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A zeta function of a smooth manifold and elliptic cohomology

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Abstract

We will propose a new definition of a zeta function of a smooth manifold, using Grothendieck's idea of crystalline cohomology which is used to express Hasse-Weil's congruent zeta function of a smooth projective variety defined over a finite field as an alternating product of characteristic polynomials of Frobenius. In order to compute our zeta function, we will use the theory of ellptic cohomology.

1 Motivation

The purpose of this note is to explain a main idea of [9]. Details are found in [9].

Our definition of a zeta function depends on the Grothendieck's idea of **Crystalline Cohomology**, hence we first recall the definition of crystalline cohomology.

Arithmetic case

In order to avoid an unneccessary complexity, we only consider the simplest case. Let X be a smooth projective variety defined over a finite prime field $\mathbf{F}_p$ of characteristic p. We assume an existence of a smooth model $\mathcal{X}$ of X defined over $\mathbf{Z}_p$. The *i-th crystalline cohomology* of rational coefficient $H^i_{crys}(X/\mathbf{Z}_p)\otimes\mathbf{Q}_p$ of X is defined to be

$$H^i_{crys}(X/\mathbf{Z}_p)\otimes\mathbf{Q}_p \stackrel{\text{def}}{=} H^i_{DR}(\mathcal{X}/\mathbf{Z}_p)\otimes\mathbf{Q}_p. \qquad (1)$$

It is well-known that $H^i_{crys}(X/\mathbf{Z}_p)\otimes\mathbf{Q}_p$ is independent a choice of a model $\mathcal{X}$. Also an endomorphism ϕ called *Frobenius* acts on $H^i_{crys}(X/\mathbf{Z}_p)\otimes\mathbf{Q}_p$.

$$\zeta_X(T) = \sum_{n=1}^{\infty} \frac{|X(\mathbf{F}_{p^n})|}{n} T^n \tag{2}$$

be the Hasse-Weil's congruent zeta function of X. $\zeta_X(T)$ can be expressed in terms of ϕ. We prepare some notations.

Notations 1.1. • $\mathrm{Tr}_\phi^+(T) = \sum_{i\equiv 0(2)} \sum_{n=1}^{\infty} \mathrm{Tr}[\phi^n | H^i_{crys}(X/\mathbf{Z}_p) \otimes \mathbf{Q}_p] T^n$

• $\mathrm{Tr}_\phi^-(T) = \sum_{i\equiv 1(2)} \sum_{n=1}^{\infty} \mathrm{Tr}[\phi^n | H^i_{crys}(X/\mathbf{Z}_p) \otimes \mathbf{Q}_p] T^n$

Now the following formula is due to Grothendieck and Berthelot.

Fact 1.1. $-T\frac{d}{dT}\log\zeta_X(T) = \mathrm{Tr}_\phi^+(T) - \mathrm{Tr}_\phi^-(T)$.

In particular, $\zeta_X(T) = \exp[-\int\{\mathrm{Tr}_\phi^+(T) - \mathrm{Tr}_\phi^-(T)\}\frac{dT}{T}]$.

Geometric case

Now we treat our geometric case. Let M be a compact oriented manifold with $w_2(M) = p_1(M) = 0$ and let $\mathcal{LM}$ be its free loop space. Also we assume M admits an almost complex structure. (In fact, this assumption is unneccesary.) Comparing to arithmetic case, M corresponds to X and $\mathcal{LM}$ corresponds to $\mathcal{X}$. Note that $\mathcal{LM}$ has a natural S^1-action by rotation of parameter. One can consider vector bundles Σ_+ and Σ_- of infinite rank over $\mathcal{LM}$ which is called as a ***plus loop spinor bundle*** and ***minus loop spinor bundle*** respectively. Between them, there exists a differential operator of the first order (*loop Dirac operator*),

$$\Gamma(\mathcal{LM}, \Sigma_+) \xrightarrow{\mathcal{D}} \Gamma(\mathcal{LM}, \Sigma_-). \tag{3}$$

Σ_+, Σ_-, and $\mathcal{D}$ have the following properties ([10]);

- Σ_+ and Σ_- admit S^1-action which is equivalent to natural one of $\mathcal{LM}$.
- $\mathcal{D}$ is S^1-equivalent.

