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On the smallness and the 1-bridge genus of knots

by

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Abstract. Let K be a knot in S^3 and $g_1(K)$ the 1-bridge genus of K. Then P. Hoidn showed that $g_1(K_1\#K_2) \geq g_1(K_1)+g_1(K_2)-1$ for any small knots K_1, K_2, where a knot is small if the exterior contains no closed essential surfaces. In the present article, we show that Hoidn's estimate is best possible, i.e., there are infinitely many pairs of small knots K_1, K_2 sucht that $g_1(K_1\#K_2) = g_1(K_1)+g_1(K_2)-1$.

1. Introduction

Let S^3 be the 3-dimensional sphere, and K a knot in S^3. We say that (V_1, V_2) is a Heegaard splitting of S^3 if $S^3 = V_1 \cup V_2$, $V_1 \cap V_2 = \partial V_1 = \partial V_2$ and both V_1 and V_2 are handlebodies. The genus of V_1 (= the genus of V_2) is called the genus of the Heegaard splitting and the surface $\partial V_1 = \partial V_2$ is called the Heegaard surface of the Heegaard splitting. Then for any knot K in S^3 it is well known that there is a Heegaard splitting (V_1, V_2) of S^3 such that K intersects V_i in a single trivial arc in V_i for both $i = 1, 2$. Hence we define the 1-bridge genus $g_1(K)$ of K as the minimal genus among all such Heegaard splittings (V_1, V_2) of S^3 (c.f. [Ho] and [MSY]).

For two knots K_1, K_2 in S^3, we denote the connected sum of K_1 and K_2 by $K_1\#K_2$. Then by a little ovservation, we immediately see the following :

Fact 1.1 *For any two knots K_1 and K_2 in S^3, $g_1(K_1\#K_2) \leq g_1(K_1)+g_1(K_2)$.*

Let $N(K)$ be a regular neighborhood of a knot K in S^3 and $E(K) = cl(S^3 - N(K))$ the exterior of K. A surface F (= a connected 2-manifold) properly embedded in $E(K)$ is essential if F is incompressible and is not parallel to $\partial E(K)$ or to a subsurface of $\partial E(K)$, and it is meridional if $\partial F \neq \emptyset$ and each component of ∂F is a meridian of K. Then we say that K is small if $E(K)$ conatins no closed essential surfaces and that K is meridionally small if $E(K)$ conatins no meridional essential surfaces. We note that if a knot in S^3 is small then it is meridionally small by [CGLS, Theorem 2.0.3].

On the problem to esitimate the lower bound of $g_1(K_1\#K_2)$, P.Hoidn showed :

Theorem 1.2 ([Ho, Theorem]) *Let K_1, K_2 be two knots in S^3. If both K_1 and K_2 are small, then $g_1(K_1\#K_2) \geq g_1(K_1) + g_(K_2) - 1$.*

In the present article, we show this esitimate is best possible :

Theorem 1.3 *There are infinitely many pairs of small knots K_1, K_2 in S^3 with $g_1(K_1\#K_2) = g_1(K_1) + g_(K_2) - 1$.*

Moreover, as a generalization of Hoidn's theorem, we show :

Theorem 1.4 *Let K_1, K_2 be two knots in S^3. If both K_1 and K_2 are meridionally small, then $g_1(K_1\#K_2) \geq g_1(K_1) + g_(K_2) - 1$.*

Remark 1.5 (1) By [Mo1, Proposition 1.6], we see that for any integer $n > 0$ there are infinitely many knots K such that (i) $g_1(K) > n$, (ii) K is meridionally small, (iii) K is not small. This shows that Theorem 1.4 properly includes Theorem 1.2. (2) Since a small knot is meridionally small as mentioned before, the estimate in Theorem 1.4 is best possible by Theorem 1.3.

Let $t(K)$ be the tunnel number of a knot K in S^3, i.e., $t(K)$ is the minimal number of mutually disjoint arcs $\gamma_1, \gamma_2, \cdots, \gamma_t$ properly embedded in $E(K)$ such that $cl(E(K) - N(\gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_t))$ is a handlebody. Then by a little observation we have :

Fact 1.6 *$t(K) \leq g_1(K) \leq t(K) + 1$ for any knot K.*

By the above inequality, we have $g_1(K) = t(K)$ or $t(K) + 1$. Let K_1 and K_2 be small knots in S^3, and suppose $g_1(K_i) = t(K_i)$ for both $i = 1, 2$. Then by Fact 1.1, Fact 1.6 and [MS Theorem], we have $g_1(K_1) + g_1(K_2) \geq g_1(K_1\#K_2) \geq t(K_1\#K_2) \geq t(K_1) + t(K_2) = g_1(K_1) + g_1(K_2)$. Hence $g_1(K_1\#K_2) = g_1(K_1) + g_1(K_2)$. This tells that to show Theorem 1.3 we need to find small knots K with $g_1(K) = t(K) + 1$.

