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THREE LECTURES ON NEWTON POLYHEDRA

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LECTURE 1. INTRODUCTION TO THE THEORY

1. The Newton polyhedra.

The Newton polyhedron of a polynomial which depends on several variables is the convex hull of the powers of the monomials appearing in the polynomial with nonzero coefficients. The Newton polyhedron generalizes the notion of degree and plays an analogous role. It is well known that the number of complex roots of a system of n equations of identical degree m in n unknowns is the same for nearly all values of the coefficients, and is equal to m^n (Bezout's theorem). Similarly, the number of complex roots of a system of n equations in n unknowns with the same Newton polyhedron is the same for nearly all values of the coefficients and is equal to the volume of the Newton polyhedron, multiplied by $n!$ (Kushnirenko's theorem, see 3.1.1).

The level line of a polynomial in two complex variables is a Riemann surface. For nearly all polynomials of fixed degree n, the topology of this surface (the number of handles g) is expressed in terms of its degree, and does not depend on the values of the coefficients of the polynomial: $g=\dfrac{(n-l)(n-2)}{2}$. In the more general case, where instead of polynomials of fixed degree we consider polynomials with fixed Newton polyhedra, all the discrete characteristics of the manifold of zeros of the polynomial (or several polynomials) are expressed in terms of the geometry of the Newton polyhedra. Among these discrete characteristics are the number of solutions of a system of n equations in n unknowns, the Euler characteristic, the arithmetic and geometric genus of complete inter- sections, and the Hodge number of a mixed Hodge structure on the cohomologies of complete intersections.

The Newton polyhedron is defined not only for polynomials but also for germs of analytic functions. For germs of analytic functions in general position, with given Newton polyhedra, one can calculate the multiplicity of the zero solution of a system of analytic equations, the Milnor number and ξ function of the monodromy operator, the asymptotics of oscillatory integrals, the Hodge number of the mixed Hodge structure in vanishing cohomologies, and in the two-dimensional and multidimensional quasihomogeneous cases, one can calculate the modality of the germ of the function.

In the answers one meets quantities characterizing both the sizes of the polyhedra (volume, number of integer points lying inside the polyhedron) as well as their combinatorics (the number of faces of various dimensions, numerical characteristics of their contacts).

In terms of the Newton polyhedron one can construct explicitly the compactification of complete intersections, and the resolution of singularities by means of a suitable toric manifold.

Thus, the Newton polyhedra connect algebraic geometry and the theory of singularities to the geometry of convex polyhedra. This connection is useful in both directions. On the one hand, explicit answers are given to problems of algebra and the theory of singularities in terms of the geometry of polyhedra. We note in this connection that even the volume of the convex hull of a system of points is a very complicated function of their coordinates. Therefore, the formulation of answers in numerical terms is so opaque that without knowing their geometric interpretation no progress is possible. On the other hand, algebraic theorems of general character (the Hodge theorem on the index of an algebraic surface, the Riemann–Roch theorem) give significant information about the geometry of polyhedra. In this way one obtains, for example, a simple proof of the Aleksandrov–Fenchel inequalities in the geometry of convex bodies.

The Newton polyhedra are also met in the theory of numbers in real analysis, in the geometry of exponential sums, in the theory of differential equations. In this lecture we present formulations of some theorems about Newton polyhedra.

2. The number of roofs of a system of equations with a given Newton polyhedron.

According to Bezout's theorem, the number of nonzero roots of the equation $f(z) = 0$ is equal to the difference between the highest and the lowest powers of the monomials appearing in the polynomial f. This difference is the volume of the Newton polyhedron (in the present case, the length of a segment). In the following two paragraphs we present the generalizations of this theorem to the case of arbitrary Newton polyhedra.

Let us start with definitions. A *monomial* in n complex variables is a product of the coordinates to integer (possibly negative) powers. Each monomial is associated with its *degree*, an integer vector, lying in n-dimensional real space, whose components are equal to the powers with which the coordinate functions enter in the monomial. A *Laurent polynomial* is a linear combination of monomials. The *support of the Laurent polynomial* is the set of powers of monomials entering in the Laurent polynomial with nonzero coefficients. (The Laurent polynomial is an ordinary polynomial if its support lies in the positive octant.) The *Newton polyhedron* of the Laurent polynomial is the convex hull of its support. It is much more convenient to consider the Laurent polynomials not in $\mathbb{C}^n$ but in the $(\mathbb{C}\setminus 0)^n$-dimensional complex space, from which all the coordinate planes have been eliminated. With each face Γ of the Newton polyhedron of the Laurent polynomial f we associate a new Laurent polynomial, which is called the *restriction of the polynomial to the face*, denoted by f^Γ and defined as follows: only those monomials appear in f^Γ that have powers lying in the face Γ, with the same coefficients that they have in f.