Therefore both Ker$\mathcal{D}$ and Coker$\mathcal{D}$ admit S^1-action and these corresppond to the Frobenius action on $H^{\cdot}_{crys}(X/\mathbf{Z}_p) \otimes \mathbf{Q}_p$. Let

- Ker$\mathcal{D} = \oplus_n H^+(n)$,
- Coker$\mathcal{D} = \oplus_n H^-(n)$,

be a weight decomposition by the S^1-action and we set

$$\chi_{\mathcal{D}}(M, q) = \sum_n \{\dim H^+(n) - \dim H^-(n)\} q^n. \tag{4}$$

This corresponds to $-T\frac{d}{dT}\log\zeta_X(T)$.

In the folowing sections, we will discuss a way of calculating this invariant using elliptic cohomology.

2 Elliptic cohomology

In this section, R denotes a commutative $\mathbf{Q}$-algebra. (In fact, in order to develop a theory of *elliptic cohomology*, it is sufficient R is a commutative ring with a unit such that 6 is invertible.)

A complex oriented cohomology theory vs a formal group

A cohomology theory $H^{\cdot}$ is said to be *complex oriented* if the cohomology ring of the classifying space $BU(1)$ of $U(1)$ is isomorphic to a formal power series ring of one variable:

$$H^{\cdot}(BU(1), R) \cong R[[T]]. \tag{5}$$

Let $\mathcal{L}$ be the universal line bundle over $BU(1)$. By universality of $\mathcal{L}$, we have a map,

$$BU(1) \times BU(1) \xrightarrow{\phi} BU(1) \tag{6}$$

such that $\phi^*\mathcal{L} = p_1^*\mathcal{L} \otimes p_2^*\mathcal{L}$, where p_1 (resp. p_2) is the first (resp. second) projection. The functoriality of $H^{\cdot}$ induces a homomorphism

$$H^{\cdot}(BU(1), R) \xrightarrow{\phi^*} H^{\cdot}(BU(1), R)\hat{\otimes} H^{\cdot}(BU(1), R), \tag{7}$$

and by (5), this is a ring homomorphism

$$R[[T]] \xrightarrow{\phi^*} R[[X, Y]]. \tag{8}$$

We set $F_H(X, Y) = \phi^*(T)$. One can easily see that $F_H(X, Y)$ satisfies the following identities.

- (commutativity) $F_H(X, Y) = F_H(Y, X)$.
- (existence of unit) $F_H(X, 0) = X$.
- (ass ociativity) $F_H(F_H(X, Y), Z) = F_H(X, F_H(Y, Z))$.
- $F_H(X, Y) = X + Y +$(higher order).

In general, a formal power series $F(X, Y) \in R[[X, Y]]$ which satisfies the above conditions is said to be a *formal group* defined over R ([8]). Here are some examples of formal groups.

Example 2.1. *1. (the formal group associated to additive group)*

$F(X, Y) = X + Y.$

2. (the formal group associated to multiplicative group)

$F(X, Y) = X + Y + XY.$

In this way, we associate a formal group defined over R to a complex oriented cohomology theory whose coefficient ring is R. It is a result of Landweber ([5]) that one can also associate an R-coefficient complex oriented cohomology theory to a formal group defined over R.

A formal group vs a complex genus

We first recall results due to Lazard and Quillen.

Fact 2.1. *(Lazard [6]) Let* $\mathcal{LA} \stackrel{\text{def}}{=} \mathbf{Z}[\{z_n\}_{n=1}^{\infty}]$. *($\mathcal{LA}$ is called as* Lazard's ring.*) Then there exists a formal group law* $F^u(X,Y)$ *defined over* $\mathcal{LA}$ *which is universal in the following sense;*

Let $F(X,Y)$ *be a formal group law defined over a ring* R. *Then there exists the unique ring homomorphism,* $\mathcal{LA} \xrightarrow{\theta} R$ *such that* $F(X,Y) = \theta(F^u(X,Y))$.

Fact 2.2. *(Quillen [7])* $\mathcal{LA}$ *is isomorphic to the complex cobordism ring* Ω^U_* *and the formal group determined by* $F_U(X,Y)$ *is isomorphic to Lazard's universal formal group.*

A ring homomorphism from Ω^U_* to R is said to be a *complex genus* whose values are in R. The above two results imply a there is a one to one correspondence between a formal group defined over R and a complex genus whose values are in R.

