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A REPORT ON HEEGAARD SPLITTINGS OF EXTERIORS OF 1-GENUS 1-BRIDGE KNOTS

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1. Introduction

Let M be a closed orientable 3-manifold (mainly, a lens space), and K a knot in the 3-sphere S^3 or M in this note.

A property embedded arc t in a solid torus V is called *trivial* if it is boundary parallel, namely, there is a disk C embedded in V such that $t \subset \partial C$ and $C \cap \partial V = \mathrm{cl}(\partial C - t)$. This disk C is called a *cancelling disk* of t.

Definition 1.1. ((1,1)-knots, (1,1)-splittings) We call K a *1-genus 1-bridge knot* in M if M is a union of two solid tori V_1 and V_2 glued along their boundary tori ∂V_1 and ∂V_2 and if K intersects each solid torus V_i in a trivial arc t_i for $i = 1$ and 2. The splitting $(M, K) = (V_1, t_1) \cup_{H_1} (V_2, t_2)$ is called a *1-genus 1-bridge splitting* of (M, K), where $H_1 = V_1 \cap V_2 = \partial V_1 = \partial V_2$. We call this splitting a $(1,1)$-*splitting* for short, and say that K is a $(1,1)$-knot. See Figure 1.

Torus knots and 2-bridge knots are $(1,1)$-knots.

Definition 1.2. ((2,0)-splitting, tunnel number one knots) We say that the pair (M, K) admits a $(2,0)$-*splitting* if M is a union of two handlebodies of genus two W_1 and W_2 and K is a 'core' in W_1. Note that $\mathrm{cl}(W_1 - N(K))$ is a compression body homeomorphic to a union of (a torus) $\times[0,1]$ and a 1-handle which has an attaching disk in (a torus) $\times\{1\}$. K is a tunnel number 1 knot if and only if (M, K) has a $(2,0)$-splitting.

An arc γ embedded in $\mathrm{int}W_1$ is called an *unknotting tunnel* if $\gamma \cap K = \partial\gamma$ and W_1 collapses to $K \cup \gamma$, see Figure 2.

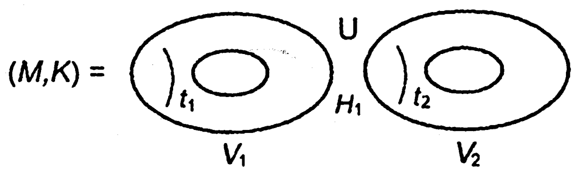

FIGURE 1

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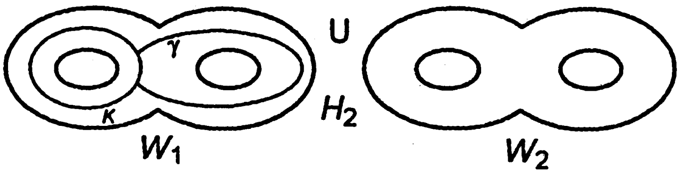

FIGURE 2

Definition 1.3. (meridionally stabilized) A $(2,0)$-splitting $(M, K) = (W_1, K) \cup_{H_2} (W_2, \emptyset)$ is called *meridionally stabilized* if there are a meridionally compressing disk D_1 of H_2 in (W_1, K) and an essential disk D_2 in W_2 such that ∂D_1 and ∂D_2 intersect each other transversely in a single point in H_2. ('meridionally' means $D_1 \cap K =$ 1 pt transversely.)

Exercise 1.4. (1) Show that a $(1,1)$-knot admits a $(2,0)$-splitting and recognize where the unknotting tunnel is.

(2) Confirm that we can obtain a $(1,1)$-splitting from a meridionally stabilized $(2,0)$-splitting.

Question. Is any $(2,0)$-splitting of a $(1,1)$-knot meridionally stabilized ?

The answer is No. This was pointed out by K.Morimoto that every torus knot has only one isotopy class of $(1,1)$-splitting torus, which is a corollary of Theorem 3 in [12] and the uniqueness of genus one Heegaard splitting. If all the $(2,0)$-splitting were meridionally stabilized for a torus knot, then the torus knot exterior would have at most two genus two Heegaard splittings derived from the unique $(1,1)$-splitting. However, there is a torus knot such that its exterior has three genus two Heegaard splittings [1]. (See also Figure 3.)

