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Additivity of Bridge Numbers of Knots

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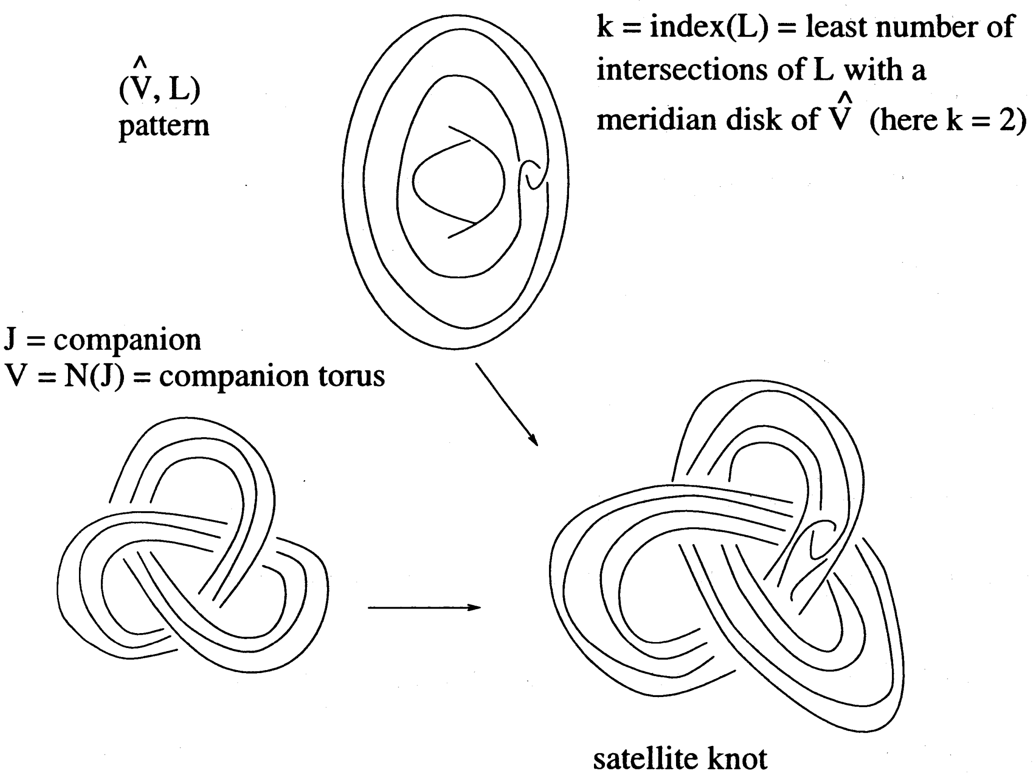
1. Satellite Construction

One of the motivations in Horst Schubert's work lay in the endeavor to understand binary operations on knots. The operation of connected sum of knots and the satellite construction are two such binary operations, usually considered separately. The definition below shows how the operation of connected sum is a special case of the satellite construction.



Definition 1. *In the special case in which the index of the pattern is 1, this construction yields the connected sum of J and L, in this case V is called a* *swallow-follow torus.*

2. History

1929: Kneser proves uniqueness of factorization of closed 3-manifolds into prime factors.

1949: Schubert proves uniqueness of prime decompositions of knots. ("Die Eindeutige Zerlegbarkeit eines Knotens in Primknoten")

1953: Schubert proves preliminary versions of torus decomposition theorems for the case of a 3-manifold that is the complement of a knot in S^3. (Habilitationsschrift: "Knoten und Vollringe")

In particular, he proves that if V, V' are distinct (solid) companion tori for a knot K, then, after isotopy, one of the following holds:

1) $V \subset V'$; or

2) $V' \subset V$; or

3) $E(K) - V \subset V'$; or

4) V, V' are opposite swallow-follow tori.

(He obtains stronger results in the case of doubles of prime knots, etc.)

1954: Schubert defines the bridge number of a knot and proves

1)$b(K) \geq kb(J)$;

2)$b(K_1 \# K_2) \geq b(K_1) + b(K_2) - 1$.