Let p, q be coprime integers, and r an arbitrally integer. Then we consider the knot obtained by adding r full twists with mutually paralle 2-strands to the (p, q)-torus knot as illustrated in Figure 1, and denote it by $K(p, q; r)$ (cf. [MSY]).

[Figure 1]

Then to get the canditates for Theorem 1.3, we show the following proposition and most of the present article will be devoted into the proof of this proposition.

Proposition 1.7 *For any p, q, r, $K(p, q; r)$ is small.*

Throughout the present article, we work in the piecewise linear category. For a manifold X and subcomplex Y in X, we denote a regular neighborhood of Y in X by $N(Y, X)$ or $N(Y)$ simply.

2. Proof of Theorem 1.4

To show Theorem 1.4, we need the following :

Theorem 2.1 ([Mo1, Corollary 1.2]) *Let K_1 and K_2 be two knots in S^3. If both K_1 and K_2 are meridionally small, then $t(K_1 \# K_2) \geq t(K_1) + t(K_2)$.*

Theorem 2.2 ([Mo2, Theorem 1.6]) *Let K_1 and K_2 be two meridionally small knots in S^3. Then $t(K_1 \# K_2) = t(K_1) + t(K_2) + 1$ if and only if $g_1(K_i) = t(K_i) + 1$ for both $i = 1, 2$.*

Suppose both K_1 and K_2 are meridionally small. Recall that $g_1(K_i) = t(K_i)$ or $t(K_i) + 1$ for $(i = 1, 2)$ by Fact 1.6.

First suppose at least one of K_1 and K_2, say K_1, satisfies the equality $g_1(K_1) = t(K_1)$. Then $t(K_2) \geq g_1(K_2) - 1$. Since both K_1 and K_2 are meridionally small, by the above Theorem 2.1, $t(K_1 \# K_2) \geq t(K_1) + t(K_2)$. Hence by Fact 1.6, $g_1(K_1 \# K_2) \geq t(K_1 \# K_2) \geq t(K_1) + t(K_2) \geq g_1(K_1) + g_1(K_2) - 1$.

Next suppose $g_1(K_i) = t(K_i) + 1$ for both $(i = 1, 2)$. Then by the above Theorem 2.2, $t(K_1 \# K_2) = t(K_1) + t(K_2) + 1$. Hence by Fact 1.6, $g_1(K_1 \# K_2) \geq t(K_1 \# K_2) = t(K_1) + t(K_2) + 1 = (g_1(K_1) - 1) + (g_1(K_2) - 1) + 1 = g_1(K_1) + g_1(K_2) - 1$. This completes the proof of Theorem 1.4. □

3. Proof of Theorem 1.3 under Proposition 1.7

To show Theorem 1.3, we need the following :

Lemma 3.1 ([Mo3, Proposition 1.7]) Let K be a knot in S^3. If $g_1(K) = t(K)+1$, then $g_1(K\#K') \leq g_1(K)$ for any 2-bridge knot K'.

For convenience to the readers, we show the above lemma here. Let (V_1, V_2) be a Heegaard splitting of a 3-sphere S^3_1 which realizes the tunnel number of K, i.e., V_1 contains K as a core of a handle of V_1 and $g(V_1) = t(K)+1 = g_1(K)$. Let $(B_1, \gamma_1 \cup \delta_1)$ and $(B_2, \gamma_2 \cup \delta_2)$ be a 2-bridge decomposition of K' in another 3-sphere S^3_2, i.e., $(B_i, \gamma_i \cup \delta_i)$ is a 2-string trivial tangle ($i = 1, 2$) and $K' = \gamma_1 \cup \gamma_2 \cup \delta_1 \cup \delta_2 \subset B_1 \cup B_2 = S^3_2$.

Let D be a meridian disk of V_1 which intersects K in a single point and $N(D)$ a regular neighborhood of D in V_1. Put $N(D) = D \times [0,1]$ and $N(D) \cap K = x \times [0,1]$, where x is a point in Int$D$. Let $N(\delta_2)$ be a regular neighborhood of δ_2 in B_2. Put $N(\delta_2) = D' \times [0,1]$ and $\delta_2 = y \times [0,1]$, where D' is a disk and y a point in IntD'.

Let $K\#K'$ be the connectd sum of K and K'. Then $K\#K'$ is a knot in the 3-sphere $S^3 = cl(S^3_1 - N(D)) \cup_{\partial N(D) = \partial N(\delta_2)} cl(S^3_2 - N(\delta_2))$. Put $W_1 = cl(V_1 - N(D))$. Then, since $N(D) \cap W_1 = \partial N(D) \cap \partial W_1 = D \times \{0,1\}$ and since $N(\delta_2) \cap B_1 = \partial N(\delta_2) \cap \partial B_1 = D' \times \{0,1\}$, we can put $U_1 = W_1 \cup_{D \times \{0,1\} = D' \times \{0,1\}} B_1$. Then U_1 is a genus $g_1(K)$ handlebody and $(K\#K') \cap U_1$ is a trivial arc in U_1 because $(K\#K') \cap W_1$ is a trivial arc in W_1 and $(K\#K') \cap B_1 \subset B_1$ is a 2-string trivial arc in B_1.

