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MULTI-DIMENSIONAL QUANTUM AZÉMA MARTINGALES

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ABSTRACT. We introduce a multi-dimensional generalization of the quantum Azéma martingale.

1. INTRODUCTION

We will study the process defined by the quantum stochastic differential equations

$$\mathrm{d}X_i(t) = A_i^+(t) + \sum_{j,k,\ell=1}^{n} \int_0^t C_{j\ell}^{ik} X_j(s)\mathrm{d}\Lambda_k^\ell(s), \tag{1.1}$$

$$\mathrm{d}V^i(t) = A_i(t) + \sum_{j,k,\ell=1}^{n} \int_0^t \overline{C_{jk}^{i\ell}} V^j(s)\mathrm{d}\Lambda_k^\ell(s), \tag{1.2}$$

where A_i^+, A_i, Λ_j^i are the creation, annihilation, and conversation processes on the boson Fock space $\Gamma\big(L^2(\mathbb{R}_+, \mathbb{C}^n)\big)$,

$$A_i^+(t) = A^+(\mathbf{1}_{[0,t[} \otimes e_i), \quad A_i(t) = A(\mathbf{1}_{[0,t[} \otimes e_i), \quad \Lambda_j^i(t) = \Lambda\big(\mathbf{1}_{[0,t[} \otimes |e_i\rangle\langle e_j|\big),$$

and $C_{i\ell}^{jk}$ are complex coefficients.

The special case $n = 1$ and $C_{11}^{11} = q-1$ leads to the quantum Azéma martingales defined by Parthasarathy [Par90] to get a quantum version of the classical Azéma martingales introduced by Emery in [Eme89]. Emery proved that the classical Azéma martingales have the chaotic representation property, i.e. the iterated integrals

$$Q^{(0)}(t) = 1, \quad Q^{(1)}(t) = M_t, \quad Q^{(n)}(t) = \int_0^t Q^{(n-1)}(s)\mathrm{d}M_s, \quad n = 2, 3, \ldots$$

are total in $L^2(\Omega, \mathcal{F}, P)$, where $(\Omega, \mathcal{F}, P)$ is the probability space on which $(M_t)_{t\in\mathbb{R}_+}$ is defined and $\mathcal{F}$ the σ-algebra generated by the process.

Other martingales having this property are, e.g., the Brownian motion and the compensated Poisson process. The Azéma martingale provided the first example of a martingale having the chaotic representation property that is not a Lévy process. The problem of the classification of all normal martingales having the chaotic representation property is still open.

Parthasarathy's [Par90] construction of the quantum Azéma martingles lead to a new proof of their chaotic representation property.

It was shown by Schürmann [Sch91, Sch93] that the the quantum Azéma martingales arise as a component of a Lévy process on an involutive bialgebra.

The goal of this paper is to construct multi-dimensional analogues of the quantum Azéma martingales and to show that they lead to classical processes having the chaotic representation property. For some results on classical multi-dimensional Azéma martingales, see [AÉ94, AÉ96].

Roughly speaking, Parthasarathy's proof of the chaotic representation property relies on three properties of the quantum Azéma martingale defined by

$$\begin{aligned} X(t) &= A^+(t) + \int_0^t (q-1)X(s)\mathrm{d}\Lambda_s, \\ V(t) &= A(t) + \int_0^t (q-1)V(s)\mathrm{d}\Lambda_s. \end{aligned}$$

He shows that the symmetric linear combinations

$$Z(t) = X(t) + V(t), \qquad t \in \mathbb{R}_+,$$

are self-adjoint for $-1 \leq q \leq 1$ (even bounded for $-1 \leq q < 1$). Furthermore, he shows that they commute for all times, i.e.

$$[Z(t), Z(s)] = 0$$

for all $s, t \in \mathbb{R}_+$. These two properties imply that there exists a classical process $(\tilde{Z}_t)_{t\in\mathbb{R}_+}$ having the same distribution as the joint spectral density of the commuting family of self-adjoint operators $\big(Z(t)\big)_{t\in\mathbb{R}_+}$ evalutated in the vacuum state. Then he shows that the iterated integrals of $Z(t)$ generate the same space from the vacuum as the quantum Browian motion $Q_t = A^+(t) + A(t)$. Since we know that the Brownian motion has the chaotic representation property, this allows one to show that the classical process $(\tilde{Z}_t)_{t\in\mathbb{R}_+}$ also has this property.

