数理解析研究所講究録 1232 巻 2001 年 160-173

Gromov–Witten Invariants and Moduli of Sheaves

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Abstract

In this paper, we shall outline a mathematical attempt to understand the Gopakumar–Vafa conjecture [GV]. We shall explain a mathematical defintion of BPS invariant, a new invariant of Calabi–Yau 3-folds from stable sheaves of dimension one. Some evidences for the Gopakumar–Vafa conjecture as an equivalence of the Gromov–Witten invariants and BPS invariants are given.

1 Gopakumar–Vafa Conjecture

Let X be a Calabi–Yau 3-fold ($\pi_1(X) = \{1\}$) and let us fix an ample line bundle $\mathcal{O}_X(1)$ on X. We denote Gromov–Witten invariants by

$$N_g(\beta) := [\overline{\mathcal{M}_{g,0}}(X,\beta)]^{virt} \in A_0(\overline{\mathcal{M}_{g,0}}(X,\beta)) \simeq \mathbb{Q},$$

and their generating functions by

$$F_g^X := \sum_{\beta \in H_2(X,\mathbb{Z})} N_g(\beta) q^\beta.$$

Based on the string duality between Type IIA and M-theory, physicists Gopakumar and Vafa [GV] introduced the following remarkable formula for the generating function of Gromov–Witten invariants.

Conjecture 1.1. ([*GV*])

(i) *There should exist* **integers** $n_h(\beta)$ *called* **BPS invariants** *such that*

$$\sum_{g\geq 0} F_g^X \lambda^{2g-2} = \sum_{k>0, h\geq 0, \beta\in H_2(X,\mathbb{Z})} n_h(\beta)\frac{1}{k}\left(2\sin(\frac{k\lambda}{2})\right)^{2h-2} q^{k\beta}. \quad (1)$$

(ii) *Let M_β be moduli of M2-branes wrapped around the curves in X. Then there exist* **support map** $\pi_\beta : M_\beta \to S_\beta$, *where S_β is a suitable moduli space parameterizing the deformation of curves (support of D-branes) in X.*

(iii) $n_h(\beta)$ *should be defined by the spin contents of the BPS states. More precisely, there exists $(sl_2)_L \times (sl_2)_R$-action on some suitable cohomology group $H^*(M_\beta)$ and $n_h(\beta)$ are defined by the following formula:*

$$n_h(\beta) := Tr_{R_h(\beta)}(-1)^{2H_R},$$

$$H^*(M_\beta) = \bigoplus_{h\geq 0}\left[(\frac{1}{2})_L \oplus 2(0)_L\right]^{\otimes h} \otimes R_h(\beta).$$

One can always define *conjectual BPS invariants* $n_h^{conj}(\beta) \in \mathbb{Q}$ recursively in terms of Gromov–Witten invariants $N_g(\beta)$ by the GV formula(1). In this approach, it is the problem to prove that $n_h^{conj}(\beta) \in \mathbb{Z}$. In [BP], Bryan and Pandharipande proved for some super-rigid curves in a Calabi–Yau 3-fold. Also, we are informed that Fukaya–Ono [FO] proved the genus 0 part of this conjecture in the symplectic category.

What we would like to do is to define BPS invariants of Calabi–Yau 3-folds independently by the moduli space of sheaves and to formulate GV conjecture as an equivalence of GW and BPS invariants. For this purpose, we have to

(i) define the moduli space of D-branes,

(ii) prove the existence of $(sl_2)_L \times (sl_2)_R$-action on a suitable cohomology on the above moduli space,

(iii) prove the Gopakumar–Vafa formula.

In this paper we present the idea of the first two steps based on our working hypothesis (table 1) and give nontrivial evidences for Gopakumar–Vafa conjecture. A mathematical definition of BPS invariants is given in section 2 and evidences are given in section 3. Especially, we can provide the answer of the problem (ii) using the intersection cohomology of the D-brane moduli spaces and the decomposition theorem due to [BBD]. This paper is a kind of survey article and the details can be found in [HST1][HST2][Ta].