On the other hand, put $W_2 = cl(B_2 - N(\delta_2))$. Then, since $N(D) \cap V_2 = \partial N(D) \cap \partial V_2 = \partial D \times [0,1]$ and since $N(\delta_2) \cap W_2 = \partial N(\delta_2) \cap \partial W_2 = \partial D' \times [0,1]$, we can put $U_2 = V_2 \cup_{\partial D \times [0,1] = \partial D' \times [0,1]} W_2$. Then U_2 is a genus $g_1(K)$ handlebody and $(K\#K') \cap U_2$ is a trivial arc in U_2 because δ_2 is a trivial arc in B_2 and $(K\#K') \cap W_2$ is a trivial arc in W_2.

Hence (U_1, U_2) is a genus $g_1(K)$ Heegaard splitting of S^3 which gives a 1-bridge decomposition of $K\#K'$. This implies $g_1(K\#K') \leq g_1(K)$ and completes the proof of Lemma 3.1. $\square$.

Now let's prove Theorem 1.3 under Proposition 1.7. Let m be an integer and consider the knot $K_1 = K(7, 17, 5m-2)$. Then by Proposition 1.7, K_1 is small, and by [MSY, Theorem 2.1], $t(K_1) = 1$ and $g_1(K_1) = 2$. Let K_2 be a (non-trivial) 2-bridge knot in S^3. Then K_2 is small and $g_1(K_2) = 1$. Then by the above Lemma 3.1, $g_1(K_1\#K_2) \leq g_1(K_1) = 2$. On the other hand, $g_1(K_1\#K_2) \geq 2$ because 1-bridge genus one knots are prime by [No, Sc] and Fact 1.6. Thus $g_1(K_1\#K_2) = 2$ and $g_1(K_1\#K_2) = g_1(K_1) + g_1(K_2) - 1$ for the small knots K_1, K_2. This completes the proof of Theorem 1.3. □

4. Preliminaries for the proof of Propositions 1.7

Put $K = K(p,q;r)$, $N(K)$ a regular neighborhood of K in S^3 and $E(K) = cl(S^3 - N(K))$ the exterior. If $r = 0$, then K is a (p,q)-torus knot and is small. Hence hereafter we assume that $r \neq 0$. Let (W_1, W_2) be a genus two Heegaard splitting of S^3 and $(D_1, D_2) \subset (W_1, W_2)$ a cancelling disk pair, i.e., D_i is a non-separating disk of W_i $(i = 1, 2)$ and $D_1 \cap D_2 = \partial D_1 \cap \partial D_2 =$ a single point. Let $N(D_1)$ be a regular neighborhood of D_1 in W_1, and regard $N(D_1)$ as a product space $D_1 \times [0,1]$ with $D_1 = D_1 \times \{\frac{1}{2}\}$. Put $D_1^0 = D_1 \times \{0\}$, $D_1^1 = D_1 \times \{1\}$, $V_1 = cl(W_1 - N(D_1))$ and $V_2 = W_2 \cup N(D_1)$. Then $V_1 \cap N(D_1) = D_1^0 \cup D_1^1$, and we can put $\partial D_2 = \gamma_1 \cup \gamma_2$, where $\partial D_2 \cap V_1 = \gamma_1$ and $\partial D_2 \cap N(D_1) = \gamma_2$.

Consider the knot K as a simple closed curve in $\partial W_1 = \partial W_2$ so that those r full twists are in $\partial N(D_1)$ as illustrated in Figure 2. Then we may assume that $D_1 \cap K = \partial D_1 \cap K =$ two points and $D_2 \cap K = \gamma_2 \cap K = |2r|$ points.

[Figure 2]

To show Proposition 1.7, we show that K is small and meridionally small simultaneously. Suppose, for a contradiction, $E(K)$ contains a meridional essential surface or a closed essential surface, say $\check{F}$. Let F be a closed surface obtained from $\check{F}$ by adding meridian disks of $N(K)$ to each component of $\partial\check{F}$. Note that $F = \check{F}$ if $\check{F}$ is closed. Hereafter we consider F instead of $\check{F}$. Then F intersects K in several points (possibly $F \cap K = \emptyset$), $F - K$ is incompressible in $S^3 - K$ and F is not a 2-sphere which bounds a 3-ball intersecting K in a trivial arc. By general position argument, we may assume that $D_1 \cap K \cap F = \emptyset$, and hence $N(D_1) \cap K \cap F = \emptyset$. Then by the incompressibility of $F - K$, we may assume that $D_1 \cap F$ consists of n arcs for some integer $n \geq 0$ and $N(D_1) \cap F$ consists of n rectangles, where each arc of $D_1 \cap F$ separates the two points $D_1 \cap K$ as illustrated in Figure 3. We assume that n is minimal among all such meridional or closed essential surfaces in $E(K)$.

