数理解析研究所講究録 1227 巻 2001 年 218-229

Graphical representations of the q-creation and the q-annihilation operators and set partition statistics

Hiroaki Yoshida

Department of Information Sciences,
Ochanomizu University, Otsuka, Bunkyo,
Tokyo 112-8610 Japan

Abstract

In this note, we introduce a certain graphical representations of the q-creation, the q-annihilation, the q-number, and the scalar operators on the q-Fock space. These are given as the cards on which some flow lines are drawn and increasing and decreasing of the number of lines correspond to creation and annihilation, respectively. By the cards arrangement technique with these cards, of which machinery is the same as of the partition function for the lattice spins model in statistical mechanics, we write the moments of the q-Poisson random variable using the number of arc crossings, which is the same set partition statistic as restricted crossings investigated by P. Biane. This result suggests another q-deformed moments-cumulants relations, which interpolates between the usual and the free cases exactly. This is the joint work with Naoko Saitoh at Ochanomizu University.

1 The q-Fock space

For a Hibert space $\mathcal{H}$ and $q \in (-1,1)$, the q-Fock space $\mathcal{F}_q(\mathcal{H})$ can be defined as follows: (see, for instance, [BS1], [BKS]) Let $\mathcal{F}^{fin}(\mathcal{H})$ be the linear span of vectors of the form $\xi_1 \otimes \cdots \otimes \xi_n \in \mathcal{H}^{\otimes n}$, where n varies in $\mathbf{Z}_{\geq 0}$ and we put $\mathcal{H}^{\otimes 0} \cong \mathbf{C}\Omega$ for some distinguished vector Ω called vacuum. We consider the sesquilinear form $\langle \cdot | \cdot \rangle_q$ on $\mathcal{F}^{fin}(\mathcal{H})$ given by the sesquilinear extension of

$$\langle \xi_1 \otimes \cdots \otimes \xi_n | \eta_1 \otimes \cdots \otimes \eta_m \rangle_q = \delta_{n,m} \sum_{\pi \in S_n} q^{i(\pi)} \langle \xi_1 | \eta_{\pi(1)} \rangle \cdots \langle \xi_n | \eta_{\pi(n)} \rangle,$$

where S_n denotes the symmetric group of permutations of n elements and $i(\pi)$ is the number of the inversions of permutation $\pi \in S_n$ defined by

$$i(\pi) = {}^{\#}\{(i,j) | 1 \leq i < j \leq n, \pi(i) > \pi(j)\}.$$

The strict positivity (see [BS2]) of $\langle \cdot | \cdot \rangle_q$ allows the following definitions:

Definition 1.1. The q-Fock space $\mathcal{F}_q(\mathcal{H})$ is the completion of $\mathcal{F}^{fin}(\mathcal{H})$ with respect to $\langle \cdot | \cdot \rangle_q$, and given the vector $\xi \in \mathcal{H}$, we define *the q-creation operator* $a^*(\xi)$ and *the q-annihilation operator* $a(\xi)$ on $\mathcal{F}_q(\mathcal{H})$ by

$$\begin{aligned} a^*(\xi)\Omega &= \xi, \\ a^*(\xi)\xi_1 \otimes \cdots \otimes \xi_n &= \xi \otimes \xi_1 \otimes \cdots \otimes \xi_n \end{aligned}$$

and

$$\begin{aligned} a(\xi)\Omega &= 0, \\ a(\xi)\xi_1 \otimes \cdots \otimes \xi_n &= \sum_{i=1}^{n} q^{i-1} \langle \xi | \xi_i \rangle \xi_1 \otimes \cdots \otimes \overset{\vee}{\xi_i} \otimes \cdots \otimes \xi_n, \end{aligned}$$

respectively, where the symbol $\overset{\vee}{\xi_i}$ means that ξ_i has to be deleted in the tensor product.