In this paper we will look for the conditions on the coefficients $C^{ik}_{j\ell}$ in the quantum stochastic differential equations (1.1) and (1.2), that guarantee that the multi-dimensional process defined by

$$Z_i(t) = X_i(t) + V^i(t)$$

has similar properties. In the end we get a multi-dimensional version of Parthasarathy's result by combining these conditions.

Note that the $Z_i(t)$ can be defined directly by one single quantum stochastic differential equation

$$Z_i(t) = A_i^+(t) + A_i(t) + \sum_{j,k,\ell=1}^{n} \int_0^t C_{j\ell}^{ik} Z_j(s) \mathrm{d}\Lambda_k^\ell(s), \tag{1.3}$$

if the coefficients satisfy

$$C_{j\ell}^{ik} = \overline{C_{jk}^{i\ell}}$$

for all $1 \leq i, j, k, \ell \leq n$.

2. Multi-dimensional Azéma processes as non-commutative Lévy processes

Let $\mathcal{B}$ be the free algebra generated by x_i, a_j^i, $i, j = 1, \ldots, n$. Setting

$$\Delta x_i = x_j \otimes a_i^j + 1 \otimes x_i, \quad \Delta a_j^i = a_k^i \otimes a_j^k, \quad \varepsilon(x_i) = 0, \quad \varepsilon(a_j^i) = \delta_{ij},$$

and extending $\Delta : \mathcal{B} \to \mathcal{B} \otimes \mathcal{B}$, $\varepsilon : \mathcal{B} \to \mathbb{C}$ as algebra homomorphisms, we get a bialgebra. Taking the free $*$-algebra $\tilde{\mathcal{B}}$ generated by these elements, and extending $\Delta : \tilde{\mathcal{B}} \to \tilde{\mathcal{B}} \otimes \tilde{\mathcal{B}}$, $\varepsilon : \tilde{\mathcal{B}} \to \mathbb{C}$ as algebra homomorphisms, we get an involutive bialgebra $\tilde{\mathcal{B}}$. For the adjoints $v^i = (x_i)^*$ and $b_i^j = \left(a_j^i\right)^*$ we have

$$\Delta v^i = v^j \otimes b_j^i + 1 \otimes v^i, \quad \Delta b_i^j = b_i^k \otimes b_k^j, \quad \varepsilon(v^i) = 0, \quad \varepsilon(b_i^j) = \delta_{ij}.$$

Recall the definition of a Lévy process on an involutive bialgebra.

Definition 2.1. [Sch93] Let $(\mathcal{B}, \Delta, \varepsilon)$ be a an involutive bialgebra. A quantum stochastic process $\{j_{st}\}_{0 \leq s \leq t \leq T}$ on $\mathcal{B}$ over some quantum probability space $(\mathcal{A}, \Phi)$ is called a *Lévy process on the involutive bialgebra* $\mathcal{B}$, if the following four conditions are satisfied.

1. (Increment property) We have
$$\begin{aligned} j_{rs} \star j_{st} &= j_{rt} \quad \text{for all } 0 \leq r \leq s \leq t \leq T, \\ j_{tt} &= \varepsilon \mathbf{1}_{\mathcal{A}} \quad \text{for all } 0 \leq t \leq T. \end{aligned}$$
2. (Independence of increments) The family $\{j_{st}\}_{0 \leq s \leq t \leq T}$ is independent (w.r.t. Φ), i.e.
   (i) $\Phi\big(j_{s_1 t_2}(b_1) \cdots j_{s_n t_n}(b_n)\big) = \Phi\big(j_{s_1 t_1}(b_1)\big) \cdots \Phi\big(j_{s_n t_n}(b_n)\big)$ for all $b_1, \ldots, b_n \in \mathcal{B}$, and all $0 \leq s_1 \leq t_1 \leq s_2 \leq \cdots \leq t_n \leq T$ and
   (ii) $[j_{st}(b_1), j_{s't'}(b_2)] = 0$ for all $b_1, b_2 \in \mathcal{B}$ and all s, t, s', t' with $0 \leq s \leq t \leq T$, $0 \leq s' \leq t' \leq T$, and $]s, t[\cap]s', t'[= \emptyset$.
3. (Stationarity of increments) The distribution $\varphi_{st} = \Phi \circ j_{st}$ of j_{st} depends only on the difference $t - s$.

4. (Weak continuity) The quantum random variables j_{st} converge to j_{ss} in distribution for $t \searrow s$.