[Figure 3]

Put $F \cap V_1 = F_1, F \cap N(D_1) = F_2$ and $F \cap W_2 = F_3$, then $F \cap W_1 = F_1 \cup F_2$. By the incompressibility of $F - K$ in $S^3 - K$ and the irreducibility of $S^3 - K$, we may assume that F_1 is incompressible in V_1. Put $K \cap V_1 = cl(K - N(D_1)) = k_1 \cup k_2 =$ two arcs in ∂V_1.

Lemma 4.1 (1) *There is no pair of a subarc α of $k_1 \cup k_2$ and an arc β properly embedded in F_1 such that $\alpha \cap \beta = \partial\alpha = \partial\beta$ and $\alpha \cup \beta$ bounds a disk in V_1.* (2) *There is no 2-gon in $(k_1 \cup k_2) \cup \partial F_1$, which bounds a disk in ∂V_1.*

Proof. (1) Suppose there is such a pair α, β, and let Δ be the disk in V_1 with $\partial\Delta = \alpha \cup \beta$. Let $N(\Delta)$ be a regular neighborhood of Δ in S^3 such that $N(\Delta) \cap F$ is a disk which is a regular neighborhood of β in F, denote it by $N(\beta, F)$. Put $c = \partial N(\beta, F)$. Then, since c is a loop in $\partial N(\Delta)$, c bounds a disk in $S^3 - K$. If c is essential in $F - K$, then $F - K$ is compressible in $S^3 - K$, a contradiction. If c is inessential in $F - K$, then F is a 2-sphere which bounds a 3-ball intersecting K in a trivial arc, a contradiction. Hence there is no such pair.

(2) If there is such a 2-gon in $(k_1 \cup k_2) \cup \partial F_1$, then we can find a subarc $\alpha \subset k_1 \cup k_2$ and an arc $\beta \subset F_1$ satisfying the condition (1), a contradiction. Hence there is no such 2-gon. □

By noting the incompressibility of $F - K$ in $S^3 - K$, we have the next two lemmas.

Lemma 4.2 *$n > 0$, where n is the number of the arcs $D_1 \cap F = D_1 \cap F_2$.*

Lemma 4.3 *Each component of $F_3 \cap D_2$ is an arc connecting γ_1 and γ_2, and there are exactly two outermost arc components each of which cuts off a disk intersect K in a single point and contains a point of $\partial\gamma_1 = \partial\gamma_2$ as in Figure 5. Hence the number of the points $\gamma_1 \cap \partial F_1$ is $(2|r| - 1)n$.*

[Figure 4]

Since F_1 is incompressible in the solid torus V_1, each component of F_1 is a ∂-parallel disk, a ∂-parallel annulus or a meridian disk of V_1. Recall the notations

D_1^0, D_1^1, k_1 and k_2. Then by several arguments we have :

Lemma 4.4 *Let G be a ∂-parallel disk component of F_1 and G' a disk in ∂V_1 to which G is parallel. Then one of the following folds :*

(1) *G' is a small regular neighborhood of k_i in ∂V_1 $(i = 1, 2)$,*

(2) *G' is a small regular neighborhood of D_1^i in ∂V_1 $(i = 0, 1)$,*

(3) *G' is a small regular neighborhood of $D_1^0 \cup k_i \cup D_1^1$ in ∂V_1 $(i = 1, 2)$.*

Lemma 4.5 *Let G be a ∂-parallel annulus component of F_1 and G' an annulus in ∂V_1 to which G is parallel. Then G' is a small regular neighborhood of $D_1^0 \cup k_1 \cup D_1^1 \cup k_2$ in ∂V_1.*

Moreover, concerning ∂-parallel disk components of F_1 in V_1, we get a stronger result than that of Lemma 4.4 as follows :

Lemma 4.6 *Let G be a ∂-parallel disk component of F_1 and G' a disk in ∂V_1 to which G is parallel. Then G' is a small regular neighborhood of k_1 or of k_2 in ∂V_1, and all such disks are mutually parallel.*

5. Sketch Proof of Proposition 1.7

Recall the notations in section 4, and recall that each component of F_1 is a ∂-parallel disk, a ∂-parallel annulus or a meridian disk in V_1. Then we have the two cases. Case I : F_1 contains no meridian disks and Case II : F_2 contains a meridian disk.

Suppose we are in Case I. In this case, by Lemmas 4.5 and 4.6, F_1 consists of mutually parallel ∂-paralle disks and mutually parallel ∂-parallel annuli. Let $\tilde{E} = E_1 \cup E_2 \cup \cdots \cup E_n$ be the disks each of which is parallel to a small regular neighborhood of k_1 in ∂V_1 and $\tilde{A} = A_1 \cup A_2 \cup \cdots \cup A_\ell$ the annuli each of which is parallel to a small regular neighborhood of $D_1^0 \cup k_1 \cup D_1^1 \cup k_2$ in ∂V_1. Note that n is the number of the arcs $D_1 \cap F$ and $2\ell = (2|r| - 1)n$ by Lemma 4.3 (see Figure 5). Let D_3 be a meridian disk of W_2 such that ∂D_3 is a longitude of V_1. Since $D_1^0 \cap k_1 \cap D_1^1$ can be homotopic to a point in ∂V_1. We may assume that $\partial D_3 \cap (D_1^0 \cup k_1 \cup D_1^1) = \emptyset$, and hence $\partial D_3 \cap \tilde{E} = \emptyset$. A schematic picture of $(\partial\tilde{E}, \partial\tilde{A}, \partial D_3, D_1^0, D_1^1, \gamma_1, k_1, k_2)$ on ∂V_1 is illustrated in Figure 5.