Thus we need a clue to classify the unknotting tunnels for $(1,1)$-knots.

Every unknotting tunnel of a tunnel number one knot in S^3 may be slid and isotoped to lie entirely in its minimal bridge sphere [6]. Further, we can observe that the unknotting tunnels γ of torus knots in S^3 are classified into two types: (1) γ determine a $(2,0)$-splitting that is meridionally stabilized; (2) γ may be slid and isotoped to lie entirely in its $(1,1)$-splitting torus, see Figure 3. Further, any $(2,0)$-splitting of a satellite knot in S^3 is meridionally stabilized [13]. Thus we present the next question instead of the above one.

Question. Can an unknotting tunnel of a $(1,1)$-knot be slid and isotoped to lie entirely in its $(1,1)$-splitting torus ?

2. MAIN THEOREM AND EXAMPLE

On the last question in the previous section, we have:

Theorem 2.1 ([4]). *Let K be a knot in the 3-sphere S^3. Suppose there are two splittings $(S^3, K) = (V_1, t_1) \cup_{H_1} (V_2, t_2) = (W_1, K) \cup_{H_2} (W_2, \emptyset)$. Then at least one of the following conditions holds.*

(1) *The $(2,0)$-splitting H_2 is meridionally stabilized.*
(2) *There is an arc γ which forms a spine of (W_1, K) and is isotopic into the torus H_1. Moreover, we can take γ so that there is a cancelling disk C_i of the arc t_i in (V_i, t_i) with $\partial C_i \cap \gamma = \partial\gamma = \partial t_i$ for $i = 1$ or 2.*
(3) *There is an essential separating disk D_2 in W_2, and an arc α in W_1 such that $\alpha \cap K$ is one of the endpoints $\partial\alpha$, and $\alpha \cap W_2$ is the other endpoint p of α and that D_2 cuts off a solid tours U from W_2 with $p \subset \partial U$ and with the torus $\partial N(U \cup \alpha)$ isotopic to H_1 in (M, K).*
(4) *The $(1,1)$-splitting H_1 admits a satellite diagram of a longitudinal slope.*

The definition of satellite diagrams is given below in Definition 2.2.

We have not investigated the behavior of unknotting tunnels in Cases (3) and (4), that is, the following is still open.

Problem.

(1) Is there an example which realizes Case (3) ?
(2) How does an unknotting tunnel of a knot in Case (4) behave ?

D.H.Choi informed me that the knots in Case (4) are the same as those treated in [3]. A knot in this class is obtained from a component of a 2-bridge link L by a Dehn surgery on the other component of L.

Definition 2.2. (a satellite diagram) We say that a $(1,1)$-splitting $(M, K) = (V_1, t_1) \cup_{H_1} (V_2, t_2)$ admits a *satellite diagram* if there is an essential simple loop l on the torus H_1 such that the arcs t_1 and t_2 have cancelling disks which are disjoint from l. We call l the *slope* of the satellite diagram. We say that the slope of the satellite diagram is *meridional* (resp. *longitudinal*) if it is meridional (resp. longitudinal) on ∂V_1 or ∂V_2.

When the slope is meridional, K is the trivial knot in M since it has a 1-bridge diagram on the 2-sphere obtained from H_1 by compressing along a meridian disk. It is shown in Theorem III in [7] that a knot with a 1-genus 1-bridge splitting is a satellite knot if and only if the splitting has a satellite diagram of the non-meridional and non-longitudinal slope.

Example 2.3. Torus knot: Any unknotting tunnel for a torus knot is one of 3 types illustrated in Figure 3 by M. Boileau, M. Rost and H. Zieschang [1]. The conclusions (1) and (2) in Theorem 2.1 occurs.