	LHS of eq.(1)	RHS of eq.(1)
Object	Stable Maps $f : \Sigma_g \to X$	Stable Sheaves $\mathcal{E}$ with $\chi(\mathcal{E}) = 1$
$\beta \in H_2(X, \mathbb{Z})$	$\beta = f_*([\Sigma_g])$	$\mathrm{Supp}(\mathcal{E}) = \cup_i Y_i$ $\beta = \sum_{i=1}^{l} l(\mathcal{E}_{v_i}) \cdot [Y_i]$
$(2 \sin \frac{\lambda}{2})^{2h-2}$	Degenerate Instanton	Jacobian
Terms with $k > 1$ in RHS of eq.(1)	Multiple Covering $\Sigma_g \overset{\deg k}{\to} X$	Strictly Semistable Sheaves

Table 1: Working hypothesis

Acknowledgement

I would like to express my gratitude to Professor Kyoji Saito for valuable advices. I would like to thank Professors Shinobu Hosono and Professor Masahiko Saito who allowed me to talk about our results. I also would like to thank Professors Jim Bryan, Ron Donagi, Toshiya Kawai, Kota Yoshioka for useful discussions. I am grateful to the organizers of Workshop "Algebraic Geometry and Integrable Systems Related to String Theory". This work was partly supported by Grand-in Aid for Scientific Research of the Ministry of Education, Science and Culture in Japan.

2 Moduli Space of D-branes

2.1 "D-brane wrapped around the cycle"

What is the mathematical definition of the "D-brane wrapped around the cycle" and the moduli space of them? Usually one may think "D-brane wrapped around the cycle" as cycles with flat $U(1)$-bundles. This translation is sufficient in many cases, but since the cycles may have singularities, it is more useful for our purpose to regard D-branes as stable sheaves (Narasimhan–Seshadri theorem, Kobayashi–Hitchin correspondence). Let us first recall the notion of stability.

Definition 2.1. A coherent sheaf $\mathcal{E}$ on a scheme X is pure of dimension k if $\dim_{\mathbb{C}} \operatorname{Supp}(\mathcal{F}) = k$ for any nontrivial coherent subsheaf $\mathcal{F} \subset \mathcal{E}$.

Definition 2.2. Let $\mathcal{E}$ be a coherent sheaf which is pure of dimension d on a projective scheme X and let

$$P(\mathcal{E}, m) := \chi(X, \mathcal{E}(m)) = \sum_{i=0}^{d} \alpha_i(\mathcal{E}) \frac{m^i}{i!}$$

be the Hilbert polynomial of $\mathcal{E}$. Then $p(\mathcal{E}, m) := P(\mathcal{E}, m)/\alpha_d(\mathcal{E})$ is called a reduced Hilbert polynomial of $\mathcal{E}$.

Definition 2.3. (Stability)
Let $\mathcal{E}$ be a coherent sheaf which is pure of dimension d on a projective scheme X. $\mathcal{E}$ is stable (resp. semistable) if for any proper subsheaf $\mathcal{F}$,

$$p(\mathcal{F}, m) < p(\mathcal{E}, m), \quad \text{for} \quad m >> 0.$$

$$(\text{resp.} \quad p(\mathcal{F}, m) \leq p(\mathcal{E}, m), \quad \text{for} \quad m >> 0).$$

One can define the moduli spaces of semistable sheaves by the Simpson's construction (see, for example [HL]):

Definition 2.4. Let X be a Calabi-Yau 3-fold and let us fix an ample line bundle L on X. Let $M_{d,\chi}(X)$ be the moduli space of semistable sheaves $\mathcal{E}$ on X with Hilbert polynomial

$$P(\mathcal{E}, m) = dm + \chi.$$

It is known that $M_{d,\chi}(X)$ is a projective scheme (Theorem 4.3.4 [HL]).

2.2 Support morphism

We must take extra care if we consider the fiber space structure of the moduli spaces, i.e., if we consider the deformation spaces of the support of the sheaves in addition. For example, let us consider the following case:

(i) n copies of D-branes wrapped around the cycle C once.