They are adjoints of each other with respect to the scalar product $\langle \cdot | \cdot \rangle_q$. Furthermore, it is very important to note that they fulfill the q-commutation relations,

$$a(\xi)a^*(\eta) - qa^*(\eta)a(\xi) = \langle \xi | \eta \rangle \cdot \mathbf{1} \qquad \xi, \eta \in \mathcal{H}.$$

We can easily see from the definition that, for $\xi \in \mathcal{H}$ with $||\xi|| = 1$, we obtain that

$$\begin{aligned} a^*(\xi)\xi^{\otimes n} &= \xi^{\otimes(n+1)}, & (n \geq 0) \\ a(\xi)\xi^{\otimes n} &= [n]_q \xi^{\otimes(n-1)}, & (n \geq 1) \end{aligned}$$

where we use the convention that $\xi^{\otimes 0} = \Omega$. Combining these relations, we have thatfor $\xi \in \mathcal{H}$ with $||\xi|| = 1$ and $n \geq 1$,

$$a^*(\xi)a(\xi)\xi^{\otimes n} = [n]_q \xi^{\otimes n}.$$

Hence we may regard $a^*(\xi)a(\xi)$ as *the q-number operator.*

2 The q-Poisson random variable

In [SY1], we introduced, for $q \in [0, 1)$, the q-deformed Poisson distribution as the orthogonalizing probability measure for a certain q-deformation of Charlier polynomials, which interpolates the usual ($q = 1$) and the free ($q = 0$) Poisson distributions.

Definition 2.1. For $q \in [0,1)$ and $\lambda > 0$, we define *the q-deformed Poisson distribution of the parameter* λ as the orthogonalizing probability measure for the sequence of polynomials

$$\begin{aligned}&P_0(X) = 1, \quad P_1(X) = X - \lambda,\\ &P_{n+1}(X) = (X - (\lambda + [n]_q))\, P_n(X) - \lambda[n]_q P_{n-1}(X) \quad (n \geq 1).\end{aligned}$$

In the subsequent paper [SY2], we gave the q-Poisson random variable on the q-Fock space, of which the distribution with respect to the vacuum expectation is the q-deformed Poisson distribution.

Definition 2.2. For $\xi \in \mathcal{H}$ with $||\xi|| = 1$ and $\lambda > 0$, we call the operator

$$\begin{aligned}&\left(a^*(\xi) + \sqrt{\lambda}\cdot \mathbf{1}\right)\left(a(\xi) + \sqrt{\lambda}\cdot \mathbf{1}\right)\\ = \;& a^*(\xi)a(\xi) + \sqrt{\lambda}\,(a^*(\xi) + a(\xi)) + \lambda \cdot \mathbf{1}\end{aligned}$$

the q-Poisson random variable of the parameter λ.

It is the sum of the q-number operator, $a^*(\xi)a(\xi)$, the q-Gaussian random variable $\sqrt{\lambda}\,(a^*(\xi) + a(\xi))$, and the scalar operator $\lambda \cdot \mathbf{1}$, which is natural q-deformation compatible with the result of Hudson-Pathasarathy in [HP].

3 The moments of the q-Poisson random variable

The nth moment of the q-Poisson random variable of the parameter λ can be given as the monic polynomial in λ of degree n without constant. Thus we can put the nth moment as in the form,

$$\phi(x^n) = \sum_{k=1}^{n} \mathcal{S}_q(n,k)\lambda^k,$$

where $\mathcal{S}_q(n,k)$ is the some constant.

In order to evaluate the constants $\mathcal{S}_q(n,k)$, we shall recall the set partition statistic, the number of restricted crossings which was introduced in [Fl] and also studied in [Bi]. Here, we will call it the number of arc crossings because we would like to use the terminology arcs.

We call the set $\pi = \{B_1, B_2, ..., B_k\}$ is the partition of $\{1,2,...,n\}$ if B_i's are disjoint sets, of which union is $\{1,2,...,n\}$. We shall call $B_i \in \pi$ a *block* of the partition π. For $n \geq 1$, we denote by $\mathcal{P}(\{1,2,...,n\})$ the set of partitions of $\{1,2,...,n\}$.

Definition 3.1. Let $\pi = \{B_1, B_2, ..., B_k\}$ be a partition in $\mathcal{P}(\{1,...,n\})$. If the block B_j has more than one elements (i.e. $|B_j| = m_j \geq 2$), put $B_j = \{b_{j,1}, b_{j,2},$

$\ldots, b_{j,m_j}\}$ where $b_{j,1} < b_{j,2} < \ldots < b_{j,m_j}$, then we make $(m_j - 1)$ connections like bridges $(b_{j,1}, b_{j,2})$, $(b_{j,2}, b_{j,3})$, ..., (b_{j,m_j-1}, b_{j,m_j}), successively. We have, of course, totally $\sum_{j=1}^{k}(|B_j| - 1)$ connections and we shall call them *arcs* of the partition π.

