数理解析研究所講究録 1230 巻 2001 年 11-32

PENTAGONAL EQUATIONS FOR OPERATORS ASSOCIATED WITH INCLUSIONS OF C^*-ALGEBRAS (PRELIMINARY VERSION)

大阪女子大学理学部 大内本夫 (MOTO O'UCHI)

OSAKA WOMEN'S UNIVERSITY

1. Introduction

The pentagonal equation (PE) first appeared in the duality theory for locally compact groups. The Kac-Takesaki operator in the theory satisfies the PE (cf. [9], [29]). S. Baaj and G. Skandalis called a unitary operator on a Hilbert space a multiplicative unitary (MU) when it satisfies PE in [2]. They constructed a pair of Hopf C^*-algebras from a regular MU. M. Enock and R. Nest constructed an MU from an irreducible regular depth 2 inclusion of factors. As for measured groupoids, T. Yamanouchi constructed an analogue of the Kac-Takesaki operator in [35]. But this operator does not satisfy the PE. J. M. Vallin showed that it satisfies an equation which is a generalization of the PE in [32]. He called a unitary operator a pseudo-multiplicative unitary (PMU) when it satisfies this generalized PE. Vallin defined the generalized PE using the Connes-Sauvageot's relative tensor products of Hilbert spaces. M. Enock and J. M. Vallin constructed a PMU from a regular depth 2 inclusion of von Neumann algebras in [10]. The basis of the PMU they studied is a (not necessarily commutative) von Neumann algebra. Recently quantum groupoids are studied by many authors. For example, see [3], [7], [18], [20], [27] and [33]. Quantum

Date: May 14, 2001.

1991 *Mathematics Subject Classification.* 46L89.

Key words and phrases. Hilbert C^*-module, inclusion of C^*-algebras, multiplicative oprator.

groupoids are related to inclusions of von Neumann algebras and PMU's. In particular, PMU's in finite-dimension were studied by G. Böhm and K. Szlachányi [3] and by J. M. Vallin [33]. They studied the PMU from the viewpoint of multiplicative isometries. Before their works, Yamanouchi studied a partial isometry which satisfies the PE in [36]. When we deal with PMU's in the theory of C^*-algebras, it is useful to formulate the generalized PE in the frame work of Hilbert C^*-modules. As for the usefulness of Hilbert C^*-modules, for example, see the works of M. A. Rieffel [25], E. C. Lance [16], B. Blackadar [4] and Y. Watatani [34]. The author defined a PMU on a Hilbert C^*-module using interior tensor products in [22]. The base algebra of the PMU defined there is a commutative C^*-algebra. (When PMU is defined on a tensor product of A-modules, we will call A a base algebra. See Definition 3.1.) An analogue of the Kac-Takesaki operator for a topological groupoid becomes a PMU in the sense of [22]. Moreover, if it is a measured groupoid, that is, if it has a quasi-invariant measure, then the PMU constructed in [22] induces the PMU studied by Vallin in [32]. The author constructed in [23] a PMU in the sense of [22] from an inclusion of finite-dimensional C^*-algebras when the inclusion satisfies certain conditions. There we had to assume a condition which implies a commutativity of the base algebra.

In this paper, we will study a PE in full generality. We will not distinguish a PE from a generalization of a PE and we will not distinguish an MU from a PMU. Therefore we will call a PE a generalization of a PE and we will call an operator a multiplicative operator when it satisfies a generalization of a PE. The aim of this paper is to give a definition of a PE in full generality in the framework of Hilbert C^*-module and to give examples of operators which satisfies this PE. Especially, we remove the assumption of the commutativity of the base algebra, which was assumed in [22] and [23]. We meet many difficulties in defining a PE in the framework of Hilbert C^*-modules. For example, we do not in general the following objects; a flip on an interior tensor product of Hilbert C^*-modules, a tensor product $I \otimes x$ as operator on an interior tensor product of Hilbert C^*-modules for an adjointable operator x on a Hilbert C^*-module and a modular involution on a Hilbert C^*-module. When the base algebra is $\mathbb{C}$, the multiplicative unitary operator (MUO) defined in this paper coincides with the MU defined by Baaj and Skandalis in [2] modulo the flip. When the base algebra is commutative, the MUO coincides with the PMU studied in [22] and [23] modulo the flip. Note that we cannot define a flip when the base algebra is not commutative.

2. Preliminaries

First, we recall some definitions and notations on Hilbert C^*-modules. For details, we refer the reader to [16]. Let A be a C^*-algebra. A Hilbert A-module is a right A-module E with an A-valued inner product $< \cdot, \cdot >$ such that E is complete with respect to the norm $||\xi|| = || < \xi, \xi > ||^{1/2}$. Note that the inner product is linear in its second variable. A Hilbert A-module E is said to be full if the closure of the linear span of $\{< \xi, \eta >; \xi, \eta \in E\}$ is all of A. Let E and F be Hilbert A-modules. We denote by $\mathcal{L}_A(E, F)$ the set of bounded adjointable operators from E to F and we denote by $\mathcal{K}_A(E, F)$ the closure of the linear span of $\{\theta_{\xi,\eta}; \xi \in F, \eta \in E\}$, where $\theta_{\xi,\eta}$ is the element of $\mathcal{L}_A(E, F)$ defined by $\theta_{\xi,\eta}(\zeta) = \xi < \eta, \zeta >$ for $\zeta \in E$. We abbreviate $\mathcal{L}_A(E, E)$ and $\mathcal{K}_A(E, E)$ to $\mathcal{L}_A(E)$ and $\mathcal{K}_A(E)$ respectively. We denote by I_E the identity operator on E. We often omit the subscript E for simplicity. A unitary operator U of E to F is an adjointable operator such that $U^*U = I_E$ and $UU^* = I_F$.

Let A and B be C^*-algebras. Suppose that E is a Hilbert A-module and that F is a Hilbert B-module. Let ϕ be a $*$-homomorphism of A to $\mathcal{L}_B(F)$. Then we can define the interior tensor product $E \otimes_\phi F$ ([16], Chapter 4). For $\xi \in E$ and $\eta \in F$, we denote by $\xi \otimes_\phi \eta$ the corresponding element of $E \otimes_\phi F$. We often omit the subscript ϕ, writing $\xi \otimes \eta = \xi \otimes_\phi \eta$ for simplicity. We have $\xi a \otimes \eta = \xi \otimes \phi(a)\eta$

for every $a \in A$. Note that $E \otimes_\phi F$ is a Hilbert B-module with a B-valued inner product such that

$$< \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2 >=< \eta_1, \phi(< \xi_1, \xi_2 >)\eta_2 >$$

for ξ_1, $\xi_2 \in E$ and η_1, $\eta_2 \in F$. Let E_i be a Hilbert A_i-module for $i = 1, 2, 3$ and let ϕ_i be a $*$-homomorphism of A_{i-1} to $\mathcal{L}_{A_i}(E_i)$ for $i = 2, 3$. Define a $*$-homomorphism $\phi_2 \otimes_{\phi_3} \iota$ of A_1 to $\mathcal{L}_{A_3}(E_2 \otimes_{\phi_3} E_3)$ by $(\phi_2 \otimes_{\phi_3} \iota)(a) = \phi_2(a) \otimes I$ for $a \in A_1$. We often omit the subscript ϕ_3, writing $\phi_2 \otimes \iota = \phi_2 \otimes_{\phi_3} \iota$ for simplicity. Then we have

$$(E_1 \otimes_{\phi_2} E_2) \otimes_{\phi_3} E_3 = E_1 \otimes_{\phi_2 \otimes \iota} (E_2 \otimes_{\phi_3} E_3).$$

We denote the above tensor product by $E_1 \otimes_{\phi_2} E_2 \otimes_{\phi_3} E_3$.