(ii) Large single D-brane wrapped around the cycle C n-times.

Mathematically, the first one corresponds to a sheaf of rank n on C and the second one corresponds to a sheaf of rank 1 on non-reduced scheme with the same topological space C (but multiplicity along C is n). Sometimes the above two objects have the same Hilbert polynomial and hence they define points of the same moduli space. Such a situation makes it very difficult to deal the support map of the moduli space of stable sheaves of dimension one on a Calabi–Yau 3-fold. In fact, in the above example there exist at least two natural scheme structures C and nC on their topological space C. Thus it is very difficult problem to define a suitable "support morphism" since the subscheme structure of the support of coherent sheaves are not unique.

Our solution to this problem is to use the Chow variety $Chow(X)$ parameterizing algebraic cycles on X. It is known that $Chow(X)$ is a projective scheme.

Let X be a smooth projective scheme over $\mathbb{C}$ and $\mathcal{E}$ be a coherent sheaf on X pure of dimension 1. Let Supp($\mathcal{E}$) be the support of $\mathcal{E}$, $Y_1, \cdots, Y_l$ be the irreducible components of Supp($\mathcal{E}$) and v_i be the generic point of Y_i. Then the stalk $\mathcal{E}_{v_i} = \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_{X,v_i}$ is an Artinian module of finite length $l(\mathcal{E}_{v_i})$. One can define an algebraic cycle $s(\mathcal{E})$ by

$$s(\mathcal{E}) := \sum_{i=1}^{l} l(\mathcal{E}_{v_i}) \cdot Y_i. \tag{2}$$

Definition 2.5. Let $M_\beta(X)$ be subspace of $M_{d,1}(X)$ with $[s(\mathcal{E})] = \beta \in H_2(X,\mathbb{Z})$, $d = \int_\beta c_1(L)$.

Let us assume that $M_\beta(X)$ is normal (In general, we take the normalization of $M_\beta(X)$ and denote it by $M_\beta(X)$ again.).

Proposition 2.1. ([*HST2*])
The natural map

$$\begin{array}{rccc} \pi_\beta : & M_\beta(X) & \rightarrow & Chow(X). \\ & \mathcal{E} & \mapsto & s(\mathcal{E}) \end{array} \tag{3}$$

becomes a morphism of projective schemes. □

Let us denote by $S_\beta(X)$ the normalization of the image of $M_\beta(X)$ in $Chow(X)$. Since $M_\beta(X)$ is normal, the morphism factors through $S_\beta(X)$ from the univesal property of the normalization, and we obtain the natural morphism

$$\pi_\beta : M_\beta(X) \to S_\beta(X). \tag{4}$$

Note that π_β is projective since $M_\beta(X)$ and $S_\beta(X)$ are projective.

Remark. If X is a smooth projective surface, then π_β coincides with the support morphism given by Le Portier [LeP].

2.3 BPS Invariants

We can prove the following theorem and give mathematical definition of BPS invariants.

Theorem 2.2. ([*HST2*])
Let $\pi_\beta : M_\beta(X) \to S_\beta(X)$ be the projective morphism defined in (4)*. Let us fix a relative ample line bundle L_1 on $M_\beta(X)$ and an ample line bundle L_2 on $S_\beta(X)$ respectively.*

Then $IH^*(M_\beta(X))$ is a representation of an $(sl_2)_L \times (sl_2)_R$ defined by the relative Lefschetz operator ω_L and by the Lefschetz operator ω_R of the base. $IH^*(M_\beta(X))$ is decomposed as an $(sl_2)_L \times (sl_2)_R$-representation as follows:

$$IH^*(M_\beta(X)) = \bigoplus_{j_1,j_2} N_{j_1,j_2}(j_1)_L \otimes (j_2)_R \tag{5}$$

$$= \bigoplus_{h \geq 0} \left[(\frac{1}{2})_L \oplus 2(0)_L \right]^{\otimes h} \otimes R_h(\beta). \tag{6}$$

where we denote by $(j)_L$ the spin-j representation of the relative Lefschetz $(sl_2)_L$-action and by $R_h(\beta)$ a (virtual) representation of the $(sl_2)_R$-action. □