Now we state our definition of *elliptic genus* and *elliptic cohomology*. Let

$$E = \{y^2 = x^3 - ax + b\}, \qquad \omega = \frac{dx}{2y}. \tag{9}$$

be a pair of an elliptic curve and its invariant diffential defined over R. We choose a formal parameter T of E at the origin to be

$$T = -\frac{x}{y}. \tag{10}$$

Let $\hat{\mathcal{O}}_{E,0}$ be the formal completion of $\mathcal{O}_E$ at the origin. The group law $E \times E \xrightarrow{\mu} E$ of E induces a homomorphism

$$\mathcal{O}_{E,0} \xrightarrow{\mu^*} \mathcal{O}_{E,0} \hat{\otimes} \mathcal{O}_{E,0}. \tag{11}$$

By the choice of a formal parameter T, $\mathcal{O}_{E,0}$ is isomorphic to $R[[T]]$. Hence (11) becomes a homomorphism

$$R[[T]] \xrightarrow{\mu^*} R[[X,Y]], \tag{12}$$

and we define a formal group $F_{(E,\omega)}(X,Y)$ assosiated to (E,ω) to be $F_{(E,\omega)}(X,Y)$ $\mu^*(T)$. The cohomology theory (resp. complex genus) which associated to a pair (E,ω) is said to be *elliptic cohomology* (resp. *elliptic genus*). Moreover if (E,ω) is defined over a ring of modular forms (of certain level), these are said to be *modular*. One can obtain the following proposition without difficulties.

Proposition 2.1. *Let ϕ be a modular elliptic genus of level Γ. Then, for an almost complex compact manifold of dimension $2n$, $\phi(M)$ is a modular form holomorphic at cusps of weight n and of level Γ.*

Particular cases of the proposition is considered in [3] and [6].

3 How to compute a zeta function (after Witten and Zagier)

In this section, we follow Witten and Zagier's argument to compute our zeta functions ([10], [11], [6].)

We first prepare some notations. For a manifold M and an indeterminate q, we set

$$S_q(TM \otimes \mathbf{C}) \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} Sym^k(TM \otimes \mathbf{C})q^k \in K_0(M)[[q]], \tag{13}$$

where $K_0(M)$ is the Grothendieck's group of M, and Sym^k is the k-th symmetric product.

Let M be an almost complex manifold with $w_2(M) = p_1(M) = 0$. Witten computed $\chi_{\mathcal{D}}(M,q)$ by formally using Atiyah-Singer's fixed point formula and he obtained

$$\chi_{\mathcal{D}}(M,q) =< \hat{A}(M)ch(\otimes_{n=1}^{\infty} S_{q^n}(TM \otimes \mathbf{C})), [M] > . \tag{14}$$

The right hand side of (14) can be calculated more explicitly.

Let $R = \mathbf{Q}[G_4, G_6]$, where G_i is the Eisenstein series of weight i. We set

$$Q_{WS}(T) = \exp[\sum_{k=2}^{\infty} \frac{2}{(2k)!} G_{2k}(q)T^{2k}] \in \mathbf{Q}[G_4, G_6][[T]]. \tag{15}$$

Let $g(T)$ be the formal inverse function of $\frac{T}{Q(T)}$ and we write

$$g'(T) = \sum_{i=0}^{\infty} a_n T^n, a_n \in R. \tag{16}$$

We define a complex genus ϕ_{WS} (which is said to be *Weierstrass-Witten genus*) to be

$$\phi_{WS}(\mathbf{P}^n(\mathbf{C})) = a_n. \tag{17}$$

Note that the rational complex cobordism ring $\Omega^U \otimes \mathbf{Q}$ is generated by $\{\mathbf{P}^n(\mathbf{C})\}_n$. For an almost complex compact manifold M of dimension $4k$ such that $w_2(M) = p_1(M) = 0$, we have

- $\chi_D(M,q) = \prod_{n=1}^{\infty}(1-q^n)^{-4k}\phi_{WS}(M)$
- $\phi_{WS}(M)$ is a modular form of weight $2k$ and of level 1.