If $\{j_{st}\}_{0\leq s\leq t\leq T}$ is a Lévy process on an involutive bialgebra, then the marginal distributions $\varphi_{t-s} = \varphi_{st} = \Phi \circ j_{st}$ form a convolution semi-group of states, i.e. $\varphi_0 = \varepsilon$ and $\varphi_s \star \varphi_t = \varphi_{s+t}$ for all s, t. It can be shown that there exists a unique hermitian linear functional $\psi : \mathcal{B} \to \mathbb{C}$ such that $\varphi_t = \exp_\star t\psi = \varepsilon + t\psi + \frac{t^2}{2}\psi \star \psi + \cdots$. Furthermore, ψ is conditionally positive, i.e. positive on $\mathcal{B}^0 = \ker \varepsilon$, and kills the unit of $\mathcal{B}$, i.e. $\psi(\mathbf{1}_\mathcal{B}) = 0$. Such functionals are called generators.

Conversely, given a generator $\psi : \mathcal{B} \to \mathbb{C}$, we can define a convolution semi-group of states by $\varphi_t = \exp_\star t\psi$ and reconstruct a Lévy process on $\mathcal{B}$ from it.

Let us briefly describe how one can construct a realization of the Lévy process on a boson Fock space. By a GNS-type construction one can complete a generator ψ to a Schürmann triple (ρ, η, ψ), i.e. a triple consisting of a unital $*$-representation ρ of $\mathcal{B}$ on some pre-Hilbert space D, a linear map $\eta : \mathcal{B} \to D$ that satisfies

$$\eta(ab) = \rho(a)\eta(b) + \eta(a)\varepsilon(b), \qquad \text{for all } a, b \in \mathcal{B}, \tag{2.1}$$

and a hermitian linear functional $\psi : \mathcal{B} \to \mathbb{C}$ that satisfies

$$\psi(ab) = \varepsilon(a)\psi(b) + \langle \eta(a^*), \eta(b)\rangle + \psi(a)\varepsilon(b), \qquad \text{for all } a, b \in \mathcal{B}. \tag{2.2}$$

The realization of the Lévy process associated to ψ on the boson Fock space $\Gamma\big(L^2(\mathbb{R}_+, \overline{D})\big)$ is then given as solution of the quantum stochastic differential equations

$$j_{st}(b) = \varepsilon(b)\mathrm{id} + \left(\int_s^t j_{s\tau} \otimes \mathrm{d}I_\tau\right)\Delta(b), \qquad \text{for all } b \in \mathcal{B}, \tag{2.3}$$

where the integrator $\mathrm{d}I$ is given by

$$\mathrm{d}I_t(b) = \mathrm{d}\Lambda_t(\rho(b) - \varepsilon(b)\mathrm{id}) + \mathrm{d}A_t^+(\eta(b)) + \mathrm{d}A_t(\eta(b^*)) + \psi(b)\mathrm{d}t.$$

For details and proofs see [Sch93].

Set

$$R^{ij}_{k\ell} = C^{ij}_{k\ell} + \delta_{ij}\delta_{k\ell}$$

and define a Schürmann triple (ρ, η, ψ) on $\tilde{\mathcal{B}}$, acting on $\mathbb{C}^n$, by

$$\begin{aligned} \rho(a^i_j)e_\ell &= \sum_{k=1}^n R^{ik}_{jl} e_k, \qquad \rho(x_i) = 0, \\ \eta(a^i_j) = \eta(b^i_j) &= 0, \qquad \eta(x_i) = e_i, \quad \eta(v^i) = 0, \\ \psi(a^i_j) &= 0, \qquad \psi(x_i) = 0, \end{aligned}$$

where $e_1, \dots, e_n$ are a basis of $\mathbb{C}^n$. There exists a unique Schürmann triple having these values on the generators of $\tilde{\mathcal{B}}$. The quantum stochastic differential equations for the associated Lévy process are

$$\begin{aligned} j_{st}(a^i_j) &= \delta_{ij}\mathrm{id} + \int_s^t \sum_{k,p,q} j_{s\tau}(a^i_k)\left(R^{kp}_{jq} - \delta_{kj}\delta_{pq}\right) \mathrm{d}\Lambda^q_p(\tau), \\ j_{st}(x_i) &= A^+_i(t) - A^+_i(s) + \int_s^t \sum_j j_{s\tau}(x_j)\left(R^{jp}_{iq} - \delta_{ij}\delta_{pq}\right) \mathrm{d}\Lambda^q_p(\tau), \end{aligned}$$

Comparing the second equation with (1.1), we see that we have

$$j_{0t}(x_i) = j_t(x_i) = X_i(t), \quad j_{0t}(v^i) = j_t(v^i) = V^i(t).$$

The theory of Lévy processes now implies that the domains of the $X_i(t)$, $V_i(t)$ obtained by solving the quantum stochastic differential equations (1.1) and (1.2) contain $\bigcap_{\alpha \in \mathbb{R}_+} \operatorname{dom} \alpha^N$, where N denotes the number operator on the boson Fock space.