Definition 2.6. (BPS invariants)
By using the decomposition (5), we can define *integers* $n_h(\beta)$ by the following formula:

$$n_h(\beta) := Tr_{R_h(\beta)}(-1)^{2H_R}. \tag{7}$$

$n_h(\beta)$ will be called *BPS invariants.*

Conjecture 2.1. *Integers $n_h(\beta)$ defined in* (7) *should be deformation invariants satisfying the Gopakumar–Vafa formula* (1)*. In particular, $n_0(\beta)$ should be the holomorphic Casson invariants defined by Thomas* [*Th*].

Since neither $M_\beta(X)$ nor the morphism π_β may not be smooth in general, we cannot prove the existence of such an action on $H^*(M_\beta(X), \mathbb{C})$ by the usual Leray's spectral sequence. However, the "perverse" Leray spectral sequence tells us the origin of the $(sl_2)_L \times (sl_2)_R$-action on intersection cohomology $IH^*(M_\beta(X))$. Not only physics but also mathematics can explain $(sl_2)_L \times (sl_2)_R$ quite naturally!

Let us give a sketch of the above theorem. Let M be a normal algebraic variety. We use the theory of perverse sheaves to show the existence of $(sl_2)_L \times (sl_2)_R$-action on intersection cohomology $IH^*(M_\beta(X))$. First of all, let us recall some definitions.

Definition 2.7. (Constructible Sheaves)
A $\mathbb{C}_M$-module $\mathcal{F}$ is called *constructible* if there exists a stratification $M = \coprod_{i=1}^r M_i$ such that restrictions $\mathcal{F}|_{M_i}$ are local systems on M_i.

We denote by $D_c^b(\mathbb{C}_M)$ the derived category of bounded complexes of $\mathbb{C}_M$-modules with constructible cohomology sheaves.

Definition 2.8. (Perverse Sheaves)
A *perverse* $\mathbb{C}_M$-module is an object $K^\bullet \in D_c^b(\mathbb{C}_M)$ such that the following conditions are satisfied:

(i) (Support condition)

$$\dim_{\mathbb{C}} \mathrm{supp} H^i(K^\bullet) \le -i, \quad i \in \mathbb{Z}.$$

(ii) (Support condition for Verdier Dual)

$$\dim_{\mathbb{C}} \mathrm{supp} H^i(\mathbb{D}_M K^\bullet) \le -i, \quad i \in \mathbb{Z},$$

where $\mathbb{D}_M$ is a Verdier dualizing functor. Let ${}^pD^{\le 0}(\mathbb{C}_M)$ (${}^pD^{\ge 0}(\mathbb{C}_M)$) be the subcategory of $D_c^b(\mathbb{C}_M)$ whose objects are complexes $K^\bullet \in D_c^b(\mathbb{C}_M)$ satisfying the support condition (Support condition for Verdier Dual), respectively. Let us set

$$Perv(\mathbb{C}_M) := {}^pD^{\le 0}(\mathbb{C}_M) \cap {}^pD^{\ge 0}(\mathbb{C}_M).$$

The category of perverse $\mathbb{C}_M$-modules is an abelian category which is both Artinian and Noetherian. The simple objects are of the form

$$\iota_{!*}L[\dim_{\mathbb{C}} V] := \mathrm{Im}(\iota_! L \to \iota_* L)[\dim_{\mathbb{C}} V],$$

where $V \hookrightarrow M$ is the immersion of locally closed subvariety of M and L is a local system on V.

