Topology of Lagrangian Submanifolds

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Y. Eliashberg gave a talk on topology of Lagrangian submanifolds at a conference held at RIMS from 9 to 12 May 2000. Here we note only a part of his talk.

The content of Sections 1 and 2, except Theorem 1.4 can be found in [1]. Theorem 1.4 is joint with L. Polterovich and is contained in [2]. Results stated in Section 3 are extracted from a joint with M. Gromov paper [3].

1 Unknotting of Lagrangian surfaces in symplectic 4-manifold

Let $(M^{2n}, \omega)$ be a symplectic manifold. An n-dimensional submanifold $L$ is called a Lagrangian submanifold if $\omega|_L = 0$.

Example $M = \mathbb{R}^{2n} = \mathbb{C}^n, \omega_0 = \sum_{i=1}^{n} dx^i \wedge dy^i$, where $(z_1, \cdots, z_n) = (x_1 + iy_1, \cdots, x_n + iy_n)$ is the standard coordinate of $\mathbb{C}^n$, is a symplectic manifold. In this case, a linear n-dimensional plane $L$ is Lagrangian if and only if $iL \perp L$. If instead we have $iL \overline{\cap} L$, then $L$ is called totally real. General totally real submanifolds are defined in an obvious manner.

We will treat $n = 2$ case of the above example. The first result we will mention is the following unknottedness theorem.

Theorem 1.1. Let $\mathbb{R}^4_+ = \{y_2 \geq 0\}$ and assume that a 2-disk $\Delta$ is embedded in $\mathbb{R}^4_+$ as $(\Delta, \partial \Delta) \subset (\mathbb{R}^4_+, \partial \mathbb{R}^4_+)$ and $\partial \Delta = \{|z_1| = 1, z_2 = 0\}$. Then, if we have $\omega|_\Delta \geq 0$, then $\Delta$ is unknotted, i.e. we can isotope $\Delta$ relative to $\partial \Delta$ to a disk in $\partial \mathbb{R}^4_+$.

The proof of this theorem relies on the method of filling with holomorphic discs and we quote the necessary result here. We first define the pseudocovexity of an oriented hypersurface $\Sigma$ of general symplectic manifold $(M^{2n}, \omega$
Let $J$ be an almost complex structure on $M$ tamed by $\omega$. Then, for every point $x$ on $\Sigma$, the tangent space $T_xM$ has a $J$-invariant $(2n - 2)$ dimensional subspace $T_x^J\Sigma$. $\bigcup_{x \in M} T_x^J\Sigma$ is a $(2n - 2)$ dimensional subbundle $T^J M$ of $TM$.

Since $\Sigma$ is oriented and each $T_x^J\Sigma$ has a natural orientation as a complex vector space, the quotient 1-dimensional bundle $T\Sigma/T^J\Sigma$ is also orientable, i.e. trivial. In particular, there is a trivial sub-line bundle $\mathbb{R}$ of $T\Sigma$ such that $T\Sigma = \mathbb{R} \oplus T^J\Sigma$. Choosing a non-vanishing section $\eta$ of $\mathbb{R}$ fixes a 1-form $\alpha$ on $\Sigma$ satisfying $\alpha|_{T^J\Sigma} = 0$ and $\alpha(\eta) > 0$.

**Definition 1.1.** $\Sigma$ is called $J$-convex, or pseudoconvex if the quadratic form $t \mapsto d\alpha(t, Jt)$ on $T^J\Sigma$ is positive definite.

With this preparation, we can state the following result.

**Theorem 1.2.** Let $\Omega$ be a domain in $\mathbb{R}^4$ such that $\partial \Omega$ is pseudo convex w.r.t. some almost complex structure $J$ tamed by $\omega_0$. Let $F$ be a surface with boundary embedded in $\partial \Omega$ such that $F$ has a unique complex point which is elliptic, and $J$ is integrable near that point. Moreover, assume that there is a $J$-holomorphic disc $\Delta$ with $\partial F = \partial \Delta$ and which is transversal to $\partial \Omega$ along $\partial \Delta$. Then $F \cup \Delta$ can be filled with a family of embedded, disjoint $J$-holomorphic discs $\{D_t\}$.