For $i = 1, 2$, let E_i be a Hilbert A-module, let F_i be a Hilbert B-module and let ϕ_i be a $*$-homomorphism of A to $\mathcal{L}_B(F_i)$. We denote by $\mathcal{L}_B((F_1, \phi_1), (F_2, \phi_2))$ the set of elements x of $\mathcal{L}_B(F_1, F_2)$ such that $x\phi_1(a) = \phi_2(a)x$ for all $a \in A$. We abbreviate $\mathcal{L}_B((F_1, \phi_1), (F_1, \phi_1))$ to $\mathcal{L}_B(F_1, \phi_1)$. We define $\mathcal{K}_B((F_1, \phi_1), (F_2, \phi_2))$ and $\mathcal{K}_B(F_1, \phi_1)$ similarly. The following proposition is useful in later arguments.

Proposition 2.1 ([22]). *For $x \in \mathcal{L}_A(E_1, E_2)$ and $y \in \mathcal{L}_B((F_1, \phi_1), (F_2, \phi_2))$, there exists an element $x \otimes_{\phi_1} y$ of $\mathcal{L}_B(E_1 \otimes_{\phi_1} F_1, E_2 \otimes_{\phi_2} F_2)$ such that $(x \otimes_{\phi_1} y)(\xi \otimes \eta) = (x\xi) \otimes (y\eta)$ for $\xi \in E_1$ and $y \in F_1$.*

We often omit the subscript ϕ_1, writing $x \otimes y = x \otimes_{\phi_1} y$ for simplicity.

3. Pentagonal equations for operators on Hilbert C^*-modules

Let A be a C^*-algebra, let E be a Hilbert A-module and let ϕ and ψ be $*$-homomorphisms of A to $\mathcal{L}_A(E)$. We assume that ϕ and ψ commute, that is, $\phi(a)\psi(b) = \psi(b)\phi(a)$ for all a, $b \in A$. We can define $*$-homomorphisms $\iota \otimes_\phi \phi$ and $\iota \otimes_\phi \psi$ of A to $\mathcal{L}_A(E \otimes_\phi E)$ by $(\iota \otimes_\phi \phi)(a) = I \otimes_\phi \phi(a)$ and $(\iota \otimes_\phi \psi)(a) = I \otimes_\phi \psi(a)$ respectively. We often omit the subscript ϕ, writing $\iota \otimes \phi = \iota \otimes_\phi \phi$ and $\iota \otimes \psi = \iota \otimes_\phi \psi$ for simplicity. We can also define $*$-homomorphisms $\iota \otimes_\psi \phi$ and $\iota \otimes_\psi \psi$ of A to $\mathcal{L}_A(E \otimes_\psi E)$.

We often omit the subscript ψ. Let W be an operator in $\mathcal{L}_A(E \otimes_\psi E, E \otimes_\phi E)$. We assume that W satisfies the following equations;

$$W(\iota \otimes_\psi \phi)(a) = (\phi \otimes_\phi \iota)(a)W, \tag{3.1}$$

$$W(\psi \otimes_\psi \iota)(a) = (\iota \otimes_\phi \psi)(a)W, \tag{3.2}$$

$$W(\phi \otimes_\psi \iota)(a) = (\psi \otimes_\phi \iota)(a)W \tag{3.3}$$

for all $a \in A$. Then, by Proposition 2.1, we can define following operators;

$$W \otimes_\psi I \in \mathcal{L}_A(E \otimes_\psi E \otimes_\psi E, E \otimes_\phi E \otimes_\psi E),$$

$$I \otimes_{\phi\otimes\iota} W \in \mathcal{L}_A(E \otimes_\phi E \otimes_\psi E, E \otimes_\psi E \otimes_\phi E),$$

$$W \otimes_\phi I \in \mathcal{L}_A(E \otimes_\psi E \otimes_\phi E, E \otimes_\phi E \otimes_\phi E),$$

$$I \otimes_{\psi\otimes\iota} W \in \mathcal{L}_A(E \otimes_\psi E \otimes_\psi E, E \otimes_{\iota\otimes\psi} (E \otimes_\phi E)),$$

$$I \otimes_{\iota\otimes\phi} W \in \mathcal{L}_A(E \otimes_{\iota\otimes\phi} (E \otimes_\psi E), E \otimes_\phi E \otimes_\phi E).$$

Since ϕ and ψ commute, there exists an isomorphism Σ_{12} of $E \otimes_{\iota\otimes\psi} (E \otimes_\phi E)$ onto $E \otimes_{\iota\otimes\phi} (E \otimes_\psi E)$ as Hilbert A-modules such that

$$\Sigma_{12}(x_1 \otimes (x_2 \otimes x_3)) = x_2 \otimes (x_1 \otimes x_3)$$

for $x_i \in E$ $(i = 1, 2, 3)$. Then we can define a pentagonal equation.

Definition 3.1. Let W be an element of $\mathcal{L}_A(E \otimes_\psi E, E \otimes_\phi E)$. Assume that W satisfies the equations (3.1), (3.2) and (3.3). An operator W is said to be multiplicative if it satisfies the pentagonal equation

$$(W \otimes_\phi I)(I \otimes_{\phi\otimes\iota} W)(W \otimes_\psi I) = (I \otimes_{\iota\otimes\phi} W)\Sigma_{12}(I \otimes_{\psi\otimes\iota} W). \tag{3.4}$$

We will call A the base algebra of the multiplicative operator W.