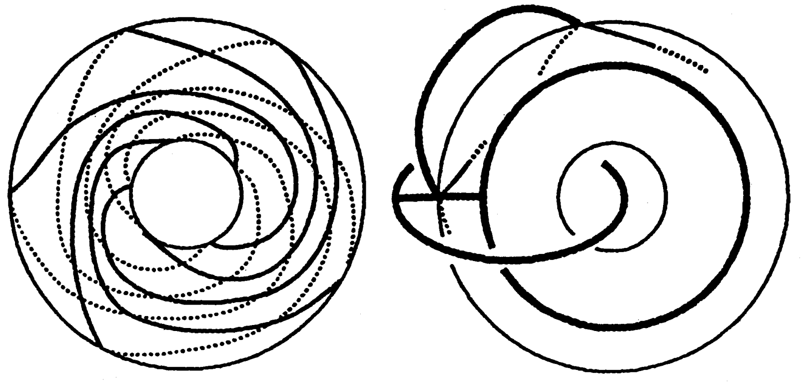

FIGURE 3

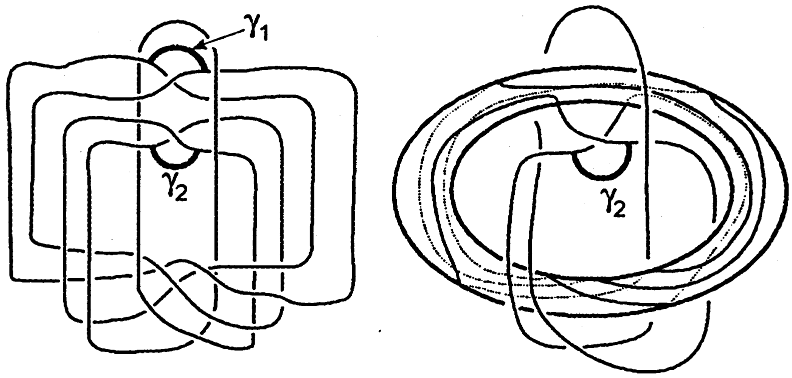

FIGURE 4

Example 2.4. Song [15] pointed out the example illustrated in Figure 4. This knot is the Morimoto-Sakuma-Yokota's knot type (5,7,2) [14]. (These knots are called twisted torus knots.) The unknotting tunnel γ_2 can be slid and isotoped into the $(1,1)$-splitting torus which is defined by the unknotting tunnel γ_1.

3. Key results to prove Theorem 2.1

Theorem 3.1 ([11]). *Suppose K in M has a 2-fold branched covering with the branch set K. Then one of the following occurs:*

(1) *either H_1 or H_2 is weakly K-reducible;*
(2) *we can isotope H_1 and H_2 so that loops of $H_1 \cap H_2 (\neq \emptyset)$ are K-essential in both H_1 and H_2.*

Note that this theorem is a version with a knot of Rubinstein-Scharlemann's results. If $M = S^3$, then the assumption is satisfied.

According to this theorem, we may consider Cases (1) and (2).

Definition 3.2. (weakly K-reducible) A $(1,1)$-splitting $(M,K)=(V_1,t_1)\cup_{H_1}(V_2,t_2)$ is called *weakly K-reducible* if there is a t_i-compressing or meridionally compressing disk D_i of H_1 in (V_i,t_i) for $i=1$ and 2 such that $\partial D_1\cap\partial D_2=\emptyset$.

A $(2,0)$-splitting $(M,K)=(W_1,K)\cup_{H_2}(W_2,\emptyset)$ is called *weakly K-reducible* if there is a K-compressing or meridionally compressing disk D_1 of H_2 in (W_1,K) and a compressing disk D_2 of H_2 in W_2 such that $\partial D_1\cap\partial D_2=\emptyset$.

Proposition 3.3 ([7]). *Suppose $(S^3,K)=(V_1,t_1)\cup_{H_1}(V_2,t_2)$ is a weakly K-reducible $(1,1)$-splitting, then one of the following occurs:*

(1) *K is the trivial knot;*

(2) *K is a 2-bridge knot.*

This proposition has been proved in the case that the ambient manifold is a lens space.