Now we explain the outline of the proof of the unknottedness theorem. First, we take a large sphere $S$ in $\mathbb{R}^4$ with the center on the $y_2$-axis which intersects with the $z_1$-plane along $\partial \Delta$, and let $B$ be the interior domain of $S$. We can take a disk $F$ in $S$ whose boundary coincides with $\partial \Delta$ and has a unique complex point which is elliptic, and moreover it is isotopic to a disk on $\partial \mathbb{R}_+^4$, relative to the boundary. On the other hand, the disk $\Delta$ can be slightly deformed by a boundary fixing isotopy so that $\omega|_\Delta > 0$. Taking $B$ large enough, we can suppose that $\Delta$ is contained in $B$. Then, there is an almost complex structure $J$ tamed by $\omega_0$ for which $\Delta$ is $J$-holomorphic. Moreover $J$ can be chosen integrable near the elliptic point of $F$. This will allow us to apply the filling with holomorphic disc technique to the triple $(\Omega = B, F, \Delta)$, and thus will supply us with the isotopy mentioned in the theorem.

Using the same technique, we can prove the next theorem.

**Theorem 1.3.** Let $\Pi_0$ and $\Pi_1$ denote the hyperplanes $\{y_2 = 0\}$ and $\{y_2 = 1\}$, and let $L_0$ be the Lagrangian cylinder $\{|z_1| = 1, x_2 = 0, 0 \leq y_2 \leq 1\}$. Suppose $L$ is another Lagrangian cylinder between $\Pi_0$ and $\Pi_1$ having the same boundary as $L_0$. Then, $L$ is Lagrangian isotopic to $L_0$ relative to the boundary in $\mathbb{R}^4 \setminus (D_+ \cup D_- \cup R_+)$, where $D_+ = \{|z_1| \leq 1, z_2 = 0\}$, $D_- = \{|z_1| \leq 1, z_2 = 1\}$, and $R_+ = \{y_2 \geq 1, x_2 = z_1 = 0\}$. 
(Outline of the proof) We again replace the plane $\Pi_0$ by a boundary $\partial \Omega$ of a large convex domain $\Omega$ such that $\partial \Omega$ intersects with the $z_1$-plane along the unit circle $C$. As before, we can take a disk $F$ whose boundary coincides with $C$ and which has a unique complex point which is elliptic. On the other hand, we can modify the cylinder $\Delta$ by a boundary fixing isotopy, as well as gluing a disk on the top of it, so that the resulting disk $\Delta$ will have the boundary $C$, on which the symplectic form is positive. Then, as before, we can choose an almost complex structure $J$ integrable near the elliptic point of $F$, tamed by $\omega_0$, with respect to which $\Delta$ is holomorphic, and then apply the filling with holomorphic disks technique to $(F, \Delta)$. This will supply the isotopy we want.

The next is the unknottedness result for Lagrangian knots in $\mathbb{R}^4$.

**Theorem 1.4.** There is no knotted Lagrangian plane in $\mathbb{R}^4$. That is, if $\phi: \mathbb{R}^2 \to (\mathbb{R}^4, \omega_0)$ is a Lagrangian embedding which coincides with the inclusion $i: \mathbb{R}^2 \to \mathbb{C}^2$ defined by $(x, y) \mapsto (x, 0, 0, y)$ outside of a compact set, then there is a compact supported Lagrangian isotopy between $\phi$ and $i$.

(outline of the proof) This theorem is a consequence of the following two results.

**Proposition 1.** If a Lagrangian knot $L$ in $\mathbb{R}^4$ is contained in some simple hypersurface $Q$, then $L$ is Lagrangian isotopic to the flat plane.