Now let us consider a system of n Laurent equations $f_1 = \cdots = f_n = 0$ in $(\mathbb{C}\setminus 0)^n$ with a common Newton polyhedron. The restricted system, $f_1^\Gamma = \cdots = f_n^\Gamma = 0$ corresponding to each face Γ of the polyhedron. The restricted system actually depends on a smaller number of variables, and in the case of general position is incompatible in $(\mathbb{C}\setminus 0)^n$. We say that a system of n equations in n unknowns with a common Newton polyhedron is *regular* if all the restrictions of this system are incompatible in $(\mathbb{C}\setminus 0)^n$. The following theorem of Kushnirenko holds: The number of solutions in $(\mathbb{C}\setminus 0)^n$, counted with their multiplicities, of a regular system of n equations in n unknowns with a common Newton polyhedron is equal to the volume of the Newton polyhedron multiplied by $n!$.

Example: The Newton polyhedron of the polynomial of degree m in n unknowns is the simplex $0 \leq x_1, \ldots, 0 \leq x_n, \sum x_, \leq m$ (we assume that the polynomial contains all monomials of degree $\leq m$). The volume of such a simplex is $m^n/n!$. The number of roots of the total system of n equations of degree m in n unknowns, according to Kushnirenko's theorem, is equal to m^n. This answer agrees with Bezout's theorem. If the polynomial does not contain all monomials of degree less than or equal to m, then the Newton polyhedron can be smaller than the simplex, so the number of solutions, calculated from Kushnirenko's theorem, can be smaller than the number m^n calculated from Bezout's theorem. Because of the absence of the monomials, certain infinitely distant points may be roots of the system of equations. Bezout's theorem, which calculates the number of roots of the system in projective space, takes into account these parasitic roots, while Kushnirenko's theorem does not.

3. How does one find the number of solutions of a system of n equations in n unknowns with different Newton polyhedra.

Here is the answer to this question for a system in general position with fixed Newton polyhedra: the number of solutions not lying on the coordinate planes is equal to the mixed volume of the Newton polyhedra, multiplied by $n!$ Below we shall give the definition of mixed volume and describe explicitly the conditions for degeneracy.

The *Minkowski sum* of two subsets of a linear space is the set of sums of all pairs of vectors, in which one vector of the pair lies in one subset and the second vector in the other. The product of a subset and a number can be determined in a similar manner. The Minkowski sum of convex bodies (convex polyhedra, convex polyhedra with vertices at integer points) is a convex body (convex polyhedron, convex polyhedron with vertices at integer points). The following theorem holds:

Minkowski theorem. *The volume of a body which is a linear combination with positive coefficients of fixed convex bodies lying in $\mathbb{R}^n$ is a homogeneous polynomial of degree n in the coefficients of the linear combination.*

Definition. The *mixed volume* $V(\Delta_1, \ldots, \Delta_n)$ *of the convex bodies* $\Delta_1, \ldots, \Delta_n$ in $\mathbb{R}^n$ is the coefficient in the polynomial $V(\lambda_1\Delta_1 + \ldots, \lambda_n\Delta_n)$ of $\lambda_1 \times \cdots \times \lambda_n$ divided by $n!$ (here $V(\Delta)$ is the volume of the body Δ).