Note finally that we can also define the same Lévy process as a solution of the backwards quantum stochastic differential equations

$$j_{st}(a^i_j) = \delta_{ij}\mathrm{id} + \int_s^t \sum_{k,p,q} \left(R^{ip}_{kq} - \delta_{ik}\delta_{pq}\right) \mathrm{d}\Lambda^q_p(\tau) j_{\tau t}(a^k_j), \tag{2.4}$$

$$j_{st}(x_i) = \int_s^t \sum_j \mathrm{d}A^+_i(\tau) j_{\tau t}(a^j_i), \tag{2.5}$$

cf. [Sch93].

3. When are our multi-dimensional Azéma martingales commutative?

In this section we find conditions on the coefficients that quarantee that the processes commute. In the one-dimensional case[Par90], no conditions were necessary, but the commutativity already followed from the form of the quantum stochastic differential equations.

Proposition 3.1. **(a):** *The two-sided ideal $\mathcal{I}$ of $\mathcal{B}$ generated by the elements*

$$\begin{array}{rl} x_i x_j - x_j x_i, & i,j = 1,\dots,n \\ a^k_i x_j + x_i a^k_j - a^k_j x_i - x_j a^k_i, & i,j,k = 1,\dots,n \\ a^k_i a^\ell_j - a^k_j a^\ell_i, & i,j,k,\ell = 1,\dots,n \end{array}$$

is a coideal.

(b): *The two-sided ideal $\tilde{\mathcal{I}}$ of $\tilde{\mathcal{B}}$ generated by the elements*

$$\begin{array}{ll} (x_i+v^i)(x_j+v^j)-(x_j+v^j)(x_i+v^i), & i,j=1,\ldots,n \\ a_i^k x_j + x_i a_j^k - a_j^k x_i - x_j a_i^k + a_i^k v^j + v^i a_j^k - a_j^k v^i - v^j a_i^k, & i,j,k=1,\ldots,n \\ b_k^i x_j + x_i b_k^j - b_k^j x_i - x_j b_k^i + b_k^i v^j + v^i b_k^j - b_k^j v^i - v^j b_k^i, & i,j,k=1,\ldots,n \\ a_i^k a_j^\ell - a_j^k a_i^\ell,\ a_i^k b_\ell^j - a_j^k b_\ell^i,\ b_k^i a_j^\ell - b_k^j a_i^\ell,\ b_k^i b_\ell^j - b_k^j b_\ell^i, & i,j,k,\ell=1,\ldots,n \end{array}$$

is a coideal.

Proof. We only show (a), the proof of (b) is similar.
We have to show

$$\Delta(\mathcal{I}) \subseteq \mathcal{I}\otimes\mathcal{B}+\mathcal{B}\otimes\mathcal{I}.$$

and $\varepsilon(\mathcal{I})=\{0\}$. It is sufficient, to verify this on the algebraic generators of $\mathcal{I}$.
We get

$$\begin{aligned} &\Delta(x_ix_j-x_jx_i) \\ =\;& (x_k\otimes a_i^k+1\otimes x_i)(x_\ell\otimes a_j^\ell+1\otimes x_j)-(x_\ell\otimes a_j^\ell+1\otimes x_j)(x_k\otimes a_i^k+1\otimes x_i) \\ =\;& x_kx_\ell\otimes(a_i^ka_j^\ell-a_j^ka_i^\ell)+x_k\otimes(a_i^kx_j+x_ia_j^k-a_j^kx_i-x_ja_i^k)+1\otimes(x_ix_j-x_jx_i), \end{aligned}$$

similarly

$$\begin{aligned} \Delta(a_i^kx_j+x_ia_j^k-a_j^kx_i-x_ja_i^k) \;=\;& a_\ell^kx_m\otimes(a_i^\ell a_j^m-a_j^\ell a_i^m)+x_ma_l^k\otimes(a_i^ma_j^\ell-a_j^ma_i^\ell) \\ & +a_\ell^k\otimes(a_i^\ell x_j+x_ia_j^\ell-a_j^\ell x_i-x_ja_i^\ell), \end{aligned}$$

and finally

$$\Delta(a_i^ka_j^\ell-a_j^ka_i^\ell) \;=\; a_r^ka_s^\ell\otimes(a_i^ra_j^s-a_j^ra_i^s).$$

