omega_1(a) & \int AG(a,x) \cdot \, dx \\ 0 & \Omega_c(x) \end{pmatrix}. \tag{52}$$

As we have shown in Sec. 3.3, there are two different classes of formal eigenfunctions [see (32) and (33)]; In the form consistent to the notation of (48), they read

$$\Omega_1(a) = v(a) + \frac{A}{2}, \quad \boldsymbol{U}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{53}$$

$$\Omega_c(\mu) = v(\mu), \quad \tilde{\boldsymbol{U}}_c(\mu) = \begin{pmatrix} \frac{m(\mu)AG(a,\mu)}{\Omega_c(\mu)-\Omega_1(a)} \\ \tilde{m}(\mu)\delta(x-\mu) \end{pmatrix}. \tag{54}$$

The first eigenfunction represents the surface wave. The second one includes an arbitrary real number μ, corresponding to the continuous spectrum, and a singular function $\delta(x-\mu)$. We must integrate (54) over $\mu \in \mathbf{R}$ to span the complete basis of V. Formally, we can generalize the transform $\tilde{T}$ of (39) as

$$\mathcal{T} = \left(\boldsymbol{U}_1 \ \int (\cdot, \delta(x-\mu))\tilde{\boldsymbol{U}}_c(\mu) \, d\mu \right) = \begin{pmatrix} 1 & \int (\cdot, \delta(x-\mu)) \frac{m(\mu)AG(a,\mu)}{\Omega_c(\mu)-\Omega_1(a)} \, d\mu \\ 0 & \int (\cdot, \delta(x-\mu)) \tilde{m}(\mu)\delta(x-\mu) \, d\mu \end{pmatrix}. \tag{55}$$

To cast this formal expression in an appropriate mathematical representation, we invoke the resolution of the identity (17). The formal correspondence is

$$\int_{-\infty}^{+\infty} (u(x), \delta(x-\mu))\delta(x-\mu) \, d\mu = \int_{-\infty}^{+\infty} dE(\mu)u = u.$$

We also define

$$F(\mu)u = \int_{-\infty}^{\mu} u(x) \, dx, \tag{56}$$

which gives

$$dF(\mu)u = u(\mu) \, d\mu.$$

Using this notation, we can write

$$\int f(\mu) \, dF(\mu)u(x) = \int f(\mu)u(\mu) \, d\mu = \int f(x)u(x) \, dx.$$

The operator $\mathcal{T}$ is now written in a rigorous form of

$$\mathcal{T} = \begin{pmatrix} 1 & \int \frac{m(\mu)AG(a,\mu)}{\Omega_c(\mu)-\Omega_1(a)} dF(\mu) \\ 0 & \int \tilde{m}(\mu) dE(\mu) \end{pmatrix} = \begin{pmatrix} 1 & \int \frac{m(x)AG(a,x)}{\Omega_c(x)-\Omega_1(a)} \cdot \, dx \\ 0 & \tilde{m}(x) \end{pmatrix} \tag{57}$$

Reflecting the non-Hermitian property of the generator $\mathcal{L}$, the operator $\mathcal{T}$ is not a unitary transform. By combing both non-resonant and resonant (nilpotent) cases [cf. (39)], this $\mathcal{T}$ is a regular transform. The inverse operator is

$$\mathcal{T}^{-1} = \begin{pmatrix} 1 & -\int \left(\frac{m(x)}{\tilde{m}(x)}\right) \left(\frac{AG(a,x)}{\Omega_c(x)-\Omega_1(a)}\right) \cdot \, dx \\ 0 & \tilde{m}(x)^{-1} \end{pmatrix}. \tag{58}$$

Using the transforms $\mathcal{T}$ and $\mathcal{T}^{-1}$, we obtain the Jordan canonical form of $\mathcal{L}$;

$$\begin{aligned}\mathcal{T}^{-1}\mathcal{L}\mathcal{T} &= \begin{pmatrix} \Omega_1 & \int \rho(\mu) dF(\mu) \\ 0 & \int \Omega_c(\mu) dE(\mu) \end{pmatrix} \\ &= \begin{pmatrix} \Omega_1 & \int \rho(x) \cdot \; dx \\ 0 & \Omega_c(x) \end{pmatrix},\end{aligned} \tag{59}$$

where

$$\rho(x) = \begin{cases} 1 & \text{if } \Omega_c(\mu) = \Omega_1(a) \\ 0 & \text{if } \Omega_c(\mu) \neq \Omega_1(a) \end{cases}.$$

The support of $\rho(x)$ can have a finite measure when the resonance condition $\Omega_c(\mu) = \Omega_1(a)$ holds on a finite interval of x.