[Figure 5]

Since we may assume that there is no 2-gon in $\partial D_3 \cup k_1 \cup k_2 \cup \partial F_1$, the arrangement of the points $\partial D_3 \cap (k_1 \cup k_2 \cup \partial F_1)$ on ∂D_3 is as in Figure 6, where the big points are the points of $\partial D_3 \cap (k_1 \cup k_2)$ and the small points are the points of $\partial D_3 \cap \partial F_1 = \partial D_3 \cap \partial \tilde{A}$. We note that there are some small points between any two successive big points because of $\tilde{A} \neq \emptyset$ by $2\ell = (2|r|-1)n > 0$.

[Figure 6]

By the incompressibility of F in $S^3 - K$ we may assume that each component of $D_3 \cap (F \cap W_2) = D_3 \cap F_3$ is an arc. Let α be an outermost arc component of $D_3 \cap F_3$ in D_3 and β the corresponding arc in ∂D_3 with $\alpha \cap \beta = \partial\alpha = \partial\beta$. Then we have the two subcases. Case I-a : $\beta \cap (k_1 \cup k_2) \neq \emptyset$ and Case I-b : $\beta \cap (k_1 \cup k_2) = \emptyset$.

Suppose we are in Case I-a. In this case, α meets a single component of $\tilde{A}$, say A_1. Then we can take an arc, say α' properly embedded in A_1, with $\alpha \cap \alpha' = \partial\alpha = \partial\alpha'$. Since $\alpha' \cup \beta$ bounds a boundary compressing disk for A_1 in V_1, together with the outer most disk for α in D_3, $\alpha \cup \alpha'$ bounds a disk, say Δ, which intersects K in a single point. Perform a 2-surgery for F along Δ, and let $\tilde{F}$ be the surface after the surgery. Then $cl(\tilde{F} - N(K))$ is a meridional essential surface properly embedded in $E(K)$, and A_1 is changed to the disk in conclusion (3) of Lemma 4.4. This contradicts Lemma 4.6. Hence Case I-a does not occur.

Suppose we are in Case I-b. In this case, α connects two components of $\tilde{A}$, say A_1, A_2. Perform a boundary compression of F_3 along the outermost disk for α. Let b be the band in V_1 produced by the boundary compression. Then $A_1 \cup b \cup A_2$ is a disk with two holes, and one component of $\partial(A_1 \cup b \cup A_2)$ bounds a disk in ∂V_1 because A_1 and A_2 are mutually parallel. Then by the incompressibility of $F - K$, we can eliminate the components A_1, A_2, and we can decrease the number n becasuse of $2\ell = (2|r|-1)n$, a contradiction. Hence Case I-b does not occur and this completes the proof of Case I.

Suppose we are in Case II. In this case, by Lemmas 4.5 and 4.6, F_1 consists of mutually parallel ∂-parallel disks and mutually parallel meridian disks. Let $\tilde{E} =$

$E_1 \cup E_2 \cup \cdots \cup E_r$ be the ∂-parallel disks each of which is parallel to a small regular neighborhood of k_1 in ∂V_1 and $\tilde{M} = M_1 \cup M_2 \cup \cdots \cup M_s$ the meridian disks. In this case $r \geq 0$ and $s > 0$. Let D_3 be a meridian disk of W_2 such that ∂D_3 is a longitude of V_1. Since $M_1, M_2, \cdots, M_s$ are all mutually parallel, we can take an annulus, say A, in ∂V_1 such that A contains $\partial \tilde{M}$ and each ∂M_i $(i = 1, 2, \cdots, s)$ is a central loop of A. Then we may assume that ∂D_3 intersect A in a single essential arc properly embedded in A and ∂D_3 intersects each ∂M_i in a single point. Then, since we may assume that $\partial D_3 \cap \tilde{E} = \emptyset$, the arrangement of the intersection $\partial D_3 \cap (k_1 \cup k_2 \cup D_1^0 \cup D_1^1 \cup \partial F_1)$ on ∂D_3 is as in Figure 7, where the big points are the points of $\partial D_3 \cap (k_1 \cup k_2)$, fat arcs are the arcs of $\partial D_3 \cap (D_1^0 \cup D_1^1)$ and the small points are the points of $\partial D_3 \cap \partial F_1 = \partial D_3 \cap \partial \tilde{M}$.