The mixed volume of n identical bodies is equal to the volume of any one of them. The mixed volume of n bodies is expressed in terms of the usual volumes of their sums in the same way as the product of n numbers is expressed in terms of the n-th powers of their sums. For example, for $n = 2$,

$$ab = \frac{1}{2}[(a+b)^2 - a^2 - b^2]$$

$$V(\Delta_1, \Delta_2) = \frac{1}{2}[V(\Delta_1 + \Delta_2) - V(\Delta_1) - V(\Delta_2)]$$

Similarly, for $n = 3$,

$$V(\Delta_1, \Delta_2, \Delta_3) = \frac{1}{3!}\left[V(\Delta_1 + \Delta_2 + \Delta_3) - \sum_{i<j} V(\Delta_i + \Delta_j) + \sum V(\Delta_i)\right].$$

Example: Suppose that Δ_1 is the rectangle $0 \le x \le a$, $0 \le y \le b$ and Δ_2 is the rectangle $0 \le x \le c$, $0 \le y \le d0$. The Minkowski sum $\Delta_1 + \Delta_2$ is the rectangle $0 \le x \le a + c$, $0 \le y \le b + d$. The mixed volume $V(\Delta_1, \Delta_2)$ is equal to $ad + be$. The number $ad + be$ is the permanent of the matrix $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$ (the definition of the permanent differs from that of the determinant only in that all the terms in the permanent have a plus sign). In the multidimensional case the mixed volume of n parallelepipeds with sides parallel to the coordinate axes is also equal to the permanent of the corresponding matrix.

Let us consider a system of n Laurent equations $f_1 = \cdots = f_n = 0$ with Newton polyhedra $\Delta_1, \ldots, \Delta_n$. Below we define the regularity condition for such systems.

Bernshtein's theorem. *The number of solutions in* $(\mathbb{C} \setminus 0)^n$ *(which take account of the multiplicity) of a regular system of n equations in n unknowns, is equal to the mixed volume of the Newton polyhedra of the equations of the system, multiplied by* $n!$.

Example: The number of roots of a general system of polynomial equations, in which the i-th variable enters in the j-th equation with a power no higher than a_{ij} is equal to the permanent of the matrix (a_{ij}), multiplied by $n!$.

Kushnirenko's theorem coincides with Bernshtein's theorem for equations with identical Newton polyhedra. We now proceed to the definition of a *regular system of equations*. We first define the truncations of a system associated with a function ξ. We take an arbitrary linear function ξ on the space $\mathbb{R}^n$ in which the Newton polyhedra lie. We denote by f^ξ the restriction of the Laurent polynomial f to that face of its Newton polyhedron on which the linear function ξ takes its maximum value. We associate the restricted system $f_\xi^1 = \cdots = f_n^\xi = 0$ with the system of equations $f_1 = \cdots = f_n = 0$ and the linear function ξ. For a nonzero function ξ the restricted system actually depends on a smaller number of variables. Therefore, in the case of general position such a system is inconsistent in $(\mathbb{C} \setminus 0)^n$. A given system of equations has only a finite number of restricted systems (if the polyhedra of all the equations coincide, then the truncations correspond to the faces of the common polyhedron). A system of n equations in n unknowns is said to be regular if its restrictions for all nonzero functions ξ are inconsistent in $(\mathbb{C} \setminus 0)^n$. It is just such a system to which Bernshtein's theorem is applicable.

We note that if for each nonzero linear function ξ, the maximum in one of the polyhedra is attained at a vertex then the regularity conditions are automatically satisfied. [In this case the truncated system contains an equation which is contained in a monomial set equal to zero; this equation has no solutions in $(\mathbb{C} \setminus 0)^n$.] For example, in the case of $n = 2$, the regularity conditions are satisfied automatically, if the two Newton polyhedra on the real plane have no parallel sides.

4. Complete intersections.

Consider in $(\mathbb{C} \setminus 0)^n$ a system of k Laurent equations $f_1 = \cdots = f_k = 0$ with Newton polyhedra $\Delta_1, \ldots, \Delta_k$

Assume that the system is nondegenerate. It turns out that the nondegeneracy condition holds for almost all Laurent polynomials with fixed Newton polyhedra. Many discrete invariants of the set of solutions are identical and are expressed in terms of the polyhedra.

Theorem. *The Euler characteristic of the nondegenerate complete intersection $f_1 = \cdots = f_k = 0$ in $(\mathbb{C}\setminus 0)^n$, $(k \leq n)$, with the Newton polyhedra $\Delta_1, \ldots \Delta_k$ is equal to $(-1)^{n-k} n! \sum V(\Delta_1, \ldots, \Delta_k, \Delta_{i_1}, \ldots \Delta_{i_{n-k}})$, where the sum is taken over all sets $1 = i_1 \leq \cdots \leq i_{n-k} \leq k$.*

We note one special case of this theorem.