One consequence, and the answer to the question that motivated Schubert, is the fact that a knot can have only finitely many different companion knots. ("Über eine Numerische Knoteninvariante")

(We here present a new proof of these theorems using Morse functions and the idea of foliations of surfaces with respect to Morse functions.)

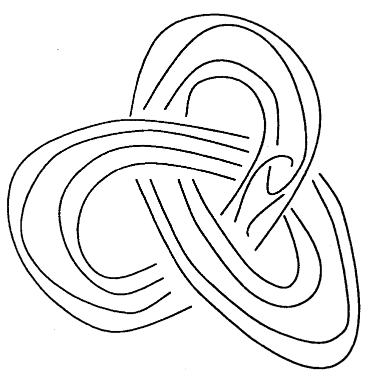
1956: Schubert gives his parametrization of 2-bridge knots. ("Knoten mit Zwei Brücken")

1960's: Haken Theory.

3. Definitions

Definition 2. *A height function on S^3 is a Morse function $h : S^3 \to \mathbf{R}$ with exactly two critical points, a minimum $-\infty$ and a maximum ∞. (This means that regular leaves are spheres.)*

Definition 3. *The bridge number of a knot $K \subset S^3$, denoted by $b(K)$, is the least number of maxima required for $h|_K$, where h is a height function on S^3. (This definition is equivalent to the usual definition.)*



Idea: To prove an inequality of the type $b(K) \geq k \cdot b(J)$, consider a height function $h : S^3 \to \mathbf{R}$ so that the number of maxima of $h|_K$ is $b(K)$, and construct a height function for $J \subset S^3$. In the picture above, this looks easy, but beware: A priori, V may be folded in on itself in a complicated fashion, when we assume that the number of maxima of $h|_K$ is $b(K)$.

Goal: Get V to be imbedded nicely with respect to $h : S^3 \to \mathbf{R}$.

Definition 4. *Say V is taut with respect to $b(K)$, if the number of critical points of $h|_T$ is minimal, where $T = \partial V$, subject to the condition that $h|_K$ has $b(K)$ maxima.*

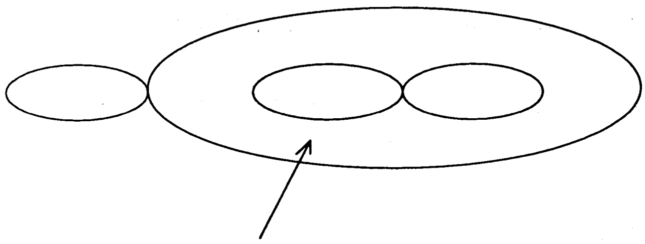
Definition 5. *Denote the singular foliation of T with respect to h by $\mathcal{F}_T$.*

Definition 6. *A saddle, σ, is a singular leaf consisting of the wedge of two circles, s_1 and s_2. It is inessential if either s_1 or s_2 bounds a disk in T, otherwise it is essential.*

4. Lemma 1

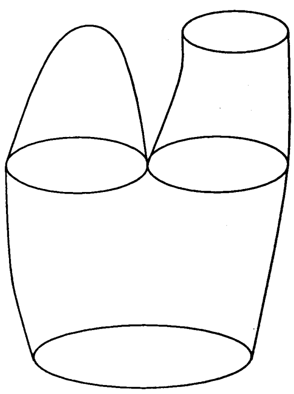
Lemma 1. *If V is taut with respect to $b(K)$, then there are no inessen saddles in $\mathcal{F}_T$.*

Idea of proof: Consider an innermost (in T) such inessential saddle σ.

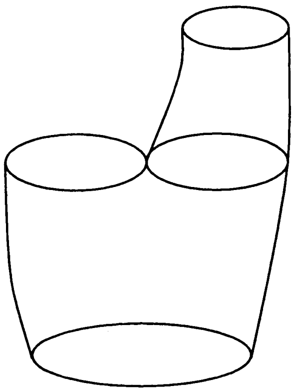


innermost inessential saddle

Case 1: The other branch of the saddle lies in the bounded region.