[Figure 7]

Let α be an outermost arc component of $D_3 \cap F_3$ in D_3 and β the corresponding arc in ∂D_3 with $\alpha \cap \beta = \partial \alpha = \partial \beta$. Perform a boudary compression for F_3 along the outermost disk for α, and let b be the band in V_1 produced by the boundary compression. Then we may assume that b connects M_1 and M_2, and by observing the upper side of b, we have the five cases (i) – (v) illustrated in Figure 8.

[Figure 8]

Suppose, for example, we are in case (i). In this case, we can find a pair of arcs α, β as in Lemma 4.1, a contradiction. In the other cases, we get contradictions similarly. Hence Case II does not occur and this completes the proof of Proposition 1.7 and Theorem 1.3. □

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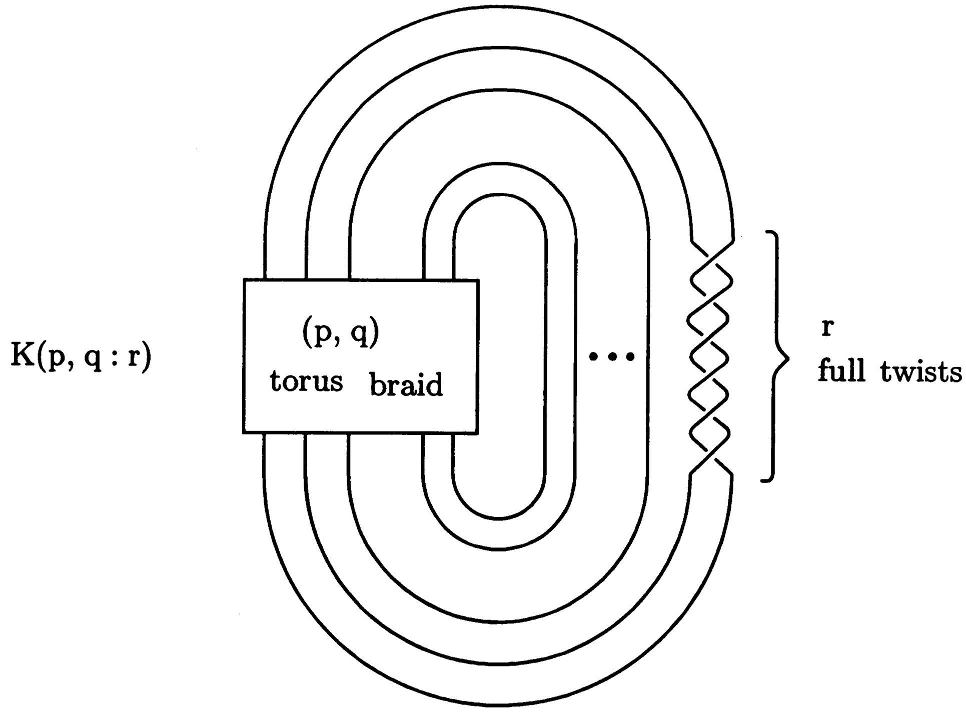