Corollary. *The Euler characteristic of a hypersurface in $(\mathbb{C}\setminus 0)^n$ defined by non degenerate equation with fixed Newton polyhedron is equal to the volume of the Newton polyhedron multiplied by $(-1)^{n-1} n!$.*

Another consequence of this theorem is Bernshtein's theorem (see par.3.1.2): for $k = n$, the nondegenerate complete intersections consist of points, and the Euler characteristic is equal to the number of points (strictly speaking, Bernshtein's theorem is slightly stronger than this corollary, since it is applicable to a degenerate regular system).

5. Genus of complete intersections.

The formulas given below for the genus of complete intersections are generalizations of the following formulas for Abelian and elliptic integrals. Let us consider a Riemann surface $y^2 = P_3(x)$ (the complex phase curve of motion of a point in a field with a cubic potential). This Riemann surface, which is diffeomorphic to the torus, exhibits a single, everywhere holomorphic, differential form dx/y (the differential of the time of motion along the phase curve). In the case of a potential of degree n, the curve $y^2 = P_n(x)$ is diffeomorphic to the sphere with g handles, where g is connected with n either by the formula $n = 2g + 1$, or the formula $n = 2g + 2$ (depending on the parity of n). The basis of holomorphic forms in this case is given by g forms of the type $x^m dx/y$, $0 \leq m < g$. The number g is the genus of the curve. The Newton polyhedron of the curve $y^2 == P_n(x)$ is a triangle with vertices $(0,0)$, $(0,2)$, and $(n,0)$. There are exactly g points with integer coordinates strictly inside this triangle. In terms of these points, one can give a basis for the space of holomorphic forms: the point (l, a), which lies inside the triangle, corresponds to the form $x^{a-1} dx/y$. We give below the generalization of this procedure for constructing a basis of holomorphic forms for the multi-dimensional case.

The nondegenerate complete intersections $f_1 = \cdots = f_k = 0$, where the f_i, are Laurent polynomials, are smooth algebraic affine manifolds. In the cohomologies of such manifolds there is an additional structure, namely, the mixed Hodge structure. The discrete invariants of such a structure are calculated in terms of the Newton polyhedra. We consider only the calculation of the arithmetic and geometric genus which are invariants of this kind.

We first recall some definitions and general statements. Suppose that Y is a nonsingular (possibly noncompact) algebraic manifold. The set of holomorphic p-forms on Y, which extend holomorphically to any nonsingular algebraic compactification, is automatically closed, and it realizes the zero class of homology of the manifold Y only if it is equal to zero. Forms of this kind form a subspace in the p-dimensional cohomologies of the manifold Y. We denote the dimension of this subspace by $h^{p,0}(Y)$. The *arithmetic genus* of the manifold Y is the alternating sum $\sum(-1)^p h^{p,0}(Y)$ of the numbers $h^{p,0}(Y)$. The geometric genus of the manifold Y is the number $h^{n,0}(Y)$, where n is the complex dimension of the manifold Y.

Now we turn to the Newton polyhedron. We shall use the *characteristic* $B(\Delta)$ *of integer polyhedra.* Here is its definition. Suppose that Δ is a q-dimensional polyhedron with vertices at integer points, lying in $\mathbb{R}^n$ and $\mathbb{R}^q$ is a q-dimensional affine subspace, containing Δ. The number $B(\Delta)$ is defined as the number of integer points lying strictly within he polyhedron Δ (in the geometry of the subspace $\mathbb{R}^q$), multiplied by $(-1)^q$.

Theorem. *The arithmetic genus of the nondegenerate complete intersection* $f_1 = \cdots = f_k = 0$ *in* $(\mathbb{C}\setminus 0)^n$, $(k \le n)$, *with the Newton polyhedra* $\Delta_1, \dots, \Delta_k$ *is*

$$1 - \sum B(\Delta_i) + \sum_{j>i} B(\Delta_i + \Delta_j) - \cdots + (-1)^k B(\Delta_1 + \cdots + \Delta_k).$$

Corollary. *The geometric genus of the nondegenerate complete intersection (for* $k < n$*) with the polyhedra of full dimensionality is*

$$(-1)^{n-k}\left((-\sum B(\Delta_i) + \sum_{j>i} B(\Delta_i + \Delta_j) - \cdots + (-1)^k B(\Delta_1 + \cdots + \Delta_k)\right).$$