The number of arc crossings for a partition $\pi \in \mathcal{P}(\{1, \ldots, n\})$ is the number:

$$c_a(\pi) = \# \left\{ (m_1, m_2, m_3, m_4) \;\middle|\; \begin{array}{l} 1 \leq m_1 < m_2 < m_3 < m_4 \leq n, \\ (m_1, m_3) \text{ and } (m_2, m_4) \text{ are arcs} \\ \text{of } \pi \end{array} \right\}.$$

Theorem 3.2.

$$\mathcal{S}_q(n, k) = \sum_{\substack{\pi \in \mathcal{P}(\{1, \ldots, n\}) \\ \pi \text{ has precisely } k \text{ blocks}}} q^{c_a(\pi)},$$

where $c_a(\pi)$ denotes the number of the arc crossings for the partition π.

In order to see the above statement, we shall adopt the cards arrangement technique which is similar as in [ER] for juggling patterns but we will use considerably different kinds of cards.

Notation 3.3. We prepare the cards C_i $(i = 0, 1, 2, \ldots)$ for the q-creation operator. The card C_i has i inflow lines from the left and $(i + 1)$ outflow lines to the right, where one new line starts from the middle point on the ground level. For each $j \geq 1$, the inflow line of the jth level goes out to the $(j + 1)$st level without any crossing. We call the card C_i *the creation card of level i* (See Fig. 3.1).

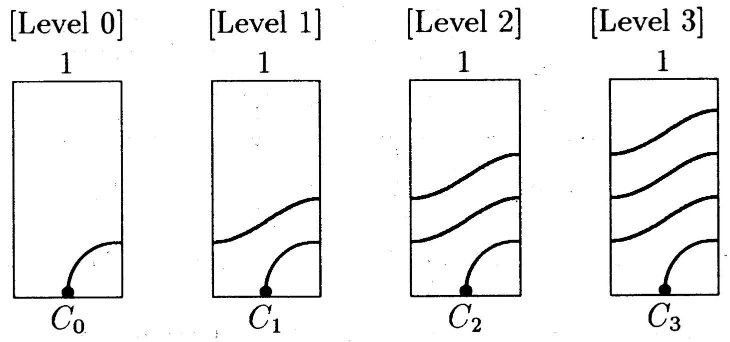


Fig. 3.1. The creation cards and the weights for the first few levels

Next we shall make the cards for the q-annihilation operator. For $i \geq 1$, we consider the cards $A_i^{(j)}$ $(j = 1, 2, \ldots, i)$ which has i inflow lines from the left and $(i - 1)$ outflow lines to the right. On the card $A_i^{(j)}$, only the inflow line of the jth level goes down to the middle point on the ground level and will be annihilated. The lines flowed into lower than jth level go in horizontally parallel and keep the levels. Hence $(j - 1)$ crossings will occur. Moreover if the line flow into the $k(> j)$th level, it will flow out to the $(k - 1)$st level without any crossing. We call the cards $A_i^{(j)}$ *the annihilation cards of level* i (See Fig. 3.2).

We shall give the weight to the card by q to the number of the crossings that occur on the card, thus the card $A_i^{(j)}$ has the weight q^{j-1} and the card C_i has 1.