Figure 1

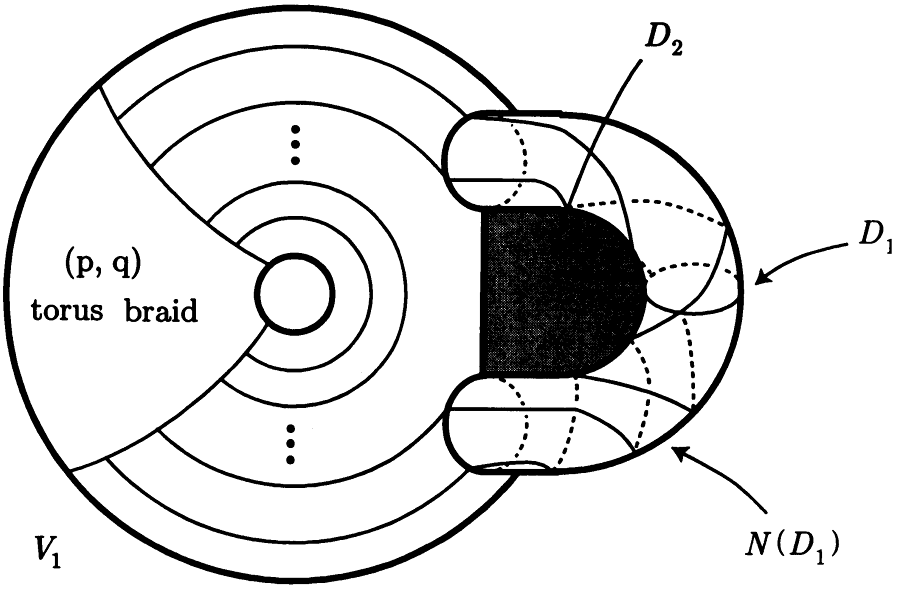


Figure 2

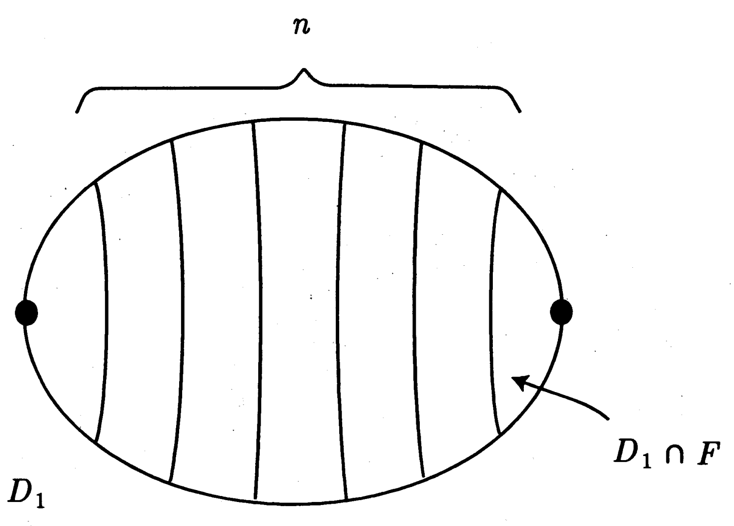
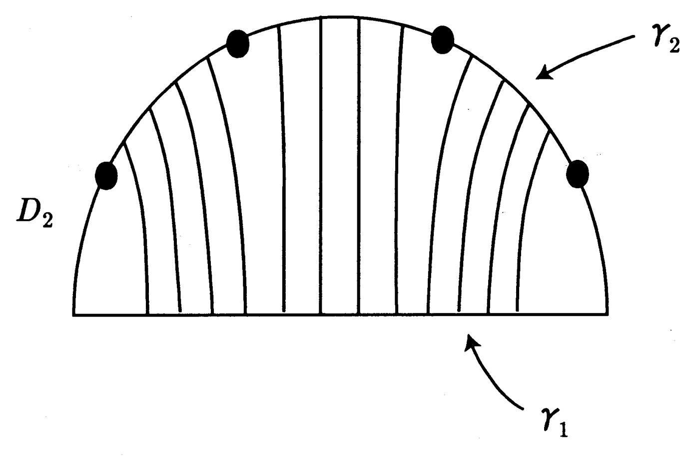


Figure 3



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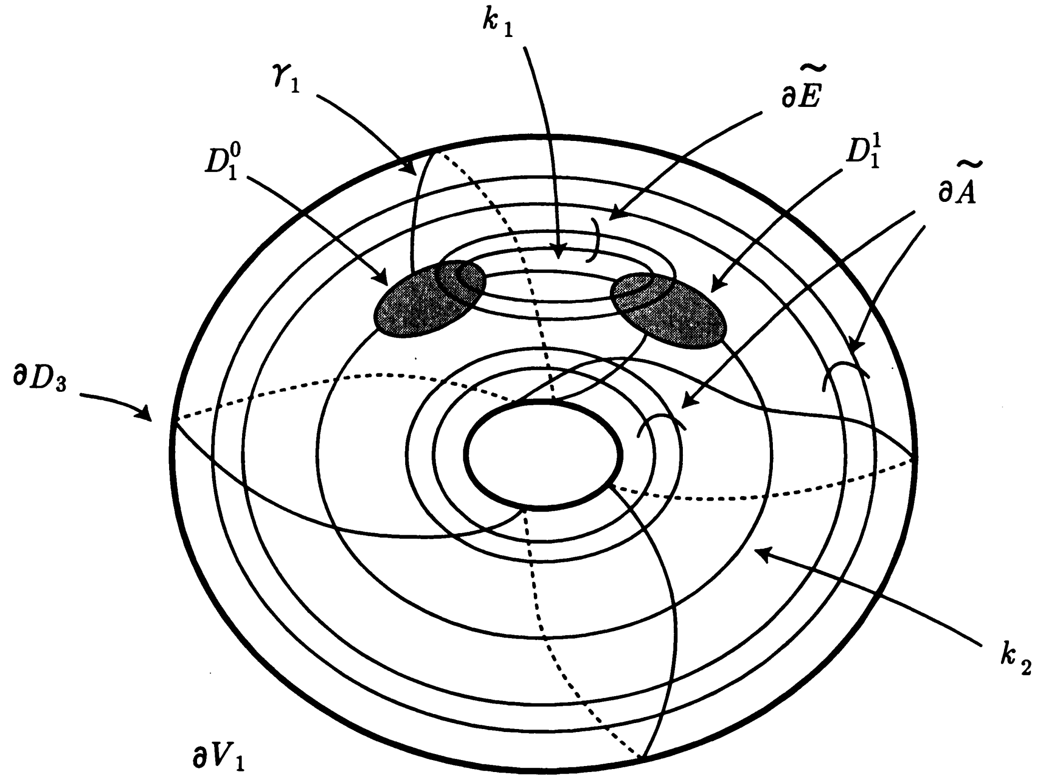