We now give a complete description of the holomorphic forms of highest dimension which could be extended holomorphically to the compactification, for the case of a nondegenerate hypersurface $f = 0$ in $(\mathbb{C}\setminus 0)^n$ with the Newton polyhedron Δ of complete dimension. For each integer point, lying strictly within the Newton polyhedron Δ, we denote by ω_a the n-form on the hypersurface $f = 0$, defined by the formula

$$\omega_a = z_1^{a_1} \times \cdots \times z_n^{a_n} \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_n}{z_n} / df; \quad a = a_1, \dots, a_n.$$

Theorem. *The forms* ω_a *lie in the space of forms extendible holomorphically on the compactification of the manifold* $f = 0$*; they are linearly independent and generate that space. In particular, the geometric genus of the hypersurface is equal to* $|B(\Delta)|$.

Example: Consider a curve in the plane determined by the equation $y^2 = P_n(x)$. The interior monomials for the Newton polyhedron of this curve have the form yx^a, where $1 \le a < n/2$. The forms ω_a corresponding to the monomials coincide [on the curve $y^2 = P_n(x)$] with the forms $x^{a-1}dx/2y$. Thus, for the curve $y^2 = P_n(x)$ we get the usual description of all Abelian differentials.

6. Affine case.

It is customary to consider systems of equations not in $(\mathbb{C}\setminus 0)^n$, but in the usual complex space $\mathbb{C}^n$. Calculations with Newton polyhedra also are carried out in this situation. The answers here are more messy. As an example we consider the calculation of the Euler characteristic of a hypersurface in $\mathbb{C}^n$. Suppose that f is a polynomial in n complex variables with a nonzero free term and Newton polyhedron Δ. We introduce the following notation: Δ^I is the intersection of the Newton polyhedron Δ with the coordinate plane R^I in $\mathbb{R}^n$, $d(I)$ is the dimension of this plane, and $V(\Delta^I)$ is the $d(I)$-dimensional volume of the polyhedron Δ^I.

Theorem. *Let f be a nondegenerate polynomial with the Newton polyhedron Δ which has a nonzero free term. Then $f = 0$ is a nonsingular hypersurface in $\mathbb{C}^n$, intersecting transversally all the coordinate planes in $\mathbb{C}^n$. The Euler characteristic of this hypersurface is equal to $\sum(-1)^{d(I)-1}d(I)!V(\Delta)$, where the summation runs over all intersections of the Newton polyhedron Δ^I with the (nonzero) coordinate planes.*

This theorem follows from the calculation of the Euler characteristic in $(\mathbb{C}\setminus 0)^n$ and from the additivity of the Euler characteristic. We also note that the formula given for a hypersurface is analogous to the formula for the Milnor number.

LECTURE 2. POLYHEDRA AND INDEX OF A POLYNOMIAL FIELD

In this lecture we will present an estimation for the index of a polynomial vector field with components of fixed degrees. We also will give examples showing that this estimation is best possible (see [1], [2]). The proof of the estimates in the nondegenerate case is closely related to the proof in Petrovskii and Oleinik [3], where the Euler characteristics of some algebraic sets are estimated. In addition, the proof is closely related to the proof in the article by Arnol'd [4] and clarifies the connection between these two methods of argument. As in [4], the index is associated with the signature of a certain quadratic form (see also [5, 6]). As in [3], a key factor in the proof is the use of the Euler–Jacobs formula.

1. Notation.

Let $V = P_1, \ldots, P_n$ be a vector field in $\mathbb{R}^n$ with polynomial components P_i. We denote by ind the sum of the indices of all singular points of the field V in $\mathbb{R}^n$. We will say that the field V has degree not equal to $m = m_1, \ldots, m_n$, provided the degrees of all the polynomials P_i, $i = 1, \ldots, n$, are equal to m_i. We will say that the field V is nondegenerate if the real singular points of the field V have multiplicity one and "lie in the finite part of the space $\mathbb{R}^n$". (We write the last condition out in more detail. Let the P_i be homogeneous polynomials of degree m_i in the variables $x_0, x_1, \ldots, x_n$, such that $\overline{P}_i(1, x_1, \ldots, x_n) \equiv P_i(x_1, \ldots, x_n)$. The last condition means that the system $\overline{P}_1 = \cdots = \overline{P}_n = x_0 = 0$ has only the trivial solution $x_0 = x_1 = \cdots = x_n = 0$.)