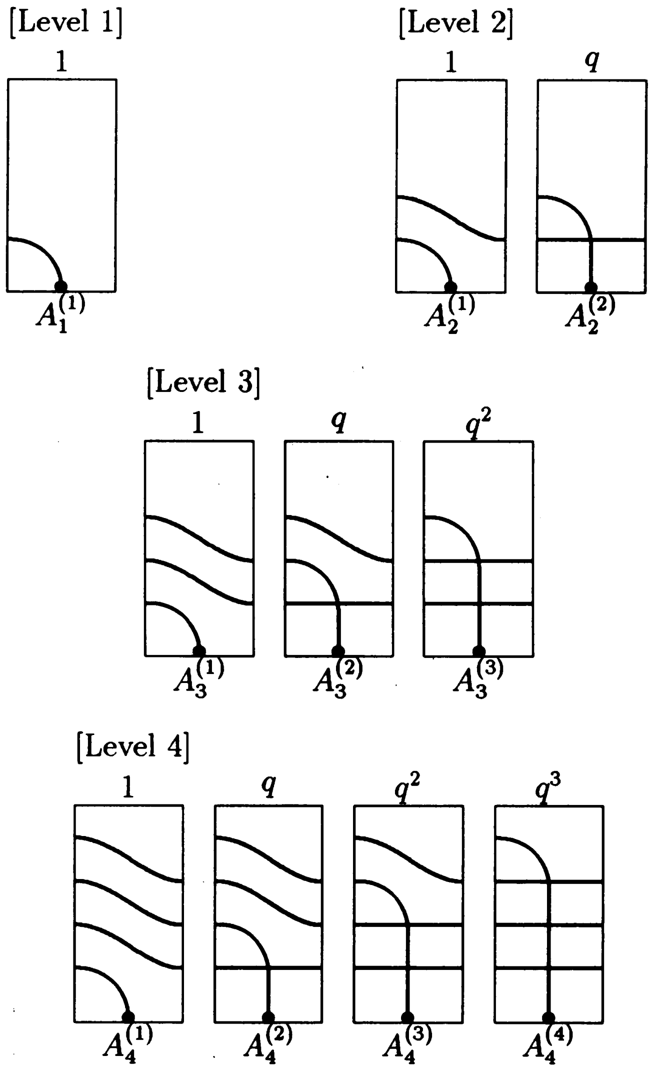


Fig. 3.2. The annihilation cards and the weights of the first few levels

Remark 3.4. The creation cards represent the relations $a^*\xi^{\otimes i} = \xi^{\otimes(i+1)}$ $(i \geq 0)$, where the number of lines correspond to the power of tensor products.

The annihilation cards reflect the relations

$$\begin{aligned} a\xi^{\otimes i} &= [i]_q \xi^{\otimes(i-1)} \\ &= \sum_{j=1}^{i} q^{j-1} (\underbrace{\xi \otimes \cdots \otimes \check{\xi} \otimes \cdots \otimes \xi}_{\text{The } j\text{th factor is deleted}}), \end{aligned}$$

where the annihilating line indicates the position of the factor in the tensor product which should be deleted.

Notation 3.5. For $i \geq 1$, we consider the cards $N_i^{(j)}$ $(j = 1, 2, \ldots, i)$ for the q-number operator. The card $N_i^{(j)}$ has i inflow lines and, the same number of, i outflow lines. Only the inflow line of the jth level goes down to the middle point on the ground and continue its flow as the first (lowest) line. The inflow lines of lower than jth level will be increased only one in upper level and ones of higher than jth level will keep their levels. Hence it will occur $(j - 1)$ crossings on the card $N_i^{(j)}$, hence, it has the weight q^{j-1}. We call the card $N_i^{(j)}$ *the number card of level* i (See Fig. 3.3).

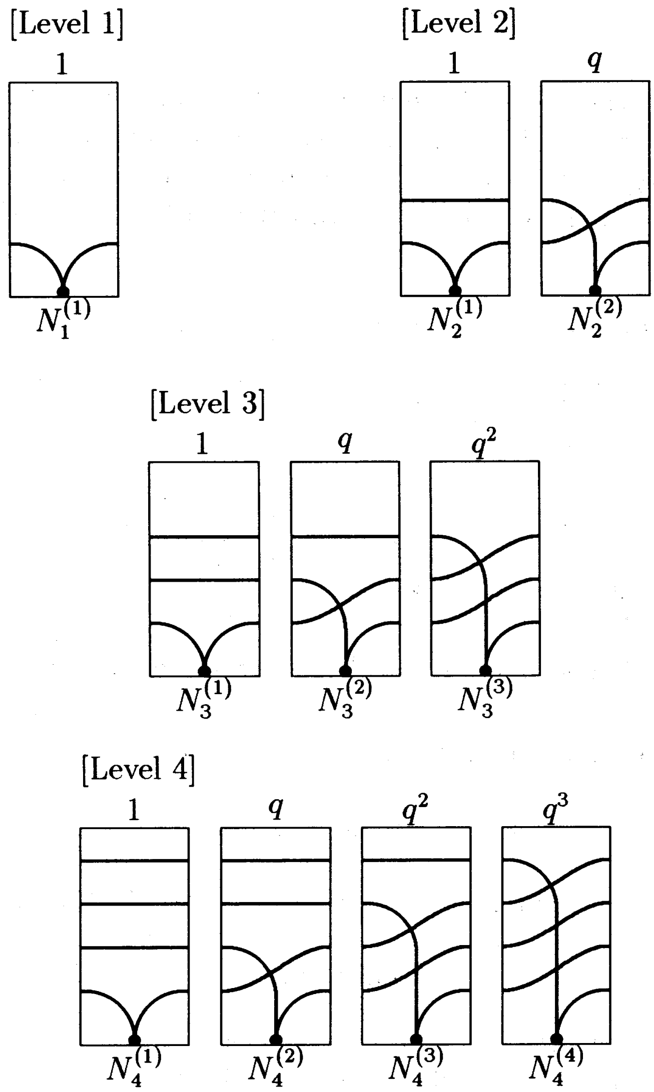
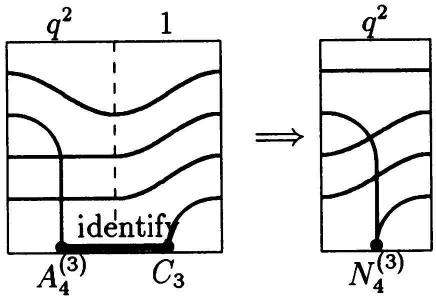