Example 3.2. Suppose that $A = \mathbb{C}$. Then $E = H$ is a usual Hilbert space and $\mathcal{L}_{\mathbb{C}}(E) = \mathcal{L}(H)$ is the C^*-albebra of bounded linear operators on H. Let $\phi = \psi = id$, where $id(\lambda) = \lambda I_H$ for $\lambda \in \mathbb{C}$. Then $E \otimes_{id} E$ is the usual tensor product $H \otimes H$. Let $\Sigma \in \mathcal{L}(H \otimes H)$ be the flip, that is, $\Sigma(\xi \otimes \eta) = \eta \otimes \xi$. Let W be an element of $\mathcal{L}(H \otimes H)$. Then the pentagonal equation (3.4) has the following form:

$$(3.5) \qquad (W \otimes I)(I \otimes W)(W \otimes I) = (I \otimes W)(\Sigma \otimes I)(I \otimes W).$$

Defin an operator $\widetilde{W}$ by $\widetilde{W} = W\Sigma$. Then W satisfies the pentagonal equation (3.5) if and only if $\widetilde{W}$ satisfies the usual pentagonal equation ;

$$(3.6) \qquad \widetilde{W}_{12}\widetilde{W}_{13}\widetilde{W}_{23} = \widetilde{W}_{23}\widetilde{W}_{13}.$$

Example 3.3. In Examaple 3.2, if $W = \Sigma$, then the equation (3.5) is the Yang-Baxter equation for the flip ([15]);

$$(\Sigma \otimes I)(I \otimes \Sigma)(\Sigma \otimes I) = (I \otimes \Sigma)(\Sigma \otimes I)(I \otimes \Sigma).$$

Example 3.4. Let G be a locally compact Hausdorff group and ν be a right Haar measure on G. Set $H = L^2(G, \nu)$. Defin an operator W on $H \otimes H$ by $(W\xi)(g, h) = \xi(h, gh)$ for $\xi \in C_c(G \times G)$ and g, $h \in G$. Then W satisfies the pentagonal equation (3.5). The operator $\widetilde{W}$ in Example 3.2 is given by $(\widetilde{W}\xi)(g, h) = \xi(gh, h)$, which is the Kac-Takesaki operator and satisfies the usual pentagonal equation (3.6).

Suppose that $A = C$ is an abelian C^*-algebra. Let E be a Hilbert C-module and ϕ be a $*$-homomorphism of C to $\mathcal{L}_C(E)$. Define a $*$-homomorphism ψ of C to $\mathcal{L}_C(E)$ by $\psi(c)\xi = \xi c$ for $\xi \in E$ and $c \in C$. In this situation, we have defined a generalized pentagonal equation and we have called a unitary operator pseudo-multiplicative if it satisfies the generalized pentagonal equation in [22]. We will describe the relation between the pentagonal equation (3.4) defined in this paper and the generalized pentagonal equation defined in [22]. We wrote $E \otimes_C E$ for $E \otimes_\psi E$ in [22]. Let $\widetilde{W}$ be a unitary operator in $\mathcal{L}_C(E \otimes_\psi E, E \otimes_\phi E)$. Suppose that $\widetilde{W}$ satisfies the following

$$\widetilde{W}(\iota \otimes_\psi \phi)(c) = (\iota \otimes_\phi \phi)(c)\widetilde{W}, \tag{3.7}$$

$$\widetilde{W}(\phi \otimes_\psi \iota)(c) = (\phi \otimes_\phi \iota)(c)\widetilde{W} \tag{3.8}$$

for $c \in C$. There exists an isomorphism σ_1 of $E \otimes_{\iota\otimes\psi} (E \otimes_\phi E)$ onto $E \otimes_{\iota\otimes\phi} (E \otimes_\psi E)$ such that $\sigma_1(\xi \otimes (\eta \otimes \zeta)) = \eta \otimes (\xi \otimes \zeta))$ and there exists an isomorphism σ_2 of $E \otimes_\psi E \otimes_\phi E$ onto $E \otimes_{\iota\otimes\phi} (E \otimes_\phi E)$ such that $\sigma_2(\xi \otimes \eta \otimes \zeta) = \eta \otimes (\xi \otimes \zeta)$. We define an operator $\widetilde{W}_{13}$ in $\mathcal{L}_C(E \otimes_{\iota\otimes\psi} (E \otimes_\phi E), E \otimes_\psi E \otimes_\phi E)$ by $\widetilde{W}_{13} = \sigma_2^*(I \otimes_{\iota\otimes\phi} \widetilde{W})\sigma_1$. In [22], the generalized pentagenal equation was defined as follows;

$$(\widetilde{W} \otimes_\phi I)\widetilde{W}_{13}(I \otimes_{\iota\otimes\psi} \widetilde{W}) = (I \otimes_{\phi\otimes\iota} \widetilde{W})(\widetilde{W} \otimes_\psi I). \tag{3.9}$$

There exists the flip Σ_ψ in $\mathcal{L}_C(E \otimes_\psi E)$ such that $\Sigma_\psi(\xi \otimes \eta) = \eta \otimes \xi$. Then we have the following;

Proposition 3.5. *Let W be an element of $\mathcal{L}_C(E \otimes_\psi E, E \otimes_\phi E)$. Set $\widetilde{W} = W\Sigma_\psi$. Then W satisfies the equation* (3.4) *if and only if $\widetilde{W}$ satisfies the equation* (3.9).

Example 3.6. Let G be a second countable locally compact Hausdorff groupoid. We denote by s (resp. r) the source (resp. range) map of G. We denote by $G^{(0)}$ the unit space of G and by $G^{(2)}$ the set of composable pairs. We set $G_u = s^{-1}(u)$ for $u \in G^{(0)}$. Let $\{\lambda_u; u \in G^{(0)}\}$ be a right Haar system of G. As for groupoids and groupoid C^*-algebras, see Renault [24]. (See also [19] and [22] for notations and definitions used here.) For an arbitrary topological space X, we denote by $C_c(X)$ the set of complex-valued continuous functions on X with compact supports and by $C_0(X)$ the abelian C^*-algebra of continuous functions on X vanishing at infinity with the supremum norm $\|\cdot\|_\infty$. Let C be the abelian C^*-algebra $C_0(G^{(0)})$ and let $\widetilde{E}$ be the linear space $C_c(G)$. Then $\widetilde{E}$ is a right C-module with the right C-action defined by $(\xi c)(x) = \xi(x)c(s(x))$ for $\xi \in \widetilde{E}$, $c \in C$ and $x \in G$. We define a C-valued inner product of $\widetilde{E}$ by

$$< \xi, \eta > (u) = \int_G \overline{\xi(x)}\eta(x)\, d\lambda_u(x)$$

for ξ, $\eta \in \widetilde{E}$ and $u \in G^{(0)}$. We denote by E the completion of $\widetilde{E}$ by the norm $\|\xi\| = \| < \xi, \xi > \|^{1/2}$. Then E is a full right Hilbert C-module. Define non-degenerate injective $*$-homomorphisms ϕ and ψ of C to $\mathcal{L}_C(E)$ by $(\phi(c)\xi)(x) = c(r(x))\xi(x)$ and $\psi(c)\xi = \xi c$ respectively for $c \in C$, $\xi \in \widetilde{E}$ and $x \in G$. Set $G^2(ss) = \{(x,y) \in G^2; s(x) = s(y)\}$. We define C-valued inner products of $C_c(G^2(ss))$ and $C_c(G^{(2)})$ by

$$< f_1, g_1 > (u) = \iint_{G^2(ss)} \overline{f_1(x,y)}g_1(x,y)\, d\lambda_u(x)d\lambda_u(y),$$

$$< f_2, g_2 > (u) = \iint_{G^{(2)}} \overline{f_2(x,y)}g_2(x,y)\, d\lambda_{r(y)}(x)d\lambda_u(y)$$