Figure 5

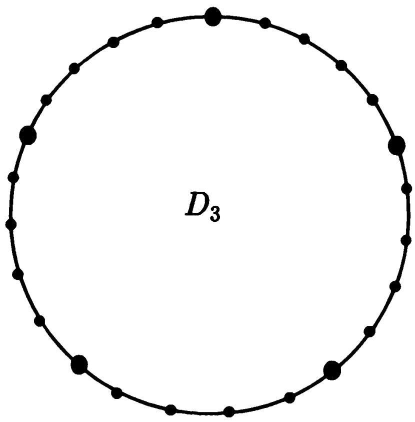

Figure 6

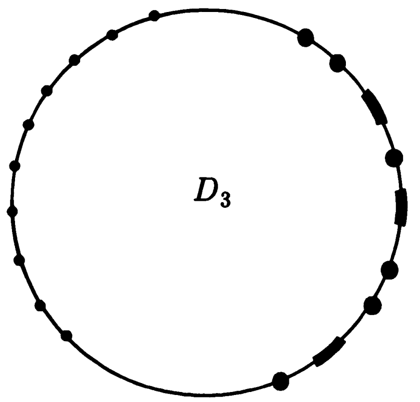

Figure 7

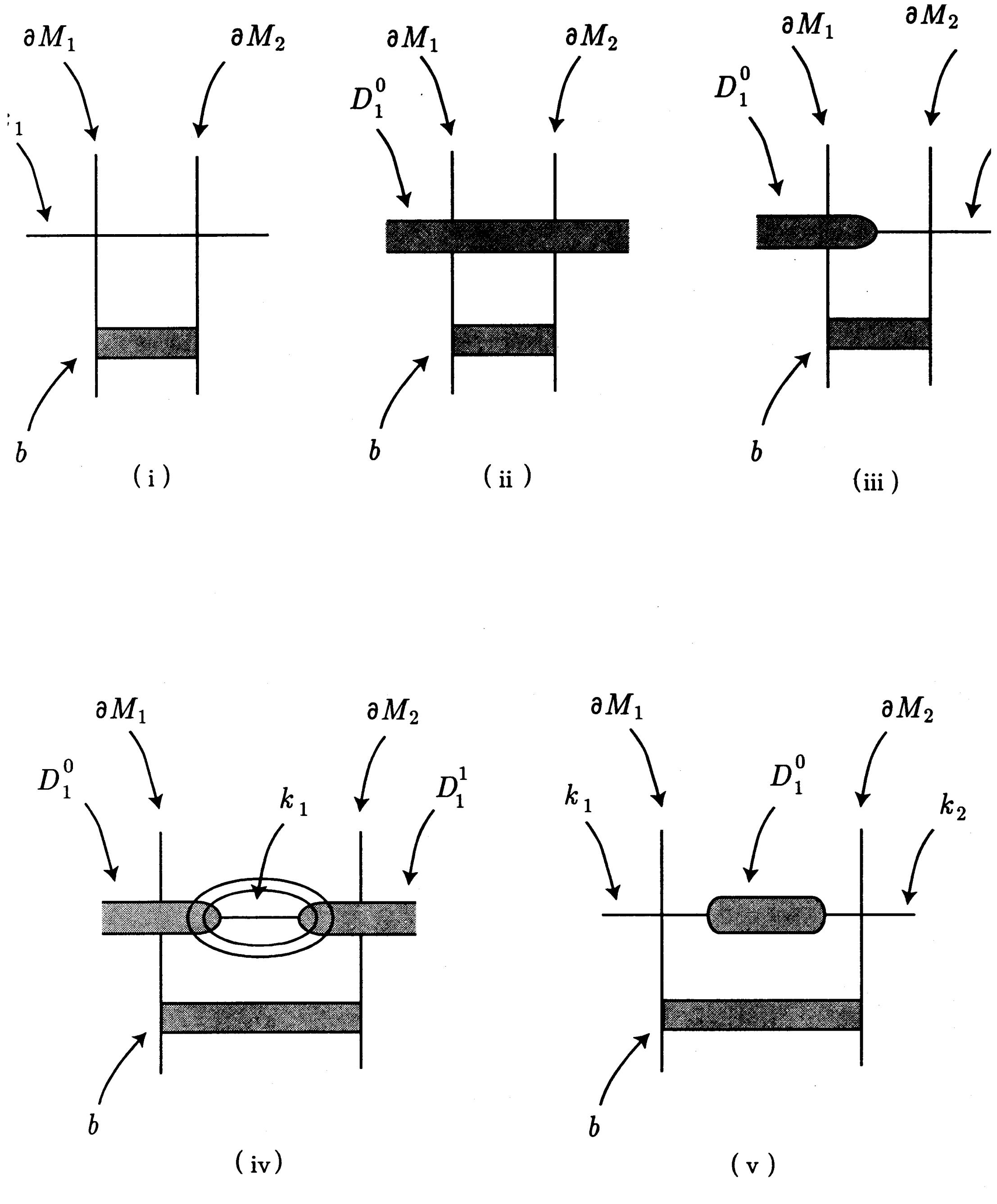

Figure 8