Theorem 2.3. (**Théorème 1.3.6 [BBD]**)
Inclusion ${}^pD^{\leq 0}(\mathbb{C}_M) \hookrightarrow D^b_c(\mathbb{C}_M)$ (${}^pD^{\geq 0}(\mathbb{C}_M) \hookrightarrow D^b_c(\mathbb{C}_M)$) *gives a right* (*left*) *adjoint functor* $\tau_{\leq 0}$ ($\tau_{\geq 0}$).

$${}^pH^0 := \tau_{\geq 0}\tau_{\leq 0} : D^b_c(\mathbb{C}_M) \to Perv(\mathbb{C}_M)$$

is a cohomology functor. ${}^pH^0$ *is called a perverse cohomology functor.* □

Definition 2.9. (**Perverse Derived Functor**)
Let $\pi : M \to S$ be a morphism of normal algebraic varieties.

$${}^pR^kf_* : Perv(\mathbb{C}_M) \to Perv(\mathbb{C}_M), \quad K^\bullet \mapsto {}^pR^kf_*K^\bullet := {}^pH^0(Rf_*K^\bullet[-k]).$$

Definition 2.10. (**Intersection Cohomology**)
Let us set $IC^\bullet_M := \iota_{!*}\mathbb{C}_{M^{smooth}}$. The intersection cohomology is defined by

$$IH^i(M) := \mathbb{H}^i(M, IC^\bullet_M) = {}^pR^i\Gamma_*IC^\bullet_M, \quad i \in \mathbb{Z}.$$

The key is the following two main theorems of the theory of perverse sheaves by Beilinson–Bernstein–Deligne.

Theorem 2.4. (**Decomposition Theorem (Theorémè 6.2.5 [BBD])**)
Let $\pi : M \to S$ *be a proper morphism and* $K^\bullet \in Perv(\mathbb{C}_M)$ *be a simple object. Then*

$$R\pi_*K^\bullet \simeq \bigoplus_k {}^pR^k\pi_*K^\bullet[-k]. \tag{8}$$

□

Theorem 2.5. (**Relative hard Lefschetz theorem (Theorémè 6.2.10 [BBD])**)
Let ω *be the first Chern class of the relative ample line bundle for the projective morphism* $\pi : M \to S$. *Then for* $k \geq 0$, *we have*

$$\omega^k\wedge : {}^pR^{-k}\pi_*K^\bullet \simeq {}^pR^k\pi_*K^\bullet. \tag{9}$$

□

There is a spectral sequence

$$E_2^{r,s} = H^r(S, {}^pR^s\pi_*IC^\bullet_{M_\beta(X)}) \Rightarrow IH^{r+s}(M_\beta(X), \mathbb{C}), \tag{10}$$

which degenerates at E_2-term because of the decomposition theorem. By applying the relative hard Lefschetz theorem to projective morphisms $\pi : M_\beta(X) \to S_\beta(X)$ and $S_\beta(X) \to \mathrm{Spec}\mathbb{C}$, we have two sl_2 action

$$\omega_L^s\wedge : E_2^{r,-s} \simeq E_2^{r,s},$$

and

$$\omega_R^r\wedge : E_2^{-r,s} \simeq E_2^{r,s},$$

which give the $(sl_2)_L \times (sl_2)_R$-action on $IH^{r+s}(M_\beta(X), \mathbb{C}) = \oplus_{r,s}E_2^{r,s}$.

3 Evidences

In this section we shall give evidences for Conjecture 2.1. For details, please see [HST1][HST2][Ta] for example.

3.1 Contractable smooth $\mathbb{P}^1$ in X

Let us consider a smooth rational curve $C \in X$ with $N_{C/X} = \mathcal{O}_C(-1) \oplus \mathcal{O}_C(-1)$. One can calculate the local contribution of Gromov–Witten invariants which counts the number of maps whose images are C. The relevant local Gromov–Witten invariants are given by

$$N_g(n \cdot C) := \int_{[\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, n[\mathbb{P}^1])]^{virt}} c_{\mathrm{top}}(R^1\pi_*\mu^* N_{C/X}), \quad n \geq 0,$$

where

$$\pi : \overline{\mathcal{M}}_{g,1}(\mathbb{P}^1, n[\mathbb{P}^1]) \to \overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, n[\mathbb{P}^1])$$

is a universal family,

$$\mu : \overline{\mathcal{M}}_{g,1}(\mathbb{P}^1, n[\mathbb{P}^1]) \to \mathbb{P}^1, \quad (f : \Sigma_g \to \mathbb{P}^1, x \in \Sigma_g) \mapsto f(x) \in \mathbb{P}^1$$

is an evaluation map.