□

Corollary 3.2. **(a):** *If the coefficients $R_{j\ell}^{ik}$ satisfy*

$$R_{j\ell}^{ik}=R_{\ell j}^{ik}$$

and

$$R_{j\ell}^{ks}R_{j\ell}^{ik}=R_{j\ell}^{ik}R_{j\ell}^{ik}$$

for all i,j, then we have

$$[X_i(s),X_j(t)]=0$$

(b): *If the coefficients $R_{j\ell}^{ik}$ satisfy*

$$R_{j\ell}^{ik}=R_{\ell j}^{ik},$$

and

for all i, j, then we have

$$[Z_i(s), Z_j(t)] = 0$$

Remark 3.3. Expressed in terms of the coefficients $C^{ik}_{j\ell}$ appearing in the quantum stochastic differential equations (1.1) and (1.2) the conditions become

..

and

..

Proof. □

4. When are our multi-dimensional Azéma martingales bounded?

Lemma 4.1. *Suppose that there exists a constant $0 \leq M < 1$ such that*

$$\left\| \left(\sum_{k,\ell} \overline{a_k} b_\ell R^{ik}_{j\ell} \right)_{1 \leq k,\ell \leq n} \right\| \leq M ||a|| \, ||b||$$

for all $a = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}, b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \in \mathbb{C}^n$. *Then the operators $j_{st}(a^i_j)$ are contractions.*

Proof. Using the first fundamental lemma [Par92, Proposition 25.1], we get the integral equations

$$\begin{aligned} \langle \mathcal{E}(f), j_t(a^i_j)\mathcal{E}(g)\rangle &= \delta_{ij}\langle \mathcal{E}(f), \mathcal{E}(g)\rangle \\ &\quad + \int_0^t \sum_{k,q,p} \langle \mathcal{E}(f), j_s(a^i_k)\mathcal{E}(g)\rangle (R^{kp}_{jq} - \delta_{kj}\delta_{pq})\overline{f_p(t)} g_q(t) \mathrm{d}s \end{aligned}$$

for the matrix elements of the operators $j_{st}(a^i_j)$, where $f, g \in L^2(\mathbb{R}_+, \mathbb{C}^n)$. If we set

$$A_{f,g}(t) = \langle \mathcal{E}(f), j_t(a^i_j)\mathcal{E}(g)\rangle$$

and

$$\begin{aligned} R_{f,g}(t) &= \left(\sum_{p,q} R^{ip}_{jq} \overline{f_p(t)} g_q(t) \right)_{1 \leq i,j \leq n}, \\ \overline{R}_{f,g}(t) &= \left(\sum_{p,q} (R^{ip}_{jq} - \delta_{ij}\delta_{pq}) \overline{f_p(t)} g_q(t) \right)_{1 \leq i,j \leq n}, \end{aligned}$$

then we get the matrix-valued differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t} A_{f,g}(t) = A_{f,g}(t)\overline{R}_{f,g}(t),$$

with the initial condition

$$A_{f,g}(0) = \Big(\langle \mathcal{E}(f), \mathcal{E}(g)\rangle \delta_{ij}\Big)_{1\leq i,j\leq n}.$$

The unique solution of this equation is

$$A_{f,g}(t) = A_{f,g}(0) \exp\left(\int_0^t \overline{R}_{f,g}(s)\mathrm{d}s\right),$$

and therefore we get

(4.1)
$$\langle \mathcal{E}(f), j_{st}(a^i_j)\mathcal{E}(g)\rangle = \left(\exp\left(\int_s^t R_{f,g}(s)\mathrm{d}s\right)\right)_{ij} \exp\left(\int_0^s \langle f(\tau), g(\tau)\rangle \mathrm{d}\tau + \int_t^\infty \langle f(\tau), g(\tau)\rangle \mathrm{d}\tau\right)$$

Using this identity, we get the desired estimate. □

Proposition 4.2. *Suppose that there exists a constant* $0 \leq M < 1$ *such that*

$$\left\| \left(\sum_{k,\ell} \overline{a_k} b_\ell R^{ik}_{j\ell}\right)_{1\leq k,\ell\leq n} \right\| \leq M ||a||\,||b||$$

for all $a = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}, b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \in \mathbb{C}^n$. *Then the operators* $X_i(t)$, $V^i(t)$ *and* $Z_i(t) = X_i(t) + V^i(t)$ *are bounded for all* $t \leq t$.