We introduce some notation.

$\Delta(m)$ is the parallelepiped in $\mathbb{R}^n$ defined by the inequalities $0 \le y_1 \le m_1 - 1, \ldots, 0 \le y_n \le m_n - 1$

$\mu = m_1 \ldots m_n$ is the number of integer points in the parallelepiped $\Delta(m)$.

$\Pi(m)$ is the number of integer points in the central section $y_1 + \cdots + y_n = \frac{1}{2}(m_1 + \cdots + m_n - n)$ of the parallelepiped $\Delta(m)$.

2. Statement of the result.

Theorem. *If V is a nondegenerate field of degree m, the number satisfies the inequality $|a| \le \Pi(m)$ and the congruence $a \equiv \mu \mod 2$. Conversely, for every number a satisfying these conditions, there exists a nondegenerate field V of degree m for which* ind $= a$.

Corollary. *The index* ind *of an isolated singular point of the field $V = P_1, \ldots, P_n$ with homogeneous components of degree $m = m_1, \ldots, m_n$ satisfies the inequality* $|\text{ind}| \le \Pi(m)$ *and the congruence* ind $\equiv \mu \mod 2$. *The number* ind *is not subject to any other restrictions.*

3. Projective transformations.

Let Γ be a hyperplane in $\mathbb{R}^n$ defined by a linear inhomogeneous equation $l(x) = l_1(x) + l_0 = 0$. We construct a projective transformation $g\colon \mathbb{R}^n \to \mathbb{R}^n$, taking the hyperplane Γ into the hyperplane at infinity; $g(x) = [1/l(x)]A(x)$, where $A(x)$ is an affine transformation. A projective transformation of a field V with components $P_1, \dots, P_n$ of degrees $m_1, \dots, m_n$ is a field $\tilde{V} = \tilde{P}_1, \dots, \tilde{P}_n$, where $\tilde{P}_i(x) = l^{m_i}(x)P_i(g(x))$ for $i = 1, \dots, n$. If a is a singular point of the field V, then the point $\tilde{a} = g^{-1}(a)$ is singular for $\tilde{V}$. The Jacobian $\det \dfrac{\partial g}{\partial x}$ is defined outside the hyperplane Γ and vanishes nowhere. We say that a transformation $x \to g(x) = [1/l(x)]A(x)$ is positive if its Jacobian is positive in the region $l(x) > 0$. For odd n, the space $\mathbb{R}P^n$ is orientable and positive transformations coincide with orientation-preserving transformations. In the general case, positive transformations correspond precisely to linear transformations of $\mathbb{R}^{n+1}$ with positive determinant.

We will be interested in how the index, of the field V changes under projective transformations. The global characteristic ind is obtained by summing the corresponding local characteristics over the set X of singular points of V, $\operatorname{ind} = \sum_{a \in X} \operatorname{ind}(V)_a$.

Given a field V and a projective transformation $x \to g(x)$, we write $X(g)$ for the set of singular points a of V for which $\tilde{a} = g^{-1}(a)$ is defined. The index of the field V is called projectively invariant if for every positive $g = [1/l(x)]A(x)$ and $a \in X(g)$ the equality $\operatorname{ind}(V)_a = \operatorname{ind}(\tilde{V})_{\tilde{a}}$ holds. The index is said to be projectively antiinvariant if $\operatorname{ind}(V)_a = \operatorname{sign} l(\tilde{a})\operatorname{ind}(\tilde{V})_{\tilde{a}}$.

A simple verification proves the following assertion.

Assertion. *1. The index is projectively invariant if* $m_1 + \dots + m_n \neq n \mod 2$.
2. The index is projectively antiinvariant if $m_1 + \dots + m_n = n \mod 2$.

We write Γ_∞ for the image of the plane at infinity under a projective transformation. For $g(x) = [1/l(x)]A(x)$, where $A(x)$ is a linear transformation and $l(x) = -l_1(x) + l_0$ the equation of the hyperplane Γ_∞ has the form $l_1(A^{-1}(x)) = 1$. We associate with the hyperplane $l_1(x) = p$, where $p > 0$, the transformation $x \to (px)/(l_1(x) + l)$, for which this plane is Γ_∞. The invariant characteristics of the singular points of the field are preserved under a projective transformation. The antiinvariant characteristics remain unchanged for the singular points lying in one of the halfspaces bounded by Γ_∞ and change sign for the singular point lying in the other half space.