Faber and Pandharipande proved the following theorem for the generating function of local Gromov–Witten invariants $N_g(n \cdot C)$:

Theorem 3.1. ([*FP*])

$$\sum_{g \geq 0, n \geq 1} N_g(n \cdot C) q^n \lambda^{2g-2} = \sum_{k \geq 1} \frac{1}{k} \left(2\sin(\frac{k\lambda}{2})\right)^{-2} q^k.$$

□

One can *define* the *conjectural local BPS invariants* $n_g^{conj}(d \cdot C) \in \mathbb{Q}$ by Gopakummar–Vafa formula (1)

$$\sum_{g \geq 0, n \geq 0} N_g(n \cdot C) q^n \lambda^{2g-2} = \sum_{k > 0, h \geq 0, n \geq 0} n_h^{conj}(n \cdot C) \frac{1}{k} \left(2\sin(\frac{k\lambda}{2})\right)^{2h-2} q^{kn}. \tag{11}$$

¿From this formula, the conjectural local BPS invariants $n_g^{conj}(n \cdot \mathbb{P}^1)$ can be given by

$$n_h^{conj}(n \cdot \mathbb{P}^1) = \begin{cases} 1 & \text{for } h = 0 \text{ and } n = 1 \\ 0 & \text{otherwise.} \end{cases} \tag{12}$$

The generalization of Theorem 3.1 to a contractable smooth rational curve C are given in [BKL]. First we recall the notion of Kollár's length.

Definition 3.1. (*Kollár's length*)
Let C be a smooth rational curve in Calabi–Yau 3-fold X and suppose that there exists a birational morphism $f : X \to Y$ with $f(C) = p \in Y$. Kollár's *length* l is defined to be the length at the generic point of C of the sheaf $\mathcal{O}_X/f^{-1}m_{Y,p}$ where the $m_{Y,p}$ is the maximal ideal sheaf of $p \in Y$.

It is known that $p \in Y$ is a compound DuVal singularity and $N_{C/X}$ is $\mathcal{O}_C(-1) \oplus \mathcal{O}_C(-1)$, $\mathcal{O}_C \oplus \mathcal{O}_C(-2)$ or $\mathcal{O}_C(1) \oplus \mathcal{O}_C(-3)$.

Let Y_0 be a generic hyperplane section and let X_0 be the proper transform of Y_0. By Reid's result, the minimal resolution Z_0 of Y_0 factors through X_0. Hence the length l can be computed by the length of $\mathcal{O}_{X_0}/f^{-1}|_{X_0}(m_{Y_0,p})$ and coincides with the multiplicity of C in the fundumental cycle of the corresponding ADE singularity.

Let $C_n \subset X_0$ be subschemes defined by the symbolic power $I_C^{(n)}$ of the ideal I_C defining $C \subset X_0$, and let k_n be the multiplicities of C_n in Hilbert scheme. The theorem by [BKL] gives the following conjectual local BPS invariants.

Theorem 3.2. ([*BKL*])
Let C is a contractable smooth rational curve in a Calabi–Yau 3-fold X. C_n deforms to k_n super-rigid rational curves with homology class $n[C]$ under a generic deformation of X. Since Gromov–Witten invariants are deformation invariants, conjectual local BPS invariants are given by

$$n_h^{conj}(n \cdot C) = \begin{cases} k_i & \text{for } h = 0, n = 1, 2, \dots, l \\ 0 & \text{otherwise.} \end{cases} \tag{13}$$

□

Let us calculate local BPS invariants $n_h(n \cdot C)$ defined by (7) and compare $n_h(n \cdot C)$ with $n_h^{conj}(n \cdot C)$. In order to have the local BPS invariant $n_g(d \cdot C)$, let us consider the subset $M_{n \cdot C}(X)$ (or more explicitly the subfunctor) of $M_{n[C]}(X)$ defined by

$$M_{n \cdot C}(X) := \{\mathcal{E} \in M_{n[C]}(X) \mid s(\mathcal{E}) = n \cdot C\} \subset M_{n[C]}(X).$$