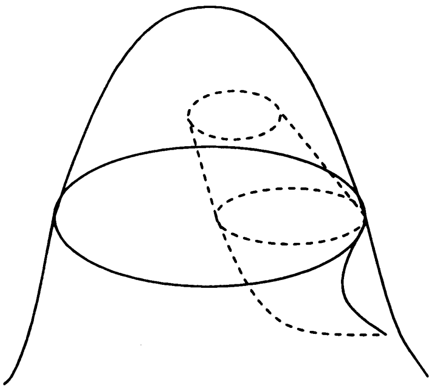


Contents of bounded region may be isotoped horizontally and downwa just below the level of σ, but above any other critical levels of $h|_K$, $h|_T$.



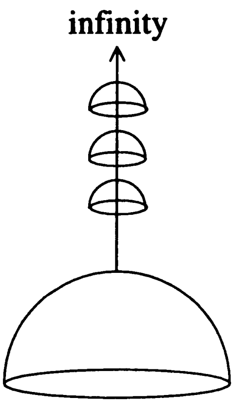
This reduces the number of critical points of $h|_T$ and contradicts tautness.

Case 2: The other branch of the saddle lies in the unbounded region.



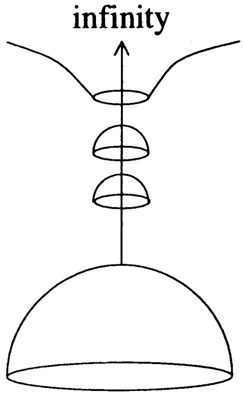
Trick to interchange bounded and unbounded regions:

Let α be a monotone arc from the maximum of disk to ∞.



Note: α may have to intersect T, though it may avoid K. But we may assume that α intersects T only in maxima.

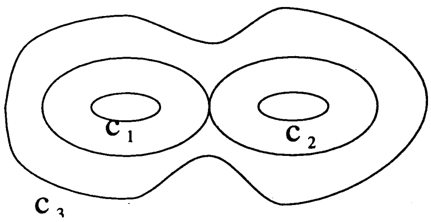
Consider a level surface L very close to ∞ and consider the ball B inside this level surface. We may "pop D_1 over ∞".



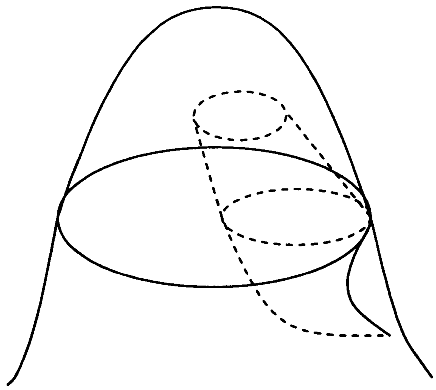
Doing this successively reduces Case 2 to Case 1.

5. Consequences of Lemma 1

Remark 1. *Let $\sigma = s_1 \cup_{point} s_2$ be a saddle in T, and let c_1, c_2, c_3 be the boundary components of a collar neighborhood of σ, with c_1 parallel to s_1, c_2 parallel to s_2. Then c_3 bounds a disk in T. (This is a consequence of the fact that $\chi(T) = 0$.)*



Definition 7. *For c_1, c_2, c_3 as above and $h(c_1) = h(c_2)$, we say that σ is a* <u>*nested*</u> *saddle, if the annulus in $L = h^{-1}(h(c_1))$ cobounded by c_1, c_2 lies in V near c_1 and c_2.*