respectively for $u \in G^{(0)}$, f_1, $g_1 \in C_c(G^2(ss))$ and f_2, $g_2 \in C_c(G^{(2)})$. Then $C_c(G^2(ss))$ and $C_c(G^{(2)})$ are dense pre-Hilbert C-submodules of $E \otimes_\psi E$ and $E \otimes_\phi E$ respectively. Define a unitary operator W in $\mathcal{L}_C(E \otimes_\psi E, E \otimes_\phi E)$ by $(W\xi)(x,y) = \xi(y, xy)$ for $\xi \in C_c(G^2(ss))$, $(x,y) \in G^{(2)}$. Set $\widetilde{W} = W\Sigma_\psi$. We have $(\widetilde{W}\xi)(x,y) = \xi(xy,y)$ for $\xi \in C_c(G^2(ss))$, $(x,y) \in G^{(2)}$. It follows from [22] that $\widetilde{W}$ satisfies the equation (3.9). By Proposition 3.5, W satisfies the pentagonal equation (3.4). When G is a measured groupoid, that is, when there exists a quasi-invariant measure on $G^{(0)}$, we discussed in [22] the relation between the operator $\widetilde{W}$ constructed above and the fundamental operator studied by Yamanouchi in [36, §2] and by Vallin in [32, §3].

4. Coproducts for Hilbert C^*-modules

It is known that multiplicative unitary operators give coproducts in several situations (cf. [2], [35], [31], [32], [10], [21], [22]). In this section, we study a coproducts for a Hilbert C^*-module associated with a multiplicative unitary operator and a fixed vector with a certain property. First we introduce a notion of coproducts for Hilbert C^*-modules. We denote by E a Hilbert A-module and by ϕ a $*$-homomorphism of A to $\mathcal{L}_A(E)$.

Definition 4.1. Let δ be an operator in $\mathcal{L}_A(E, E \otimes_\phi E)$. We say that E is a coproduct of (E, ϕ) if δ satisfies the following equations;

$$\delta\phi(a) = (\phi \otimes \iota)(a)\delta \qquad \text{for all } a \in A \tag{4.10}$$

$$(\delta \otimes I_E)\delta = (I_E \otimes \delta)\delta \tag{4.11}$$

The triplet (E, ϕ, δ) is called a Hopf Hilbert A-module.

Suppose that δ is coproduct for E. For ξ, $\eta \in E$, we define a product $\xi\eta$ in E by $\xi\eta = \delta^*(\xi \otimes \eta)$. It follows from (4.11) that this product is associative. Then E is a right A-algebra with this product. Note that we have $||\xi\eta|| \leq ||\delta|| ||\xi|| ||\eta||$. Therefore, if $||\delta|| \leq 1$, then E is a Banach algebra.

Let ψ be a $*$-homomorphism of A to $\mathcal{L}_A(E)$ such that ϕ and ψ commute and let $W \in \mathcal{L}_A(E \otimes_\psi E, E \otimes_\phi E)$ be a multiplicative unitary operator. For an element ξ_0 of E, we say that ξ_0 has the property (E1) if it satisfies the following conditions;

(i) $||\xi_0|| = 1$.

(ii) $W(\xi_0 \otimes_\psi \xi_0) = \xi_0 \otimes_\phi \xi_0$.

(iii) For every $\xi \in E$, there exists an element $\pi_{\xi_0}(\xi)$ of $\mathcal{L}_A(E)$ such that

$$< \eta, \pi_{\xi_0}(\xi)\zeta > = < W(\xi_0 \otimes_\psi \eta), \xi \otimes_\phi \zeta >$$

for every η, $\zeta \in E$.

We fix an element ξ_0 with the property (E1). Define an operator $\delta = \delta_{\xi_0}$ in $\mathcal{L}_A(E, E \otimes_\phi E)$ by $\delta(\eta) = W(\xi_0 \otimes \eta)$. Then we have $||\delta|| \leq 1$ and $\delta^*(\xi \otimes \eta) = \pi_{\xi_0}(\xi)\eta$. Since W satisfies the pentagonal equation, we can show that (E, ϕ, δ) is a Hopf Hilbert A-module. We denote by $\xi \bullet \eta$ the product of ξ and η associated with δ. Then we have $\pi_{\xi_0}(\xi)\eta = \xi \bullet \eta$. Moreover the map π_{ξ_0} of E to $\mathcal{L}_A(E)$ is a representation of the Banach algebra $(E, \bullet)$. We denote by $B(\xi_0)$ the closed linear subspace generated by elements of the form $\pi_{\xi_0}(\xi)$ with $\xi \in E$. Then $B(\xi_0)$ is a Banach subalgebra of $\mathcal{L}_A(E)$. We denote by $C^*(B(\xi_0))$ the C^*-subalgebra of $\mathcal{L}_A(E)$ generated by $B(\xi_0)$.

For an element η_0 of E, we say that η_0 has the property (E2) if it satisfies the following conditions;

(i) $\|\eta_0\| = 1$.

(ii) $W(\eta_0 \otimes_\psi \eta_0) = \eta_0 \otimes_\phi \eta_0$.

(iii) For every $\xi \in E$, there exists an element $\widehat{\pi}_{\eta_0}(\xi)$ of $\mathcal{L}_A(E)$ such that

$$< \eta, \widehat{\pi}_{\eta_0}(\xi)\zeta >=< W^*(\eta_0 \otimes_\phi \eta), \xi \otimes_\psi \zeta >$$

for every η, $\zeta \in E$.

We fix an element η_0 with the property (E2). Define an operator $\widehat{\delta} = \widehat{\delta}_{\eta_0}$ in $\mathcal{L}_A(E, E \otimes_\psi E)$ by $\widehat{\delta}(\eta) = W^*(\eta_0 \otimes \eta)$. Since W satisfies the pentagonal equation, we can show that $(E, \psi, \widehat{\delta})$ is a Hopf Hilbert A-module. We denote by $\xi \diamond \eta$ the product of ξ and η associated with $\widehat{\delta}$. Then we have $\widehat{\pi}_{\eta_0}(\xi)\eta = \xi \diamond \eta$. Moreover the map $\widehat{\pi}_{\eta_0}$ of E to $\mathcal{L}_A(E)$ is a representation of the Banach algebra $(E, \diamond)$. We denote by $\widehat{B}(\eta_0)$ the closed linear subspace generated by elements of the form $\widehat{\pi}_{\eta_0}(\xi)$ with $\xi \in E$. Then $\widehat{B}(\eta_0)$ is a Banach subalgebra of $\mathcal{L}_A(E)$. We denote by $C^*(\widehat{B}(\eta_0))$ the C^*-subalgebra of $\mathcal{L}_A(E)$ generated by $\widehat{B}(\eta_0)$.