4. Examples.

In the construction of our examples, a principal role will be played by the simplest field $V(m)$ of degree $m = m_1, \dots, m_n$ with components $P_i = \prod_{0 \le k \le m_i - 1}(x_i - k)$, $i = 1, \dots, n$. We note that all the singular points of $V(m)$ coincide with the integral points of the polyhedron $\Delta(m)$ defined by the inequalities $0 \le x_i \le m_i - 1$, $i = 1, \dots, n$. The signs of the Jacobian at the singular points alternate in "chessboard order". Moreover, at the singular points lying on the single section $\sum x_i = k$, the sign of the Jacobian is constant. Upon passing to the next section $\sum x_i = k + 1$, this sign is replaced by the opposite sign. In this section we often encounter the number $\frac{1}{2}\sum(m_i - 1)$, which we denote by ρ.

Consider the case $m_0 + \cdots + m_n \not\equiv n \mod 2$. In this case $\Pi(m) = 0$, and by Theorem 1 every nondegenerate field V has zero index.

We consider the case $m_1 + \cdots + m_n \equiv n \mod 2$. In this case, the characteristic ind is antiinvariant. The central section $\sum x_i = \rho$ of the rectangular box $\Delta(m)$ contains exact $\Pi(m)$ singular points of the field $V(m)$. The sections $\sum x_i = \rho - k$ and $\sum x_i = \rho + k$ contain the same number of singular points with Jacobians of the same sign. We carry out a projective transformation $x \to g(x)$ for which the plane Γ_∞ has the equation $\sum x_i = \rho - 1/2$, e.g., the transformation $x \to (\rho - 1/2)x/(\sum x_i + 1)$. The sections $\sum x_i = \rho - k$ and $\sum x_i = \rho - k$ for $k > 0$ lie in different half spaces bounded by Γ_∞. The indices of the inverse images of the singular points of the field $V(m)$ lying in these sections cancel one another. Therefore, the absolute value of the index of the field $\tilde{V}$ is equal to the number of singular points of $V(m)$ lying on the section $\sum x_i = \rho$, i.e., it is equal to $\Pi(m)$. Here is an explicit formula for the component $\tilde{P}_i$ of the field $\tilde{V}$: $P_i(x) = \prod_{0 \le k \le m_i} [(p - \frac{1}{2})x_i - k(\sum x_i + 1)]$.

The index of the field $\tilde{V}$ does not change if the position of the plane Γ_∞ is perturbed slightly. We define Γ_∞ by the equation $\sum a_i x_i = t$, where the a_i are scalars close to unity which are independent over the rationals, and t is close to $\rho - 1/2$. We now begin to let t get bigger. This will not change the index of the field $\tilde{V}$ until Γ_∞ passes through a singular point of $V(m)$ from the direction of the central section $\sum x_i = \rho$. When this occurs, $\tilde{V}$ becomes degenerate and its index changes by one. If the number t is increased a bit more, the field $\tilde{V}$ again becomes nondegenerate and its index again changes by one. Continuing the motion of the plane Γ_∞, we obtain examples of nondegenerate fields $\tilde{V}$ of degree m with any index satisfying the conditions $|\text{ind}| \le \Pi(m)$, $\text{ind} \equiv \mu \mod 2$. The leading homogeneous components of the field $\tilde{V}$ form a field with an isolated singular point at zero with the same index.

5. Signature and index.

5.1. A Finite Set with Involution. Let A be a finite set containing μ elements, $\tau\colon A \to A$ an involution of A, and let X be the set of fixed points of τ. We consider the algebra L_τ over the field R consisting of all complex value functions on A for which $f\tau = \tilde{f}$. Let φ be a fixed function in L_τ which is nowhere zero. The number of points of the set X at which φ is positive is denoted by φ^+, the number of points at which it is negative, by φ^-. We consider the bilinear form ω_φ on L_τ defined by $\omega_\varphi(f, g) = \sum_{a \in A} \varphi(a) f(a) g(a)$. The signature of a quadratic form K is denoted by σK.