Fig. 3.3. The number cards and the weights of the first few levels

Remark 3.6. It is obvious that the number card $N_i^{(j)}$ can be obtained by composition of the annihilation card $A_i^{(j)}$ and the creation card C_{i-1} with identifying the two middle points on the ground. For instance, by gluing the cards $A_4^{(3)}$ and C_3, we have the card $N_4^{(3)}$ as follows:



Notation 3.7. Furthermore, we shall make the scalar cards K_i $(i = 0, 1, 2, ...)$ for the scalar operator. The card K_i has i horizontally parallel lines and the short pole-like segment of line at the middle point on the ground without any crossing, thus it has the weight 1. We call the card K_i *the scalar card of level* i (See Fig. 3.4).

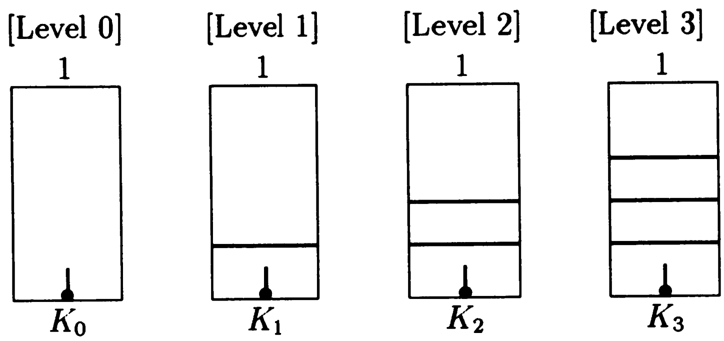


Fig. 3.4. The scalar cards and the weights of the first few levels

In order to evaluate the nth moments, we expand

$$x^n = \left(N + \sqrt{\lambda}a^* + \sqrt{\lambda}a + \lambda \cdot \mathbf{1}\right)^n$$

and consider in a monomial wise. Now we consider the monomial

$$y = f_n f_{n-1} \cdots f_2 f_1,$$

where each f_j is one of N, $(\sqrt{\lambda}a^*)$, $(\sqrt{\lambda}a)$, and $(\lambda \cdot \mathbf{1})$. Here we should be aware of that the position numbers are labeled from the right. Put the sets as

$$\begin{aligned} N(y) &= \{j : f_j = N\}, & C(y) &= \{j : f_j = (\sqrt{\lambda}a^*)\}, \\ A(y) &= \{j : f_j = (\sqrt{\lambda}a)\}, \quad \text{and} & K(y) &= \{j : f_j = (\lambda \cdot \mathbf{1})\}, \end{aligned}$$

and denote $k_N = {}^\#N(y)$, $k_C = {}^\#C(y)$, $k_A = {}^\#A(y)$, and $k_K = {}^\#K(y)$. It is trivial that we have the disjoint union of

$$\{1, 2, ..., n\} = N(y) \cup C(y) \cup A(y) \cup K(y)$$

and $k_N + k_C + k_A + k_K = n$.