Theorem 3.4 ([9]). *Every $(2,0)$-splitting for a 2-bridge knot is meridionally stabilized.*

Proposition 3.5 ([9]). *$(S^3,K)=(W_1,K)\cup_{H_2}(W_2,\emptyset)$ is a weakly K-reducible $(2,0)$-splitting if and only if one of the following occurs:*

(1) *K is the trivial knot;*

(2) *H_2 is meridionally stabilized.*

We can have the similar result in the case that the ambient manifold is a lens space, see [5].

In the case that neither H_1 nor H_2 is weakly K-reducible, a clue to argue is essential loops $H_1\cap H_2$ (Theorem 3.1 (2)). Here the next proposition is useful.

Proposition 3.6 ([10]). *Suppose K in M has a 2-fold branched covering with the branch set K. If H_1 is contained in the interior of W_1 and there is K-compressing or meridionally compressing disk D of H_2 in (W_1,K) with $D\cap H_1=\emptyset$. Then either*

(1) *$M=S^3$ and K is the trivial knot or*

(2) *H_2 is weakly K-reducible.*

When $M=S^3$, the assumption is satisfied. This proposition is proved under a more general situation in [10].

Thus there is an obstruction that M has a 2-fold branced covering with the branch set K to obtain a result in the general case (i.e., M is a lens space).

Problem. Can we delete the assumption that M has a 2-fold branched covering in Theorem 3.1 and Proposition 3.6 ?

In [2], they have a result when a $(1,1)$-knot in a lens space has 2-fold branched covering with the branch set K.

4. GENERAL SETTING

We have obtained some results in case that M is a lens space (other than $S^2 \times S^1$) and under the assumption that satisfies Theorem 3.1 and Proposition 3.6.

Let $(M, K) = (V_1, t_1) \cup_{H_1} (V_2, t_2)$ be a $(1,1)$-splitting and $(M, K) = (W_1, K) \cup_{H_2} (W_2, \emptyset)$ a $(2,0)$-splitting.

Proposition 4.1 ([4]). *Suppose H_1 and H_2 intersect each other four or more collection of loops which are K-essential both in H_1 and H_2. Then at least one of the following holds.*

(1) *We can isotope H_1 and H_2 so that they intersect each other in non-empty collection of smaller number of loops which are K-essential both in H_1 and H_2.*
(2) *H_1 or H_2 is weakly K-reducible.*
(3) *K is a torus knot.*
(4) *K is a non-composite satellite knot.*

Proposition 4.2 ([4]). *Suppose H_1 and H_2 intersect each other in precisely three loops which are K-essential both in H_1 and H_2. Then at least one of the following holds.*

(1) *We can isotope H_1 and H_2 so that they intersect each other in non-empty collection of smaller number of loops which are K-essential both in H_1 and H_2.*
(2) *H_1 is weakly K-reducible.*
(3) *H_1 admits a satellite diagram.*

Proposition 4.3 ([4]). *Suppose H_1 and H_2 intersect each other in precisely two loops which are K-essential both in H_1 and H_2. Then at least one of the following holds.*

(1) *We can isotope H_1 and H_2 so that they intersect each other in non-empty collection of smaller number of loops which are K-essential both in H_1 and H_2.*
(2) *H_1 or H_2 is weakly K-reducible.*
(3) *K is a torus knot.*
(4) *K is a satellite knot.*
(5) *There is an essential separating disk D_2 in W_2 and an arc α in W_1 such that $\alpha \cap K$ is one of the endpoints $\partial\alpha$, and $\alpha \cap W_1$ is the other endpoint p of α and that D_2 cuts off a solod torus U from W_2 with $p \subset \partial U$ and with the torus $\partial N(U \cup \alpha)$ isotopic to H_1 in (M, K).*

Proposition 4.4 ([4]). *Suppose H_1 and H_2 intersect each other in a single loop which is K-essential both in H_1 and H_2. Then at least one of the following holds.*

(1) *H_2 is weakly K-reducible.*
(2) *K is a torus knot.*

(3) *There is an arc γ which forms a spine of (W_1, K) and is isotopic into H_1. Moreover, we can take γ so that there is a cancelling disk C_i of the arc t_i in (V_i, t_i) with $\partial C_i \cap \gamma = \partial\gamma = \partial t_i$ for $i = 1$ or 2.*

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