**Proposition 2.** For every Lagrangian knot $L$ in $\mathbb{R}^4$, there is a simple hypersurface $Q$ containing it.

We first explain the word simple hypersurface. Let $R$ be a oriented hypersurface in $(\mathbb{R}^4, \omega_0)$. Then, the symplectic form $\omega_0$ restricted to $R$ defines an oriented 1-dimensional distribution on $R$ by $\text{Ker}\omega_0$. $R$ integrates into a 1-dimensional foliation. We call this foliation characteristic.

**Definition 1.2.** A hypersurface $Q$ in $\mathbb{R}^4$ is called simple if each leaf of its characteristic foliation is diffeomorphic to $\mathbb{R}$ and outside a compact set of $Q$, each leaf coincide with a part of one of parallel straight lines of a given direction.

The proof of proposition 1 is carried out by constructing a 2-dimensional foliation $\{M_t\}_{t \in \mathbb{R}}$ on $Q$ such that each leaf is a Lagrangian diffeomorphic to $\mathbb{R}^2$, $M_0 = L$ and $M_t$ are embedded standard $\mathbb{R}^2$s for $t < -1, t > 0$. It can be done using the characteristic foliation. As for the proof of proposition 2, we need the filling with holomorphic disks technique. Namely, one first takes a 2-dimensional foliation whose leaves consist of trajectories of the
characteristics foliation which intersect at $-\infty$ a line, parallel to a given direction. The constructed foliation is not flat at $+\infty$, but can be flatten via an appropriate Hamiltonian isotopy. We first fix some notations. Let $(u, v, x, y)$ be the coordinate for $\mathbb{R}^4$, $Q_0$ be the hyperplane \{v = 0\}, $L_0$ be the standard Lagrangian plane \{(u, 0, 0, y)\} and $\Sigma_0 = L_0 \cap C$. Let $C = \{(x-u)^2 + y^2 \leq 1\}$ and $K = \{(x-u)^2 + y^2 \leq 1/2\}$ be two cylinders contained in $\mathbb{R}^3 = \{(u, x, y)\}$. There is a convex domain $V_\delta$ defined by $V_\delta = \{-\delta \phi(u, x, y) < v < \delta \phi(u, x, y)\}$ where $\delta > 0$ and $\phi(u, x, y) = 1 - (x-y)^2 - y^2$. It satisfies $\partial V_\delta \supset \partial C$. Then, by a suitable dilatation, we can suppose that our Lagrangian knot $L$ coincides with $L_0$ outside of $K$ and is contained in $V_\delta$. We now isotope $C \cap \{-1 \leq u \leq 1\}$ to a set like the figure below.

We denote this map by $\Phi$. This can be done so that the images of the disks $\{t\} \times D^2$ are symplectic. We call the image of the discs by $N$. Then, there is a symplectic embedding $\chi$ from a neighbourhood of $N$ to $V$ such that $\chi(\Sigma_0) = V \cap L$ and $\chi$ is the identity outside $K$. We can define an almost complex structure $J$ on $\mathbb{R}^4$ tamed by $\omega_0$ such that the image of the disks $\{t\} \times D^2$ by the map $\chi \circ \Phi$ are $J$-holomorphic and flat near $\partial V$ and outside of a compact set in $\mathbb{R}^4$. Then, since $\partial C$ is contained in a pseudo convex boundary, examining the Maslov class of the generator of the first homology group of $\partial C$, we see that we can extend $\chi \circ \Phi$ to the whole cylinder $C$ in a way that images of the discs $\{t\} \times D^2$, $t \in \mathbb{R}$ are $J$-holomorphic and for $|t|$ larger than 1, the map on $\{t\} \times D^2$ is the identity. If we call this map $F$, then $Q = (Q_0 - C \cap \{-1 \leq u \leq 1\}) \cup F(\{-1 \leq u \leq 1\})$ is the required simple hypersurface.
2 Invariants of $S^2$-knots in $\mathbb{R}^4$ via symplectic geometry

Let $f : S^2 \hookrightarrow \mathbb{R}^4$ be an embedding, and $\alpha := [f]$ the isotopy class of $f$. Let us denote by $D(a, b)$ the polydisc $\{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1| \leq a, |z_2| \leq b\}$.