For the monomial y, we define the level of the kth factor f_k, $L(k)$ $(k \geq 2)$, by

$$L(k) = \sum_{j=1}^{k-1} \chi(j), \qquad \text{with } L(1) = 0,$$

where $\chi(j)$ is the step function defined by

$$\chi(j) = \begin{cases} 1, & \text{if } j \in C(y), \\ -1, & \text{if } j \in A(y), \\ 0, & \text{if } j \in N(y) \cup K(y). \end{cases}$$

Then, we can easily obtain that the monomial y has non-zero vacuum expectation if and only if the following conditions:

(a) $L(k) \geq 0$ for every $1 \leq k \leq n$,

(b) $L(k) \geq 1$ if $k \in N(y)$, and

(c) $\sum_{j=1}^{n} \chi(j) = 0$.

Especially, we have that $\phi(y) \neq 0$ implies $k_A = k_C$.

Let y be the monomial of non-zero vacuum expectation as above, then we have

$$\phi(y) = \left(\prod_{k \in A(y) \cup N(y)} [L(k)]_q \right) \lambda^{(k_K + k_C)},$$

where $[x]_q$ denotes the q-number. Because the level $L(k)$ reflects the fact

$$(f_{k-1} f_{k-2} \cdots f_1)\Omega \in \mathbb{C}\xi^{\otimes L(k)}.$$

For the monomial y of non-zero vacuum expectation, we will make the set of partitions, $\Psi_n(y)$, of the ordered set $\{1, 2, ..., n\}$ as follows: The partition has $(k_K + k_C)$ blocks where each number in the set $K(y)$ makes a block of size 1 and each of the rest k_C blocks starts from a number in $C(y)$ and ends by a number in $A(y)$, and the intermediate numbers are in the set $N(y)$. Of course, we consider the numbers in each block are arranged in increasing order. That is,

$$\Psi_n(y) = \left\{ \pi = \{B_1, B_2, ..., B_k\} \;\middle|\; \begin{array}{l} k = k_C + k_K,\ \pi \text{ is a partition such that} \\ \text{each block } B_j = \{b_{j,1}, b_{j,2}, ..., b_{j,m_j}\}, \\ \text{where } m_j = |B_j| \text{ and } b_{j,\ell} < b_{j,\ell+1}, \\ \text{satisfies that if } m_j = 1 \text{ then } b_{j,1} \in \\ K(y), \text{ and if } m_j \geq 2 \text{ then } b_{j,1} \in C(y), \\ b_{j,m_j} \in A(y), \text{ and } b_{j,\ell} \in N(y) \text{ for } \ell \neq \\ 1, m_j. \end{array} \right\}.$$

Proposition 3.8. *Let $y = f_n f_{n-1} \cdots f_2 f_1$ be a monomial of non-zero vacuum expectation, where each f_j is one of N, $(\sqrt{\lambda}a^*)$, $(\sqrt{\lambda}a)$, and $(\lambda \cdot \mathbf{1})$, and $\Psi_n(y)$ be the subset of $\mathcal{P}(\{1,2,...,n\})$ as described above. Then we have that*

$$\phi(y) = \left(\sum_{\pi \in \Psi_n(y)} q^{c_a(\pi)} \right) \lambda^{(k_K + k_C)},$$

where $c_a(\pi)$ is the number of arc crossings for the partition π.

Proof. For the monomial y of non-zero vacuum expectation, we will arrange the cards along with the following rule: If $k \in C(y)$, that is, if $f_k = (\sqrt{\lambda}a^*)$ then we will put the creation card of level $L(k)$ at the kth position with the $\sqrt{\lambda}$-multiplicated weight, which is unique. If $k \in A(y)$, that is, if $f_k = (\sqrt{\lambda}a)$ then we will put the annihilation card of level $L(k)$ at the kth position. At this time, $L(k)$ cards are available for our arrangement and the weights should be also multiplicated by $\sqrt{\lambda}$. If $k \in N(y)$, that is, if $f_k = N$ then we will use the number card of level $L(k)$ with the original weight, where $L(k)$ cards are also available. If $k \in K(y)$, that is, if $f_k = (\lambda \cdot \mathbf{1})$ then the scalar card of level $L(k)$ with the weight λ will be used at the kth position.