nested saddle

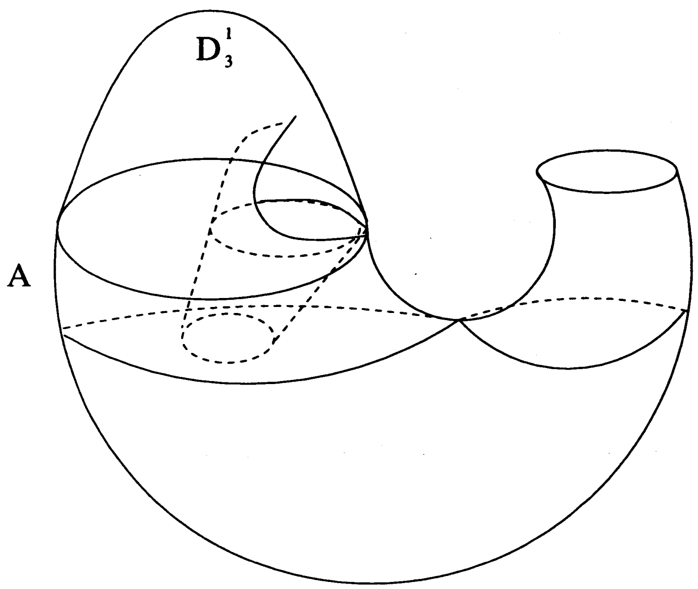
Remark 2. *If V is taut with respect to $b(K)$, then the highest saddle of T is not nested. (If it was, s_1 and s_2 would bound disks in $S^3 - V$, though not in T, hence V would be unknotted.)*

6. Lemma 2

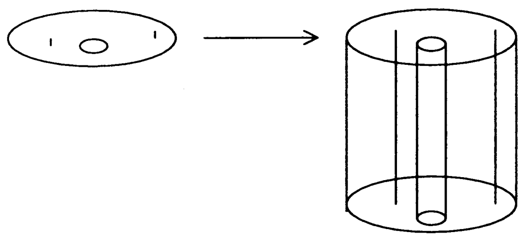
Lemma 2. *If V is taut with respect to $b(K)$, then $\mathcal{F}_T$ has no nested saddles.*

Idea of proof: Consider an adjacent pair (in T) of saddles, σ_1, σ_2 with σ_1 nested, σ_2 not nested. See the figure below.

First isotope everything "under" D_3^1 horizontally and downward, then further isotope everything "under" $(new D_3^1) \cup A \cup (level\ disk\ spanned\ by\ s_2)$



horizontally and downward. (After the isotopy, the portion above the level disk will look like level disk $\times I$.)

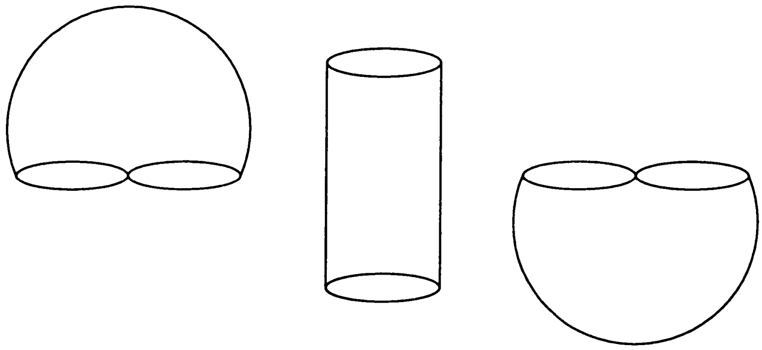


Again, the number of critical points of $h|_T$ is reduced, contradicting tautness.

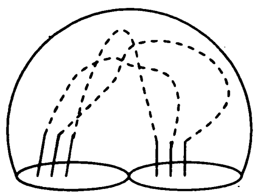
7. Theorem 1

Theorem 1 (Schubert). $b(K) \geq k \cdot b(J)$.

Idea of proof: V is constructed from three types of pieces.



In each "knee", at least k strands enter (and exit) hence there are at least k maxima.



$b(K) \geq k(\#maxima\ \ of\ \ T \geq k \cdot b(J))$.

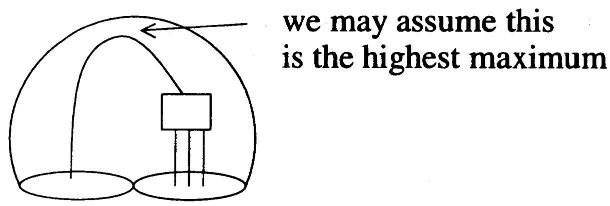
8. Theorem 2

Theorem 2 (Schubert). $b(K_1 \# K_2) = b(K_1) + b(K_2) - 1$.