We say that the class $\alpha$ admits a $(a, b)$-realization for $a > 1$, $b > 0$ if $\alpha$ can be represented by an embedded sphere $S = \Delta \cup D \subset \mathbb{R}^4$ where $D = \{|z_1| \leq 1, z_2 = b\}$ and $\Delta$ is a 2-disk satisfying the following properties: $(\Delta, \partial \Delta) \subset (\mathbb{C}^2 \setminus \text{Int}$\(D(a, b), \partial D(a, b))$ intersects $\partial D(a, b)$ transversely along the circle $\partial \Delta = \{|z_1| = 1, z_2 = b\}$, and $\omega|_{\Delta} > 0$.

![Diagram](image)

**Lemma 2.1.** For any isotopy class $\alpha$ of embeddings $S^2 \hookrightarrow \mathbb{R}^4$, there exist $a > 1$, $b > 0$ such that $\alpha$ admits a $(a, b)$-realization.

The following theorem asserts that a symplectic 2-disc cannot be knotted not only in the half-space but even in the complement of a sufficiently large polydisc.

**Theorem 2.2.** If $[f]$ admits a $(3, 2)$-realization, then it is trivial.
We sketch the proof of this theorem. Set the following notations:

\[
\Omega = \{ x_2 \leq \varepsilon |z_1|^2 /(1 - \varepsilon)^2 \} \text{ where } z_2 = x_2 + iy_2 \\
\Sigma = \partial \Omega \cap D(a, b) \\
A_{c,d} = \{ |z_1| \leq c, |y_2| \leq d \} \\
\Sigma_{c,d} = A_{c,d} \cap \Sigma \\
G = D(a, b) \setminus (A_{1,\varepsilon} \cap \Omega) \\
S = \{ y_2 = 0, |z_1| \leq 1 - \varepsilon \} \cap \Sigma.
\]

Deform $\Delta$ into the following form, and denote the resulting disc by $\tilde{\Delta}$.

![Diagram of $\tilde{\Delta}$ and $\Sigma$](image)

The disc $\tilde{\Delta}$ intersects $\Sigma$ transversely along $\partial \tilde{\Delta} = \{ z_1 | = 1 - \varepsilon, z_2 = \varepsilon \}$. We can assume that $\omega|_{\tilde{\Delta}} > 0$ and $\tilde{\Delta}$ is holomorphic near $\partial \tilde{\Delta}$ (with respect to the standard complex structure on $\mathbb{C}^2$). Let us choose an almost complex structure $J$ on $\mathbb{R}^4$ such that:

- $J$ is tamed by $\omega$.
- $J$ is standard on $G$, near $\Sigma$ and at infinity.
- $\tilde{\Delta}$ is $J$-holomorphic.

Then, the theorem can be deduced from the following:

**Lemma 2.3.** The pair $(S, \tilde{\Delta})$ can be filled with $J$-holomorphic discs.

Let $q \in S$ be the elliptic point of $S$, and $\{ \Delta_t \}_t$ be a Bishop family of $J$-holomorphic disks developing from $q$. To show the lemma, it is sufficient
to prove that \( \text{Int}\Delta_t \cap \Sigma_{1,\epsilon} = \emptyset \). We want to eliminate the following case.

Notice that no disk can be tangent to a strictly pseudoconvex hypersurface from a convex side.

Suppose that some disc \( \Delta_t \) is tangent to \( \Sigma_{2,1} \) at a point \( p \) from the concave side. Observe that for any \( t \) we have

\[
\int_{\Delta_t} \omega < \int_S \omega = \pi(1 - \epsilon)^2 \quad \text{by Stokes’ theorem.}
\]

On the other hand, holomorphic curves have the following monotonicity property:

**Lemma 2.4.** Let \( C \) be a properly embedded holomorphic curve in the open ball \( B \) of radius \( r \) in \( \mathbb{C}^n \). Suppose that \( C \) contains the center of \( B \). Then \( \text{Area} \ C \geq \pi r^2 \).