Proof. Using the first fundamental lemma [Par92, Proposition 25.1] and the backward stochastic differential equation (2.5) for $X_i(t) = j_t(x_i)$, we get

$$\langle \mathcal{E}(f), X_i(t)\mathcal{E}(g)\rangle = \int_0^t \langle \mathcal{E}(f), j_{st}(a^j_i)\mathcal{E}(g)\rangle \overline{f_j(s)}\mathrm{d}s$$

Using (4.1), this implies that the $X_i(t)$ are bounded, and therefore also the $V_i(t)$ and the $Z_i(t)$. □

[FSS01]

5. Chaos completeness

Define the iterated integrals

$$Q_{i_1,\dots,i_r}(t) = \int_{0\leq t_1\leq\cdots t_r\leq t} \mathrm{d}Q_{i_1}\cdots \mathrm{d}Q_{i_1},$$

where $Q_i(t) = A_i^+(t) + A_i(t)$, and

$$I_{i_1,\dots,i_r}(t) = \int_{0\leq t_1\leq\cdots t_r\leq t} \mathrm{d}Z_{i_1}\cdots \mathrm{d}Z_{i_1}$$

then we have

$$Q_{i_1,\dots,i_r}(t)\Omega = I_{i_1,\dots,i_r}(t)\Omega,$$

for all $r\in\mathbb{N}$, $i_1,\dots,i_r\in\{1,\dots,n\}$, and $t\geq 0$.

This implies that we can copy Parthasarathy's proof of the chaos completeness whenever the Z_i define a bounded and commutative self-adjoint operator process.

Theorem 5.1. *Suppose that the coefficients satisfy ...*

Then there exists a classical process $\left(\left(\tilde{Z}_1(t),\dots,\tilde{Z}_n(t)\right)\right)_{t\in\mathbb{R}_+}$ *having the same joint distribution as the quantum stochastic process* $\left(\left(Z_1(t),\dots,Z_n(t)\right)\right)_{t\in\mathbb{R}_+}$ *and havin the chaotic representation property, i.e. the iterated integrals*

$$\tilde{I}_{i_1,\dots,i_r}(t) = \int_{0\leq t_1\leq\cdots t_r\leq t} \mathrm{d}\tilde{Z}_{i_1}\cdots \mathrm{d}\tilde{Z}_{i_1}$$

with $r\in\mathbb{N}$ *and* $i_1,\dots .i_r\in\{1,\dots,n\}$ *are total in the* L^2*-space over the underlying probability space (where we assume that the* σ*-algebra is generated by the process).*

References

[AÉ94] S. Attal and M. Émery. Équations de structure pour des martingales vectorielles. In *Séminaire de Probabilités, XXVIII*, pages 256–278. Springer, Berlin, 1994.

[AÉ96] S. Attal and M. Émery. Martingales d'Azéma bidimensionnelles. *Astérisque*, (236):9–21, 1996. Hommage à P. A. Meyer et J. Neveu.

[Eme89] M. Emery. On the Azéma martingales. In *Séminaire de Probabilités XXIII*, volume 1372 of *Lecture Notes in Math.* Springer-Verlag, Berlin, 1989.

[FSS01] U. Franz, R. Schott, and M. Schürmann. Lévy processes and Brownian motion on braided spaces. Preprint, 2001.

[Par90] K.R. Parthasarathy. Azéma martingales and quantum stochastic calculus. In R.R. Bahadur, editor, *Proc. R.C. Bose Memorial Symposium*, pages 551–569. Wiley Eastern, 1990.

[Par92] K.R. Parthasarathy. *An Introduction to Quantum Stochastic Calculus*. Birkhäuser, 1992.

[Sch91] M. Schürmann. The Azéma martingales as components of quantum independent increment processes. In J. Azéma, P.A. Meyer, and M. Yor, editors, *Séminaire de Probabilités XXV*, volume 1485 of *Lecture Notes in Math.* Springer-Verlag, 1991.

[Sch93] M. Schürmann. *White Noise on Bialgebras*, volume 1544 of *Lecture Notes in Math.* Springer-Verlag, Berlin, 1993.

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