Theorem 3.3. *Let $C \subset X$ be a contractable smooth rational curve on a Calabi–Yau 3-fold X and let l be the Kollár's length for C. Then $M_{n \cdot C}$ is isomorphic to the component of $Hilb(X)$ containing $n \cdot C$ and our local BPS invariants coincides with conjectual BPS invariants*:

$$n_h(n \cdot C) = \begin{cases} k_i & \text{for } h = 0, n = 1, 2, \dots, l \\ 0 & \text{otherwise.} \end{cases} \tag{14}$$

The proof of this theorem is given in [HST1] when $N_{C/X} \simeq \mathcal{O}_C(-1) \oplus \mathcal{O}_C(-1)$ and in [Ta] for general cases. The key facts to prove this theorem are that there exists a nontrivial homomorphism $\mathcal{O}_X \to \mathcal{E}$ for $\mathcal{E} \in M_{n \cdot C}$ by the condition $\chi(\mathcal{E}) = 1$, $\mathcal{E} \simeq \mathcal{O}_{C_n}$ for all $\mathcal{E} \in M_{n \cdot C}$ by stability condition and one can prove that $M_{n \cdot C}$ is isomorphic to the component of $Hilb(X)$ containing C_n by the construction of Simpson's moduli space.

3.2 Super-rigid elliptic curve in X

Let $E \subset X$ be a super-rigid elliptic curve, i.e., a smooth elliptic curve $E \in X$ with the normal bundle $N \simeq L \oplus L^{-1}$ where L is a non-torsion element of the Picard group of E.

Pandharipande [P] showed the following:

Theorem 3.4. *([P])*

$$N_g(n \cdot E) = \begin{cases} \frac{\sigma(n)}{n} = \sum_{i|n} \frac{1}{i} & for\ g = 1, n \geq 1 \\ 0 & otherwise. \end{cases} \tag{15}$$

Therefore

$$n_h^{conj}(n \cdot E) = \begin{cases} 1 & for\ h = 1, n \geq 1 \\ 0 & otherwise. \end{cases} \tag{16}$$

□

$M_{n \cdot E}$ and local BPS invariants $n_h(n \cdot E)$ are given as follows:

Theorem 3.5. *([HST2])*

$$M_{n \cdot E} \simeq E.$$

Thus we have

$$n_h(n \cdot E) = \begin{cases} 1 & for\ h = 1, n \geq 1 \\ 0 & otherwise. \end{cases} \tag{17}$$

□

More precisely, we have proved that the moduli $M_{n \cdot E}$ of stable sheaf on X with $\chi = 1$ and support E coincides with the moduli of stable sheaf on E of rank d with $\chi = 1$. The latter moduli space is well-understood by the work of Atiyah and elements are of the form $W_n \otimes L$ where L is a line bundle of degree 0 and W_n are stable bundles of degree 1 defined recursively by the unique nontrivial extension

$$0 \to \mathcal{O}_E \to W_n \to W_{n-1} \to 0, \quad W_1 := \mathcal{O}_E(p_0). \tag{18}$$

3.3 Rational elliptic surface in a Calabi–Yau 3-fold

Let $\pi : S \to \mathbb{P}^1$ be a rational elliptic surface, and let σ and F be a section and fiber of p, respectively.