4.3 Spectral representation of the propagator

The propagator $e^{-it\mathcal{L}}$ is defined by solving the initial value problem for (11)

$$\begin{cases} i\partial_t \psi = \mathcal{L}\psi, \\ \psi(0) = \psi_0 \end{cases} \tag{60}$$

and writing the solution as

$$\psi(t) = e^{-it\mathcal{L}}\psi_0.$$

Defining $\psi = \mathcal{T}\chi$, we transform (60) into

$$\begin{cases} i\partial_t \chi = \mathcal{T}^{-1}\mathcal{L}\mathcal{T}\chi, \\ \chi(0) = \mathcal{T}^{-1}\psi_0. \end{cases} \tag{61}$$

Using the spectral resolution (59), the solution of (61) is given by

$$\begin{aligned} e^{-it\mathcal{T}^{-1}\mathcal{L}\mathcal{T}} &= \begin{pmatrix} e^{-it\Omega_1} & -\int ite^{-it\Omega_1}\rho(\mu) dF(\mu) \\ 0 & \int e^{-it\Omega_c(\mu)} dE(\mu) \end{pmatrix} \\ &= \begin{pmatrix} e^{-it\Omega_1} & -\int ite^{-it\Omega_1}\rho(x) \cdot \; dx \\ 0 & e^{-it\Omega_c(x)} \end{pmatrix}. \end{aligned} \tag{62}$$

The solution of (60) is given by

$$\psi(t) = \mathcal{T}\left[e^{-it\mathcal{T}^{-1}\mathcal{L}\mathcal{T}}\right]\mathcal{T}^{-1}\psi_0.$$

Using (57) and (58), we obtain

$$\begin{aligned} e^{-it\mathcal{L}} &= \mathcal{T}\begin{pmatrix} e^{-it\Omega_1} & -\int ite^{-it\Omega_1}\rho(x) \cdot \; dx \\ 0 & e^{-it\Omega_c(x)} \end{pmatrix}\mathcal{T}^{-1} \\ &= \begin{pmatrix} e^{-it\Omega_1} & X \\ 0 & e^{-it\Omega_c(x)} \end{pmatrix}, \end{aligned} \tag{63}$$

$$X = \int \left([1-\rho(x)] \frac{[e^{-it\Omega_c(x)} - e^{-it\Omega_1(a)}]AG(a,x)}{\Omega_c(x) - \Omega_1(a)} - ite^{-it\Omega_1} AG(a,x)\rho(x) \right) \cdot dx,$$

and we have used the relations

$$\begin{cases} \dfrac{m(x)}{\tilde{m}(x)} = 1 - \rho(x) \\ \dfrac{\rho(x)}{\tilde{m}(x)} = AG(a,x)\rho(x) \end{cases}.$$

The off-diagonal part X of the matrix operator (63) represents the mode interactions originating from the non-Hermitian property of the generator. The X consists of two parts; one is the contribution from the non-resonant flow in the region of the support of $1-\rho(x)$, and the other is from the resonant flow in that of $\rho(x)$. The latter produces secular behavior (represented by the factor $ite^{-it\Omega_1}$).

References

[1] K. Yosida, *Functional Analysis* (Springer-Verlag, Berlin, 1995).

[2] By changing the scale of y, we can normalize k to 1. On the contrary, to take $k \neq 1$, we translate $y \to ky$, $v \to kv$, $w \to kw$ and $e^{-|x-\xi|}/2 \to e^{-k|x-\xi|}/(2k)$ in the later calculations; see (8), (11) and (9).

[3] S. Chandrasekhar, *Hydrodynamic and hydromagnetic stability* (Clarendon, Oxford, 1961).

[4] There are rich examples of relevant phenomena; The Rossby waves of perturbations in geological jet streams, the diocotron waves in non-neutral plasmas, and the drift waves in neutral plasmas.