For a modular form f of level 1, let $a_0(f)$ be the contant term of the Fourier expansion of f. We define a *zeta function* of M to be the Mellin transform of $\phi_{WS}(M) - a_0(\phi_{WS}(M))$ (Compare **Fact 1.1**):

$$\zeta_M(s) = \int_0^{\infty} [\phi_{WS}(M) - a_0(\phi_{WS}(M))](it)t^s \frac{dt}{t}. \tag{18}$$

In general, without the conditions $w_2 = p_1 = 0$, we define a zeta function of a smooth manifold by (16). Here are some examples.

Example 3.1. *1.* $\zeta_{\mathbf{P}^4(\mathbf{C})}(s) = -\frac{2^7\pi^4}{4!}\zeta(s)\zeta(s-3)$,

2. $\zeta_{\mathbf{P}^6(\mathbf{C})}(s) = -\frac{2^8 3\pi^6}{6!}\zeta(s)\zeta(s-5)$.

4 Comments and remarks

We will briefly explain a relationship between ϕ_{WS} and a series of linear representations of Monster. Details are found in [3].

It is well-known (cf.[2]) as a *Moonshine conjecture* that there is a misterious relationship between *Monster* and $j(q) - 744$, where $j(q)$ is the elliptic modular function.

Let consider the Fourier expansion of $j(q) - 744$,

$$j(q) - 744 = q^{-1} + 196884q + 21493760q^2 + 864299970q^3 + \cdots. \tag{19}$$

Note that the first coefficient 1 is the dimension of trivial representation of Monster. It is known the dimension of the smallest non-trivial irreducible representation of Monster is 196883, and this is nothing but $a_1(j(q)-744) - a_{-1}(j(q)-744)$. (Remember that $a_i(\cdot)$ denotes the i-th Fourier coefficient.) So it is natural to conjecture that $j(q) - 744$ is the generating function (in some sense) of dimension of irreducible representation of Monster. This conjecture was solved by Borchers using a vertex operator algebra. ([1], [4])

Hirzeburch proposed a problem to construct a series of irreducible representation by a geometric way. ([3], **Prize Question**) His plan is as follows.

1. Construct a 24 dimensional compact oriented smooth spin manifold M with $p_1 = 0 \in H^4(M, \mathbf{Q})$ and $\phi_{WS}(M) = E_4^3 - 744\Delta$, where E_i is the normalized Eisenstein series of weight i and Δ is the normalized cusp form of weght 12.

2. Find such a manifold which admits an action of Monster.

Such a manifold satisfies an identity

$$q^{-1} \cdot \hat{A}(M, \otimes_{n=1}^{\infty} S_{q^n}(TM \otimes \mathbf{C})) = j(q) - 744, \tag{20}$$

where $\hat{A}$ denotes $\hat{A}$-genus. This identity implies

- $\hat{A}(M) = 1$ and $\hat{A}(M, TM \otimes \mathbf{C}) = 0$.
- $\hat{A}(M, Sym^2(TM \otimes \mathbf{C})) = 196884$.

Since we have a decomposition,

$$Sym^2(TM \otimes \mathbf{C}) = E \oplus \mathbf{1}, \tag{21}$$

where $\mathbf{1}$ is the trivial bunble, the smallest non-trivial irreducible representation of Monster may be realized as the cohomology group of E.

Let $M(1)_R$ be the graded ring of modular forms of full level which are holomorphic at the cusp whose Fourier coefficients are valued in a commutative ring R. It is easy to see that compact smooth oriented manifolds whose dimension is divisible by 4 and which satisfy conditions $w_2 = 0$ and $p_1 = 0 \in H^4(\mathbf{Q})$ form a subring of oriented codordism ring. We denote this subring by Ω^0. Then ϕ_{WS} becomes a ring homomorphism

$$\Omega^0 \xrightarrow{\phi_{WS}} M(1)_{\mathbf{Z}}. \tag{22}$$

If this is surjective, we obtain a manifold which satisfies the condition 1. We have obtained the following proposition.

Proposition 4.1. *After tensoring $\mathbf{Z}[\frac{1}{6}]$, (22) becomes surjective.*

In fact, we have constructed a compact smooth 24 dimensional manifold M satisfying the conditions and $\phi_{WS}(M) = 144(E_4^3 - 744\Delta)$. But we do not know whether (22) is surjective or not. A problem to find a manifold which admits an action of Monster seems much more difficult.

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