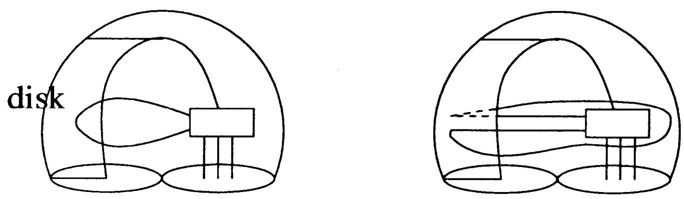
Idea of proof: We may assume that $b(K_1) \geq b(K_2)$ (by choosing the correct swallow-follow torus). If $K \cap (level\ \ meridian\ \ disks) \geq 2$ for all such disks, then

$$b(K) \geq 2b(K_1) \geq b(K_1) + b(K_2) - 1.$$

Otherwise, we may assume that the single strand of K starting at the bottom of some "knee" reaches the highest maximum of K in that "knee".

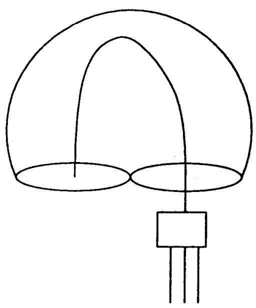


There is a disk D cobounded by the monotone portion of this strand, a monotone arc in T and two level arcs. And D can be made disjoint from K.

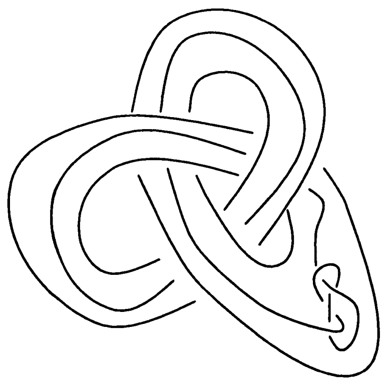


intersection is avoided by rerouting via monotone arc

This allows for an isotopy after which K intersects the "knee" in a single strand.



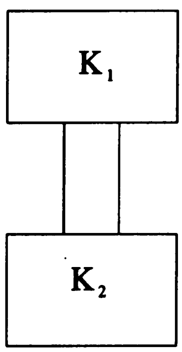
Performing further isotopies of this sort yields a nicely imbedded copy of the swallow-follow torus with the copy of the "swallowed knot in one "knee". This means that we can expand the "knee" downward and move the swallowed knot below the rest of the swallow-follow torus. Furthermore, by shrinking the swallow-follow torus to its core we obtain a height function for the "followed" knot.



Thus

$$b(K) \geq b(K_1) + b(K_2) - 1.$$

Conversely,



Thus

$$b(K) \leq b(K_1) + b(K_2) - 1.$$

9. Generalizations

If $K \subset V_1 \subset V_2 \subset \cdots \subset V_n$ is a nested sequence of companion tori, with $V_i = N(J_i)$, for some knot J_i, k_{i+1} the index of J_i in V_{i+1}, k_1 the index of K in V_1 and $T_i = \partial V_i$, then $b(K) \geq k_1 k_2 \ldots k_n b(J_n)$. And similarly, if $K = K_1 \# \ldots \# K_n$, then $b(K) \geq b(K_1) + \cdots + b(K_n) - n + 1$.

This follows by considering the following generalization of the notion of tautness:

Definition 8. $\{V_i\}_{i=1}^n$ *is taut with respect to* $b(K)$ *if the ordered n-tuple*

(#critical points of $h_{T_n}, \ldots,$ *#critical points of* h_{T_1}*)*

is minimal.

By applying the same procedure as above using this notion of tautness in succession (working from the outside in) implies that V_n, $V_{n-1}, \ldots, V_1$ are imbedded nicely and the same calculations as above can be performed.