We apply this lemma to \( C = \Delta_t, \ B = B_{1-\epsilon}(p) \). By assumption, \( B \cap \Delta_t \) is contained in \( G \), and \( J \) is standard on \( G \). Therefore

\[
\pi(1 - \epsilon)^2 \leq \text{Area}(\Delta_t \cap B) \leq \int_{\Delta_t} \omega.
\]

This contradicts the inequality \( \int_{\Delta_t} \omega < \pi(1 - \epsilon)^2 \).
3 Legendrian linking problem

Let $V$ be a manifold and $PT^*(V)$ the projetivized cotangent bundle, i.e., the space of all tangent hyperplanes in $T(V)$. The manifold $PT^*(V)$ has a contact structure $\eta \subset T(PT^*(V))$ such that lift of each hypersurface $W \subset V$ to $PT^*(V)$, denote by $L_W \subset PT^*(V)$, is a Legendrian submanifold for $\eta$. Moreover, let $W \subset V$ be a smooth submanifold of positive codimension. Put

$$L_W := \left\{ (w, H_w) \in PT^*(V) \mid T_w(W) \subset H_w \subset T_w(V) \right\}.$$ 

Then $L_W$ is also a Legendrian submanifold for $\eta$. Let $W_1$ and $W_2$ be submanifolds properly immersed into $V$ such that they intersect transversely. Here "properly" means "being closed as a subset in $V". Then $L_{W_1} \cap L_{W_2} = \emptyset$. Let $L_1(t)$ and $L_2(t)$ be compact supported contact isotopies of $L_{W_1}$ and $L_{W_2}$ such that $L_1(1)$ and $L_2(1)$ have disjoint projections to $V$. We denote by $\#(L_1(t) \times_{\text{reg}} L_2(t))$ the minimal number of crossings between all (compact supported) contact isotopies $L_1(t)$ and $L_2(t)$ which intersect transeversely and move $L_1(0)$ and $L_2(0)$ to $L_1(1)$ and $L_2(1)$.

**Theorem 3.1.** Suppose $W_1 \cap W_2$ is compact, then we have

$$\#(L_1(t) \times_{\text{reg}} L_2(t)) \geq \frac{1}{2} \text{rank } H_*(W_1 \cap W_2),$$

where $W_1 \cap W_2$ denote the set $\{(w_1, w_2) \in W_1 \times W_2 \mid w_1 = w_2\}$.

Let $V = W \times \mathbb{R}$, $W_1 \subset W \times \mathbb{R}$, and the projection $W_1 \to W$ has non-zero degree. Here we assume $W$ and $W_1$ connected orientable manifolds of the same dimension. One can drop the orientability condition if works with coefficient $\mathbb{Z}_2$. Moreover let $W_2 \subset W$ be a compact submanifold which lies on the left of $W_1$, i.e., $W_1 \cap \{(w_2, t_2 + t) \in W \times \mathbb{R} \mid (w_2, t_2) \in W_2, t \leq 0\} = \emptyset$.

**Theorem 3.2.** If the projection of $L_2(1)$ to $V$ lies on the right of the projection $L_1(1)$, then we have

$$\#(L_1(t) \times_{\text{reg}} L_2(t)) \geq \text{rank } H^*(W_2).$$

The proofs of these theorems rely on the generating functions and the stable Morse theory.
Postscript. In this lecture note we could note only a part of Eliashberg’s talk. He mentioned many other topics on symplectic field theory (SFT), symplectic cobordisms, compactness properties, generalized Viterbo’s theorem, Lagrangian skeletons, Lagrangian tori in $\mathbb{R}^4$ and so on.

References