In the following examples, we consider a finite groupoid, an r-discrete groupoid and a compact groupoid. Let G be a second countable locally compact Hausdorff groupoid. We keep the notations in Example 3.6 except for $C_0(G^{(0)})$. Here we denote by A the C^*-algebra $C_0(G^{(0)})$. Let $W \in \mathcal{L}_A(E \otimes_\psi E, E \otimes_\phi E)$ be the multiplicative unitary operator constructed in Example 3.6. Then we have $(W\xi)(x, y) = \xi(y, xy)$ for $\xi \in C_c(G^2(ss))$ and $(x, y) \in G^{(2)}$. Note that we have $(W^*\xi)(x, y) = \xi(yx^{-1}, x)$ for $\xi \in C_c(G^{(2)})$ and $(x, y) \in G^2(ss)$. We denote by $C^*_r(G)$ the reduced groupoid

C^*-algebras. (As for the definition of the reduced groupoid C^*-algebra, see [19], [22].)

Example 4.2. Let G be a finite groupoid and let $\{\lambda_u\}$ be a right Haar system such that λ_u is a counting measure on G_u. Then we have $A = C(G^{(0)})$ and $E = C(G)$. The A-valued inner product of E is given by $< \xi, \eta > (u) = \sum_{x \in G_u} \overline{\xi(x)}\eta(x)$. We have $E \otimes_\psi E = C(G^2(ss))$ and the A-valued inner product of $E \otimes_\psi E$ is given by

$$< \xi, \eta > (u) = \sum_{s(x)=s(y)=u} \overline{\xi(x,y)}\eta(x,y).$$

We have $E \otimes_\phi E = C(G^{(2)})$ and the A-valued inner product of $E \otimes_\phi E$ is given by

$$< \xi, \eta > (u) = \sum_{\substack{s(x)=r(y) \\ r(y)=u}} \overline{\xi(x,y)}\eta(x,y).$$

We set $M = \max\{|G_u|; u \in G^{(0)}\}$, where $|G_u|$ is the number of elements of G_u. We define an element ξ_0 of E by $\xi_0(x) = M^{-1/2}$ for all $x \in G$. Then ξ_0 has the properties (E1) and (E2). We have $\pi_{\xi_0}(\xi)\zeta = \xi * \zeta$, where $\xi * \zeta$ is the convolution product defined by

$$(\xi * \zeta)(x) = \sum_{y \in G_{s(x)}} \xi(xy^{-1})\zeta(y).$$

Therefore we have $B(\xi_0) = C_r^*(G)$. Since we have $\widehat{\pi}_{\xi_0}(\xi) = M^{1/2}\theta_{\xi,\xi_0}$, we have $C^*(\widehat{B}(\xi_0)) = \mathcal{K}_A(E)$. We define an element η_0 of E by $\eta_0 = \chi_{G^{(0)}}$, where $\chi_{G^{(0)}}$ is the characteristic function of $G^{(0)}$. Then η_0 has the properties (E1) and (E2). Since we have $\pi_{\eta_0}(\xi) = \theta_{\xi,\eta_0}$, we have $C^*(B(\eta_0)) = \mathcal{K}_A(E)$. We have $\widehat{\pi}_{\eta_0}(\xi) = m(\xi)$, where $m(\xi)$ is the multiplication operator on E defined by $(m(\xi)\zeta)(x) = \xi(x)\zeta(x)$. Therefore we have $\widehat{B}(\eta_0) = C(G)$.

Example 4.3. Let G be an r-discrete groupoid [24, I.2.6]. Note that $G^{(0)}$ is open and closed in G and that G_u is discrete for every $u \in G^{(0)}$. Let $\{\lambda_u\}$ be a right Haar system such that λ_u is the counting measure on G_u. Since we have $||\xi||_\infty \leq ||\xi||_E$ for $\xi \in C_c(G)$, E is a subspace of $C_0(G)$. Fix an element f of A such that $||f||_\infty = 1$. We define an element η_0 of E by $\eta_0 = f\chi_{G^{(0)}}$. Then η_0 has the properties (E1) and (E2). We have $\pi_{\eta_0}(\xi) = \theta_{\xi,\eta_0}$. If the support of f is $G^{(0)}$, then we have $C^*(B(\eta_0)) = \mathcal{K}_A(E)$. We have $\widehat{\pi}_{\eta_0}(\xi) = m(\phi(\overline{f})\xi)$, where $m(\eta)$ is the multiplication operator on E. If f is real-valued, then we have $\widehat{\pi}_{\eta_0}(\xi)^* = \widehat{\pi}_{\eta_0}(\overline{\xi})$. Therefore, if f is real-valued and the support of f is $G^{(0)}$, then we have $\widehat{B}(\eta_0) = C_0(G)$.

Example 4.4. Let G be a compact groupoid and let $\{\lambda_u\}$ be a right Haar system such that $\lambda_u(G) = 1$ for all $u \in G^{(0)}$. We define an element ξ_0 of E by $\xi_0(x) = 1$ for all $x \in G$. Then ξ_0 has the properties (E1) and (E2). Note that $C(G)$ is a dense subspace of E. For ξ, $\zeta \in C(G)$, we have $\pi_{\xi_0}(\xi)\zeta = \xi * \zeta$, where $\xi * \zeta$ is the convolution product defined by

$$(\xi * \zeta)(x) = \int \xi(xy^{-1})\zeta(y)\, d\lambda_{s(x)}(y).$$

Therefore we have $B(\xi_0) = C_r^*(G)$. Since we have $\widehat{\pi}_{\xi_0}(\xi) = \theta_{\xi,\xi_0}$, we have $C^*(\widehat{B}(\xi_0)) = \mathcal{K}_A(E)$.

5. Operators associated with inclusions of C^*-algebras

Let A_1 be a C^*-algebra and let A_0 be a C^*-subalgebra of A_1. In this section, we do not assume that A_1 and A_0 are unital. Let E_1 be a Hilbert A_0-module and let ϕ_1 be a $*$-homomorphism of A_1 to $\mathcal{L}_{A_0}(E_1)$. We denote by ϕ_0 the restriction of ϕ_1 to A_0. Define $E_2 = E_1 \otimes_{\phi_0} E_1$ and define a $*$-homomorphism ϕ_2 of A_1 to $\mathcal{L}_{A_0}(E_2)$ by $\phi_2 = \phi_1 \otimes \iota$. In general, we define $E_n = E_{n-1} \otimes_{\phi_0} E_1$. We denote by A the C^*-algebra $\mathcal{L}_{A_0}(E_1, \phi_1)$ and by E the normed space $\mathcal{L}_{A_0}((E_1, \phi_1), (E_2, \phi_2))$. We define on E a structure of a right A-module by $(xa)(\xi) = x(a\xi)$ for $x \in E$, $a \in A$ and $\xi \in E_1$ and define on E an A-valued inner-product by $< x, y >= x^*y$ for x, $y \in E$. Then E becomes a Hilbert A-module. We define $*$-homomorphisms ϕ and ψ of A to $\mathcal{L}_A(E)$ by $(\phi(a)x)(\xi) = (a \otimes I)x(\xi)$ and $(\psi(a)x)(\xi) = (I \otimes a)x(\xi)$ respectively for $a \in A$, $x \in E$ and $\xi \in E_1$. We denote by i the inclusion map of A into $\mathcal{L}_{A_0}(E_1)$.