Theorem 3.6. ([*HST1*][*HST2*])

$$\sum_{g\geq 0, r\geq 0} n_r(\sigma + gF)\left(2\sin\frac{\lambda}{2}\right)^{2r-2} q^g$$
$$= \frac{1}{(e^{-\sqrt{-1}\lambda/2} - e^{\sqrt{-1}\lambda/2})^2}\prod_{n\geq 1}\frac{1}{(1-e^{-\sqrt{-1}\lambda}q^n)^2(1-e^{\sqrt{-1}\lambda}q^n)^2(1-q^n)^8}. \quad (19)$$

□

Remark. It is important that our theorem also holds for other elliptic surfaces in a Calabi–Yau manifold. In particular, if we consider the case of K3 surface, we have the same results as that of Kawai–Yoshioka [KY], i.e., the both generating functions of BPS states become $\chi_{10,1}(\tau, \nu)$. They considered the Abel–Jacobi map and counted the number of BPS states from D0-D2 system. On the other hand, we use the relative Lefshetz action on the relative Jacobian and counted the spin contents of BPS states from M2 brane. The coincidence of these results is very natural since the original physical theory should be equivalent.

If we allow some physical arguments (holomorphic anomaly equation), we have the nontrivial evidence for Gopakumar–Vafa conjecture. Let us write the generating functions of Gromov–Witten invariants as

$$Z_{g;n}(q) := \sum_d N_{g,d;n}q^d, \quad N_{g,d;n} := \sum_{(\beta,\sigma)=d,(\beta,F)=n} N_g(\beta), \quad n \geq 1, \quad (20)$$

where $N_g(\beta) \in \mathbb{Q}$ are genus g Gromov–Witten invariants for $\beta \in H_2(S, \mathbb{Z})$ defined by

$$N_g(\beta) := \int_{[\overline{\mathcal{M}}_{g,0}(S,\beta)]^{virt}} c_{\text{top}}(R^1\pi_*\mu^* N_{S/X}). \quad (21)$$

Especially, $Z_{0;1}(q)$ is given by

$$Z_{0;1}(q) = E_4(q)\prod_{k\geq 1}\frac{1}{(1-q^k)^{12}}. \quad (22)$$

Proposition 3.7. (Holomorphic anomaly equation [HST1])

(i) $Z_{g;n}(q)$ *has the following expression*

$$Z_{g;n}(q) = \frac{P_{2g+6n-2}(E_2(q), E_4(q), E_6(q))}{\prod_{k\geq 1}(1-q^k)^{12n}}, \tag{23}$$

where $P_{2g+6n-2}(E_2(q), E_4(q), E_6(q))$ *is a homogeneous polynomial of weight* $2g+6n-2$ *and* $E_*(q)$ *are Eisenstein series of weight* $*$.

(ii) $P_{2g+6n-2}(E_2, E_4, E_6)$ *satisfies the following equation:*

$$\frac{\partial P_{2g+6n-2}}{\partial E_2} = \frac{1}{24}\sum_{g=g'+g''}\sum_{s=1}^{n-1} s(n-s)P_{2g'+6s-2}P_{2g''+6(n-s)-2} + \frac{n(n+1)}{24}P_{2(g-1)+6n-2}. \tag{24}$$

□

If $n=1$, we can solve the holomorphic anomaly equation easily and the generating function of $Z_{g;1}(q)$ is expressed as

$$\sum_{g\geq 0} Z_{g;1}(q)\lambda^{2g} = Z_{0;1}(q)\exp\left(2\sum_{k\geq 1}\frac{\zeta(2k)}{k}E_{2k}(q)\left(\frac{\lambda}{2\pi}\right)^{2k}\right). \tag{25}$$

By the famous Jacobi's triple product formula, we have

$$\lambda^{-2}\exp\left(2\sum_{k\geq 1}\frac{\zeta(2k)}{k}E_{2k}(q)\left(\frac{\lambda}{2\pi}\right)^{2k}\right) = \frac{1}{(e^{-\sqrt{-1}\lambda/2}-e^{\sqrt{-1}\lambda/2})^2}\prod_{n\geq 1}\frac{(1-q^n)^4}{(1-e^{\sqrt{-1}\lambda}q^n)^2(1-e^{-\sqrt{-1}\lambda}q^n)^2}. \tag{26}$$

Multiplying $Z_{0;1}(q)$ both sides, we can easily verify the Gopakumar–Vafa conjecture, which was given in [HST1] where $n_h(\sigma+gF)$ are obtained with some intuitions (of course we had no mathematical proof of $(sl_2)_L \times (sl_2)_R$ decomposition).

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