We will consider all the admissible cards arrangements then it is trivial that there are $\prod_{k \in A(y) \cup N(y)} L(k)$ ways of cards arrangements. We also obtain easily that the sum of the products of the weights for all the admissible cards arrangements can be given as

$$\lambda^{k_K} \left(\sqrt{\lambda}\right)^{(k_C + k_A)} \left(\prod_{k \in A(y) \cup N(y)} [L(k)]_q \right) = \lambda^{(k_K + k_C)} \left(\prod_{k \in A(y) \cup N(y)} [L(k)]_q \right),$$

because we know $k_A = k_C$. By the rule for our cards arrangements, it follows that the partitions determined by the connected lines in the pattern of the above cards arrangements and the set $\Psi_n(y)$ are in a one-to-one correspondence.

It is not so difficult to see that the crossings appeared in the patterns of the cards arrangements are nothing else but the arc crossings for the partitions determined by the connected lines.

We remind, now, how to give the weights to the cards, then it follows that

$$\prod_{k \in A(y) \cup N(y)} [L(k)]_q = \sum_{\pi \in \Psi_n(y)} q^{c_a(\pi)},$$

which ends the proof. □

Example 3.8. We consider the monomial

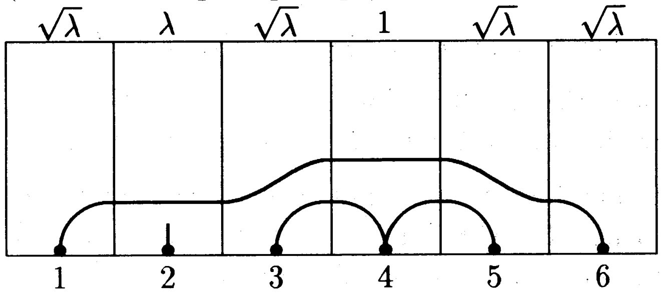
$$\left(\sqrt{\lambda}a\right) \left(\sqrt{\lambda}a\right) N \left(\sqrt{\lambda}a^*\right) (\lambda \cdot \mathbf{1}) \left(\sqrt{\lambda}a^*\right) = f_6 f_5 f_4 f_3 f_2 f_1,$$

which has non-zero expectation with the sequence of the levels $\{L(k)\}_{k=1}^{6} = \{0,1,1,2,2,1\}$. This monomial yields the four ways of the cards arrangement

$$(C_0, K_1, C_1, N_2^{(\alpha)}, A_2^{(\beta)}, A_1^{(1)}), \qquad \text{where } \alpha = 1,2 \text{ and } \beta = 1,2.$$

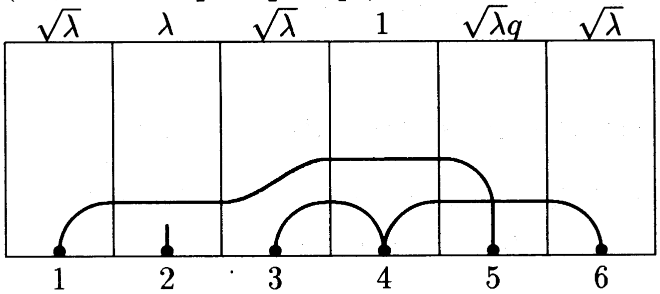
We shall list all the admissible ways of the cards arrangements with the corresponding partitions and the products of the weights.

(i) $(C_0, K_1, C_1, N_2^{(1)}, A_2^{(1)}, A_1^{(1)})$:



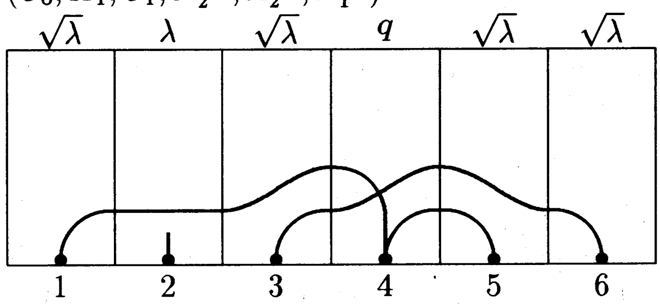
$\pi = \{\{1,6\},\{2\},\{3,4,5\}\}$, Product of weights $= \lambda^3$

(ii) $(C_0, K_1, C_1, N_2^{(1)}, A_2^{(2)}, A_1^{(1)})$:



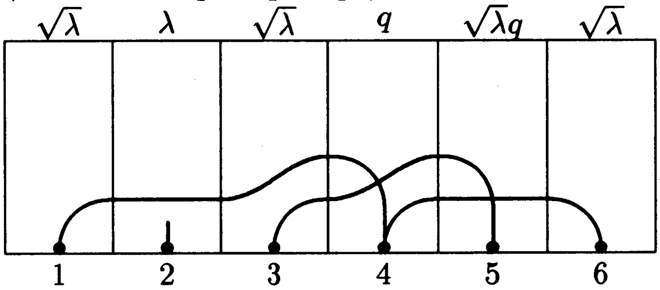
$\pi = \{\{1,5\},\{2\},\{3,4,6\}\}$, Product of weights $= q\lambda^3$

(iii) $(C_0, K_1, C_1, N_2^{(2)}, A_2^{(1)}, A_1^{(1)})$:



$\pi = \{\{1,4,5\},\{2\},\{3,6\}\}$, Product of weights $= q\lambda^3$

(iv) $(C_0, K_1, C_1, N_2^{(2)}, A_2^{(2)}, A_1^{(1)})$:



$\pi = \{\{1,4,6\},\{2\},\{3,5\}\}$, Product of weights $= q^2\lambda^3$

The vacuum expectation of this monomial is $[2]_q[2]_q[1]_q\lambda^3 = (1 + 2q + q^2)\lambda^3$, which is given by the sum of the products of the weights listed above.

Remark 3.9. The numbers $\mathcal{S}_q(n,k)$ has been already investigated in [Bi] as a certain q-deformation of the Stirling number of the second kind. He introduced them as the coefficients of the generating function for the continued fraction, which corresponds to the orthogonal polynomials for our q-deformed Poisson distribution. Hence our card arrangement method is nothing else but the certain graphical interpretation for his q-deformation, which is intermediated by the q-cration and q-annihilation operators.

4 A q-deformed moments-cumulants relation

Having the result of the previous section in mind, we should like to propose the following q-deformed moment-cumulant relation:

$$\mu_n = \sum_{\substack{\pi \in \mathcal{P}(\{1,\ldots,n\}) \\ \pi = \{B_1, B_2, \ldots, B_k\}}} q^{c_a(\pi)} \prod_{i=1}^{k} \alpha_{|B_i|}, \quad n \geq 1,$$

where $c_a(\pi)$ is the number of arc crossings of the partition π, which interpolates the relations for usual and for free exactly.

Theorem 4.1. *The nth moments of our q-deformed Poisson random variable of parameter λ can be given as the above relation by putting $\alpha_i = \lambda$ $(i \geq 1)$.*

This result is compatible with the characterization for the usual Poisson random variable that the Poisson distribution has the constant cumulant in all orders. Furthermore, the above moments-cumulants relation characterizes the q-Gaussian law by putting $\alpha_i = 0$ for $i \geq 3$, which interpolates the usual Gaussian law and the semi-circle one.

References

[Bi] Biane, P.: Some properties of crossings and partitions. Discrete Math. **175**, 41–53 (1997)

[BS1] Bożejko, M. and Speicher, R.: An example of a generalized Brownian motion. Commun. Math. Phys. **137**, 519–531 (1991)

[BS2] Bożejko, M. and Speicher, R.: Interpolations between bosonic and fermionic relations given by generalized Brownian motions. Math. Z. **222**, 135–160 (1996)

[BKS] Bożejko, M.,Kümmerer, B., and Speicher, R.: q-Gaussian processes: Non-commutative and classical aspects. Commun. Math. Phys. **185**, 129–154 (1997)

[ER] Ehrenborg, R. and Readdy, M.: Juggling and application to q-analogues. Discrete Math. **157**, 107–125 (1996)

[Fl] Flajolet, P.: Combinatorial aspects of continued fractions. Discrete Math. **32**, 41–53 (1980)

[HP] Hudson, R. L. and Parthasarathy, K. R.: Quantum Ito's formula and stochastic evolution. Commun. Math. Phys. **93**, 301–323 (1984)

[SY1] Saitoh, N. and Yoshida, H.: A q-deformed Poisson distribution based on orthogonal polynomials. J. Phys. A: Math. Gen. **33**, 1435–1444 (2000)

[SY2] Saitoh, N. and Yoshida, H.: q-deformed Poisson random variables on q-Fock space. J. Math. Phys. **41**, 5767–5772 (2000)