Proposition 5.1. *There exists an A_0-linear bounded map U of $E \otimes_i E_1$ to E_2 such that $U(x \otimes \xi) = x(\xi)$ for $x \in E$ and $\xi \in E_1$. Moreover the following equalities hold:*

$$< U\alpha, U\beta > = < \alpha, \beta > \quad \text{for } \alpha,\ \beta \in E \otimes_i E_1,$$

$$U(\phi(a) \otimes I) = (a \otimes I)U \quad \text{for } a \in A,$$

$$U(\psi(a) \otimes I) = (I \otimes a)U \quad \text{for } a \in A,$$

$$U(I \otimes \phi_1(a)) = \phi_2(a)U \quad \text{for } a \in A.$$

The proof is straightforward and we omit it. Note that U may not be adjointable. We can define the following A_0-linear bounded operators;

$$I \otimes_{\phi \otimes \iota} U : E \otimes_\phi E \otimes_i E_1 \longrightarrow E \otimes_{i \otimes \iota} E_2,$$

$$U \otimes_{\phi_0} I : E \otimes_{i \otimes \iota} E_2 \longrightarrow E_3,$$

$$I \otimes_{\psi \otimes \iota} U : E \otimes_\psi E \otimes_i E_1 \longrightarrow E \otimes_{\iota \otimes i} E_2,$$

$$I \otimes_{\iota \otimes \phi_0} U : E_1 \otimes_{\iota \otimes \phi_0} (E \otimes_i E_1) \longrightarrow E_3.$$

There exists an isomorphism S of $E \otimes_{\iota \otimes i} E_2$ onto $E_1 \otimes_{\iota \otimes \phi_0} (E \otimes_i E_1)$ as Hilbert A_0-modules such that $S(x \otimes (\xi \otimes \eta)) = \xi \otimes (x \otimes \eta)$ for $x \in E$ and $\xi,\ \eta \in E_1$. Define an A_0-linear bounded operator V of $E \otimes_\phi E \otimes_i E_1$ of E_3 by

$$V = (U \otimes_{\phi_0} I)(I \otimes_{\phi \otimes \iota} U),$$

and define an A_0-linear bounded operator $\widetilde{V}$ of $E \otimes_\psi E \otimes_i E_1$ of E_3 by

$$\widetilde{V} = (I \otimes_{\iota \otimes \phi_0} U) S (I \otimes_{\psi \otimes \iota} U).$$

We summarize the properties of V and $\widetilde{V}$ in the following proposition. The proof is easy and we omit it.

Proposition 5.2. *The operators V and $\widetilde{V}$ satisfies the following equalities;*

$$< V\alpha, V\beta > =< \alpha, \beta > \quad \text{for } \alpha,\ \beta \in E \otimes_\phi E \otimes_i E_1,$$

$$< \widetilde{V}\alpha, \widetilde{V}\beta > =< \alpha, \beta > \quad \text{for } \alpha,\ \beta \in E \otimes_\psi E \otimes_i E_1,$$

$$V(x \otimes y \otimes \xi) = (x \otimes_{\phi_0} I_{E_1})y(\xi) \quad \text{for } x,\ y \in E \text{ and } \xi \in E_1,$$

$$\widetilde{V}(x \otimes y \otimes \xi) = (I_{E_1} \otimes_{\phi_0} x)y(\xi) \quad \text{for } x,\ y \in E \text{ and } \xi \in E_1.$$

In the rest of this section, we will prove the following theorem.

Theorem 5.3. *Let U, V and $\widetilde{V}$ be as above. Suppose that U is unitary and suppose that there exists an element W of $\mathcal{L}_A(E \otimes_\psi E, E \otimes_\phi E)$ such that $V^*\widetilde{V} = W \otimes I_{E_1}$. Then W is a multiplicative unitary operator.*

Since U is unitary, V and $\widetilde{V}$ are also unitary operators. By straightforward calclulation, we have, for every $a \in A$,

$$V(\phi(a) \otimes I_E \otimes I_{E_1}) = (a \otimes I_{E_1} \otimes I_{E_1})V,$$

$$V(I_E \otimes \psi(a) \otimes I_{E_1}) = (I_{E_1} \otimes I_{E_1} \otimes a)V,$$

$$V(\psi(a) \otimes I_E \otimes I_{E_1}) = (I_{E_1} \otimes a \otimes I_{E_1})V,$$

$$\widetilde{V}(I_E \otimes \phi(a) \otimes I_{E_1}) = (a \otimes I_{E_1} \otimes I_{E_1})\widetilde{V},$$

$$\widetilde{V}(\psi(a) \otimes I_E \otimes I_{E_1}) = (I_{E_1} \otimes I_{E_1} \otimes a)\widetilde{V},$$

$$\widetilde{V}(\phi(a) \otimes I_E \otimes I_{E_1}) = (I_{E_1} \otimes a \otimes I_{E_1})\widetilde{V}.$$

Therefore W satisfies the equations (3.1), (3.2) and (3.3). For $n \geq 2$, we set

$$E^{\otimes_\phi n} = E \otimes_\phi \cdots \otimes_\phi E \quad (n \text{ times })$$

and we define $E^{\otimes_\psi n}$ similary. It follows from Proposition 5.1 that we have $U(\phi \otimes \iota)(a) = (i \otimes \iota)(a)U$ for $a \in A$. Therefore we can define the following operators;

$$I_E \otimes I_E \otimes U \in \mathcal{L}_{A_0}(E^{\otimes_\phi 3} \otimes_i E_1, E^{\otimes_\phi 2} \otimes_{i\otimes\iota} E_2),$$

$$I_E \otimes U \otimes I_{E_1} \in \mathcal{L}_{A_0}(E^{\otimes_\phi 2} \otimes_{i\otimes\iota} E_2, E \otimes_{i\otimes\iota\otimes\iota} E_3),$$

$$U \otimes I_{E_1} \otimes I_{E_1} \in \mathcal{L}_{A_0}(E \otimes_{i\otimes\iota\otimes\iota} E_3, E_4).$$

We define an element U_3 in $\mathcal{L}_{A_0}(E^{\otimes_\phi 3} \otimes_i E_1, E_4)$ by

$$U_3 = (U \otimes I_{E_1} \otimes I_{E_1})(I_E \otimes U \otimes I_{E_1})(I_E \otimes I_E \otimes U).$$

Since U is unitary by the assumption, U_3 is also a unitary operator. To prove Theorem 5.3, it is enough to prove the following proposition.

Proposition 5.4. *Set*

$$W_1 = (W \otimes_\phi I)(I \otimes_{\phi\otimes\iota} W)(W \otimes_\psi I),$$

$$W_2 = (I \otimes_{\iota\otimes\phi} W)\Sigma_{12}(I \otimes_{\psi\otimes\iota} W).$$

Then the following equation holds;

$$\begin{aligned} &U_3(W_1 \otimes_i I_{E_1})(x \otimes y \otimes z \otimes \xi) \\ &= U_3(W_2 \otimes_i I_{E_1})(x \otimes y \otimes z \otimes \xi) \\ &= (I_{E_1} \otimes I_{E_1} \otimes x)(I_{E_1} \otimes y)z\xi. \end{aligned}$$

for x, y, $z \in E$ and $\xi \in E_1$.

In the rest of this section, we will prove Proposition 5.4. Let

$$S_\psi : E \otimes_\psi E \otimes_{\iota\otimes i} E_2 \longrightarrow E_1 \otimes_{\iota\otimes\iota\otimes\phi_0} (E \otimes_\psi E \otimes_i E_1)$$

be an isomorphism defined by $S_\psi(x \otimes \xi \otimes \eta) = \xi \otimes (x \otimes \eta)$ for $x \in E \otimes_\psi E$ and $\xi,\ \eta \in E_1$, and let

$$S_\phi : E \otimes_\phi E \otimes_{\iota\otimes i} E_2 \longrightarrow E_1 \otimes_{\iota\otimes\iota\otimes\phi_0} (E \otimes_\phi E \otimes_i E_1)$$

be an isomorphism defined by $S_\phi(x \otimes \xi \otimes \eta) = \xi \otimes (x \otimes \eta)$ for $x \in E \otimes_\phi E$ and $\xi,\ \eta \in E_1$. Set $U^{(13)} = (I \otimes_{\iota\otimes\phi_0} U)S$.

Lemma 5.5. *We have the following equalities for x, y, $z \in E$ and $\xi \in E_1$. ;*

(5.12)

$$U_3((W \otimes_\phi I) \otimes_i I_{E_1}) = (\widetilde{V} \otimes_{\phi_0} I_{E_1})(I_{E\otimes_\psi E} \otimes_{\phi\otimes\iota} U),$$

(5.13)

$$((I \otimes_{\phi\otimes\iota} W) \otimes_i I_{E_1})((W \otimes_\psi I) \otimes_i I_{E_1})(x \otimes y \otimes z \otimes \xi)$$

$$= (I_E \otimes_{\iota\otimes i\otimes\iota} V^*)(I_E \otimes_{\phi\otimes\iota} U^{(13)})S_\phi^*(I_{E_1} \otimes_{\phi_0\otimes\iota\otimes\iota} V^*)(I_{E_1} \otimes I_{E_1} \otimes x)(I_{E_1} \otimes y)z\xi,$$

(5.14)

$$(I_E \otimes_{\iota\otimes i\otimes\iota} V^*)(I_E \otimes_{\phi\otimes\iota} U^{(13)})S_\phi^*(I_{E_1} \otimes_{\phi_0\otimes\iota\otimes\iota} V^*)$$

$$= (I_{E\otimes_\psi E} \otimes_{i\otimes\iota} U^*)(\widetilde{V}^* \otimes_{\phi_0} I_{E_1})$$

Lemma 5.6. *We have the following equalities for $x \in E$ and $\xi_i \in E_1$ $(i = 1, 2, 3)$;*

$$W_2 \otimes I_{E_1} = (I_E \otimes_{\iota\otimes\phi\otimes\iota} V^*\widetilde{V})(\Sigma_{12} \otimes_i I_{E_1})(I_E \otimes_{\psi\otimes\iota\otimes\iota} V^*\widetilde{V}), \tag{5.15}$$

$$U_3(I_E \otimes_{i\otimes\iota\otimes\iota} V^*) = U \otimes I_{E_1} \otimes I_{E_1}, \tag{5.16}$$

$$(I_E \otimes_{\iota\otimes\phi\otimes\iota} \widetilde{V})(\Sigma_{12} \otimes I_{E_1})(I_E \otimes_{\iota\otimes\iota\otimes i} V^*)(x \otimes (\xi_1 \otimes \xi_2 \otimes \xi_3)) = U^*(\xi_1 \otimes \xi_2) \otimes x\xi_3, \tag{5.17}$$

Proof of Proposition 5.4. Let x, y, z be elements of E and let ξ be an element of E_1. It follows from Lemma 5.5 that we have

$$
\begin{aligned}
&U_3(W_1 \otimes_i I_{E_1})(x \otimes y \otimes z \otimes \xi) \\
&= U_3((W \otimes_\phi I) \otimes_i I_{E_1})((I \otimes_{\phi\otimes\iota} W) \otimes_i I_{E_1})((W \otimes_\psi I) \otimes_i I_{E_1})(x \otimes y \otimes z \otimes \xi) \\
&= U_3((W \otimes_\phi I) \otimes_i I_{E_1})(I_{E\otimes_\psi E} \otimes_{i\otimes\iota} U^*)(\widetilde{V}^* \otimes_{\phi_0} I_{E_1})(I_{E_1} \otimes I_{E_1} \otimes x)(I_{E_1} \otimes y)z\xi \\
&= (I_{E_1} \otimes I_{E_1} \otimes x)(I_{E_1} \otimes y)z\xi.
\end{aligned}
$$

It follows from (5.16) and (5.17) that we have

$$
\begin{aligned}
&U_3(I_E \otimes_{\iota\otimes\phi\otimes\iota} V^*\widetilde{V})(\Sigma_{12} \otimes_i I_{E_1})(I_E \otimes_{\iota\otimes\iota\otimes i} V^*)(x \otimes \xi_1 \otimes \xi_2 \otimes \xi_3) \\
&= (U \otimes I_{E_1} \otimes I_{E_1})(U^*(\xi_1 \otimes \xi_2) \otimes x\xi_3) \\
&= (I_{E_1} \otimes I_{E_1} \otimes x)(\xi_1 \otimes \xi_2 \otimes \xi_3).
\end{aligned}
$$

for $\xi_i \in E_1$ $(i = 1, 2, 3)$. Then by using (5.15) we have

$$
\begin{aligned}
&U_3(W_2 \otimes_i I_{E_1})(x \otimes y \otimes z \otimes \xi) \\
&= U_3(I_E \otimes_{\iota\otimes\phi\otimes\iota} V^*\widetilde{V})(\Sigma_{12} \otimes_i I_{E_1})(I_E \otimes_{\psi\otimes\iota\otimes\iota} V^*\widetilde{V})(x \otimes y \otimes z \otimes \xi) \\
&= U_3(I_E \otimes_{\iota\otimes\phi\otimes\iota} V^*\widetilde{V})(\Sigma_{12} \otimes_i I_{E_1})(I_E \otimes_{\iota\otimes\iota\otimes i} V^*)(x \otimes \{(I_{E_1} \otimes y)z\xi\}) \\
&= (I_{E_1} \otimes I_{E_1} \otimes x)(I_{E_1} \otimes y)z\xi.
\end{aligned}
$$

□

6. Inclusions of index finite-type

In this section, we study a multiplicative unitary operator associated with an iclusion of C^*-algebras when the inclusion is of index-finite type in the sense of Watatani [34]. Let A_1 be a C^*-algebra with the identity 1, let A_0 be a C^*-subalgebra of A_1 which contains 1 and let $P_1 : A_1 \longrightarrow A_0$ be a faithful positive conditional expectation. We assume that P_1 is of index-finite type, that is, there exists a family $u_i \in A_1$ $(i = 1, \cdots, n)$ such that

$$\sum_{i=1}^{n} u_i P_1(u_i^* a) = \sum_{i=1}^{n} P_1(a u_i) u_i^* = a$$

for every $a \in A_1$ [34, 1.2.2, 2.1.6]. Then the index of P_1 is given by Index $P_1 = \sum_i u_i u_i^*$ which is an element of the center of A_1. We denote by E_1 a right A_0-module A_1 whose right A_0-action is the product in A_1. Define an A_0-valued inner product E_1 by $< a, b >= P_1(a^* b)$ for a, $b \in E_1$. It follows from [34, 2.1.5] that there exists a positive number λ such that

$$\lambda \|a\|_{A_1} \leq \|a\|_{E_1} \leq \|a\|_{A_1}$$

for every $a \in E_1 = A_1$, where $\|\cdot\|_{A_1}$ and $\|\cdot\|_{E_1}$ denote the norms of A_1 and E_1 respectively. Therefore E_1 is complete and is a Hilbert A_0-module. Define a unital injective $*$-homomorphism $\phi_1 : A_1 \longrightarrow \mathcal{L}_{A_0}(E_1)$ by $\phi_1(a)b = ab$ for $a \in A_1$ and $b \in E_1$, where ab is the product in A_1. Then we can construct A, E, ϕ and ψ as in Section 5. Moreover we can construct the operators U, V and $\widetilde{V}$.

We denote by A_2 the C^*-algebra $\mathcal{K}_{A_0}(E_1)$ (cf. [34, 2.1.2, 2.1.3]). Note that we have $\mathcal{K}_{A_0}(E_1) = \mathcal{L}_{A_0}(E_1)$. In fact, we have $I = \sum_{i=1}^{n} \theta_{u_i, u_i}$ in $\mathcal{L}_{A_0}(E_1)$. We identify $\phi(A_1)$ with A_1 and we have inclusions $A_0 \subset A_1 \subset A_2$, which is the basic construction ([34, 2.2.10], see also [11, Chapter 2]). Let $P_2 : A_2 \longrightarrow A_1$ be the dual conditional expectation of P_1, that is, $P_2(\theta_{a,b}) = (\text{ Index } P_1)^{-1} ab^*$ for a, $b \in A_1$ [34, 2.3.3]. Note that P_2 and $P_1 \circ P_2$ are of index-finite type [34, 1.7.1, 2.3.4]. We denote by F_2 a right A_0-module A_2 whose right A_0-action is the product in A_2. Define an A_0-valued inner product of F_2 by $< \xi, \eta >= P_1 \circ P_2(\xi^* \eta)$ for ξ, $\eta \in F_2 = A_2$. Then F_2 is a Hilbert A_0-module. Define a unital injective $*$-homomorphism $\widetilde{\phi}_2 : A_1 \longrightarrow \mathcal{L}_{A_0}(F_2)$ by $\widetilde{\phi}_2(a)\xi = a\xi$ for $a \in A_1$ and $\xi \in F_2$, where $a\xi$ is the product in A_2. Define a linear map $\Phi : E_2 \longrightarrow F_2$ by

$$\Phi(a \otimes b) = \theta_{a,b^*}\phi_1((\text{ Index } P_1)^{1/2})$$

for $a,\ b \in E_1$. Then Φ is an isomorphism between the Hilbert A_0-modules. Moreover we have $\Phi(\phi_2(a_1)\xi) = \widetilde{\phi}_2(a_1)\Phi(\xi)$ for $a_1 \in A_1$ and $\xi \in E_2$.

We denote by $A_0' \cap A_2$ the C^*-algebra $\{a \in A_2;\ ab = ba \text{ for every } b \in A_0\}$ and denote by $\overline{\text{lin}}\, A_1(A_0' \cap A_2)$ the closed linear subspace of A_2 generated by elements ab with $a \in A_1$ and $b \in A_0' \cap A_2$. For $a \in A_1$, we denote by $C(a)$ the norm closure of the convex hull of the set consisting of elements uau^* with unitary elements u of A_0. We consider the following two conditions:

(P1) $\quad A_2 = \overline{\text{lin}}\, A_1(A_0' \cap A_2)$.

(P2) $\quad A_0' \cap C(a) \neq \emptyset \quad$ for every $a \in A_1$.

Remark. It seems that (P1) is equivalent to the condition that the inclusion $A_0 \subset A_1$ is of depth 2. The latter condition is assumed by Enock and Vallin in [10]. But I cannot prove the equivalence yet.

In the following theorem, we show that the conditions (P1) and (P2) imply the assumptions of Theorem 5.3. Thus we have a multiplicative unitary operator when these conditions are satisfied.

Theorem 6.1. (1) *The operator U is uniatry if and only if the condition* (P1) *is satisfied.*

(2) *Suppose that U is unitary and that the condition* (P2) *is satisfied. Then there exists an elment W of $\mathcal{L}_A(E \otimes_\psi E, E \otimes_\phi E)$ such that $V^*\widetilde{V} = W \otimes I_{E_1}$.*

Corollary 6.2. *Suppose that the conditions* (P1) *and* (P2) *are satisfied. Then there exists a multiplicative unitary operator W in $\mathcal{L}_A(E \otimes_\psi E, E \otimes_\phi E)$ such that $V^*\widetilde{V} =$*

Corollary 6.3. *Suppose that A_0 is finite-dimensional and that the codition* (P1) *is satisfied. Then there exists a multiplicative unitary operator W in $\mathcal{L}_A(E \otimes_\psi E, E \otimes_\phi E)$ such that $V^* \widetilde{V} = W \otimes I_{E_1}$.*

The following proposition is useful to prove Theorem 6.1.

Proposition 6.4. (1) *There exists a bijection q_1 of $A_0' \cap A_1$ onto A such that $q_1(a)b = ba$ for $a \in A_0' \cap A_1$ and $b \in E_1$, where ba is the product of A_1.*

(2) *There exists a bijection q_2 of $A_0' \cap A_2$ onto E such that $q_2(a)b = \Phi^{-1}(ba)$ for $a \in A_0' \cap A_2$ and $b \in E_1$, where ba is the product of A_2.*

7. Crossed products by finite groups

Let A_0 be a unital C^*-algebra, let G be a finite group and let α be an action of G on A. We denote by A_1 the crossed product $A_0 \rtimes_\alpha G$. Then we have the inclusion $A_0 \subset A_1$ and the canonical conditional expectation P_1 of A_1 onto A_0. Note that Index $P_1 = |G|$. In this section, we will show that the above inclusion satisfies the condition (P1) and the assumption of Theorem 5.3. Therefore we have a multiplicative unitary operator W associated with the inclusion $A_0 \subset A_0 \rtimes_\alpha G$. We can give a formula for W.

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Department of Applied Mathematics, Faculty of Science, Osaka Women's University, Sakai City, Osaka 590-0035, Japan

E-mail address: `ouchi@appmath.osaka-wu.ac.jp`