Lemma 1. *The dimension of the algebra L_τ is equal to μ. The quadratic form $K_\varphi(f) = \omega_\varphi(f, f)$ takes real values and is nondegenerate. The signature σK_φ is equal to $\varphi^+ - \varphi^-$. In particular, σK_φ for $\varphi = 1$ is equal to the number of fixed points of τ.*

Proof. Under the action of τ, the set A decomposes into invariant sets A^k consisting of one or two points. Let L_τ^k denote the subalgebra of L_τ consisting of functions with support A^k, $l_\tau = \sum L_\tau^k$. The subspaces L_τ^k are orthogonal with respect to the form ω_φ. If A^k consists of a single point a, $a \in X$, then $\dim L_\tau^k = 1$ and the signature of the restriction of the form K_φ to L_τ^k is equal to sign $\varphi(a)$. For two-point sets $A^k = \{a, \tau a\}$ $\dim L_\tau^k = 2$. In this case, the restriction of the form K_φ

to L_τ^k is equal to $\varphi(a)f^2(a)+\varphi(\tau a)f^2(\tau a) = 2\mathrm{Re}[\varphi(a)f^2(a)]$. As is easily seen, the signature of such a form is zero. The lemma is proved.

Corollary. *1. The congruence* $\varphi^+ - \varphi^- \equiv \mu \mod 2$ *holds. 2. Let* L_0 *be any linear subspace of the algebra* L_τ *on which the form* K_φ *is identically equal to zero. Then the estimate* $|\varphi^+ - \varphi^-| \leq \mu - 2\dim L_0$ *is valid. If the null subspace is maximal, this estimate is best possible.*

Proof. Indeed, the signature of a nondegenerate form always has the same parity as the dimension of the space. In addition, the estimate $|\sigma K| \leq \mu - 2\dim L_0$ holds for every nondegenerate form K on $\mathbb{R}^\mu$ with nullspace L_0. For a maximal subspace L_0, this estimate is an equality.

We will apply the lemma 1 and corollary to the case when A is the set of complex singular points of a real vector field and $\tau\colon A \to A$ is the involution given by complex conjugation.

5.2. We consider a real vector field V in $\mathbb{R}^n$ with polynomial components $V = P_1, \dots, P_n$. Let $A \subset \mathbb{C}^n$ be the set of complex solutions of the system

$$P_1 = \cdots = P_n = 0 \tag{1}$$

and let the involution $\tau\colon A \to A$ be complex conjugation. The set X of fixed points τ under coincides with the set of real solutions of system (1). We assume that all the complex solutions $a \in A$ of system (1) have multiplicity one. This means that the Jacobian $j(x) = \det \dfrac{\partial P}{\partial x}$ of (1) does not vanish at the points of A. Let P_0 be any polynomial with real coefficients which does not vanish at the points of A.

We obtain the following assertion by applying the preceding lemma.

Assertion. *The signature* σK_φ *of the quadratic form* $K_\varphi(f) = \sum_{a\in A}\varphi(a)f^2(a)$ *is equal to the number of real singular points of the field* V *if* $\varphi = 1$. *If* $\varphi = 1/j$, *where* j *is the Jacobian of system (1), then* $\sigma K_\varphi = \mathrm{ind}$.

In order to estimate the number ind, it is now necessary to describe the algebra L_τ and exhibit a nullspace for the form K_φ which is as large as possible for $\varphi = 1/j$. The null subspace is obtained with the aid of the Euler–Jacobi formula, which we recall.

Consider a system of n polynomial equations of degrees $m_1, \dots, m_n$, in n complex unknowns,

$$P_1 = \cdots = P_n = 0.$$

We assume that the set of roots of the system contains exactly $\mu = m_1 \dots m_n$ elements. In this case, the Jacobian of the system $j = \det \dfrac{\partial P}{\partial x}$ does not vanish on the set A. Then for every polynomial Q of degree less than $\sum m_i - n$, we have the following Euler-Jacobi formula: $\sum_{a\in A} \dfrac{Q(a)}{j(a)} = 0$.

A purely algebraic proof of this formula can be found in [7]. An analytic proof and generalization to nondegenerate systems of equations with fixed Newton polyhedra is given in [8].

6. Convenient systems of equations.

A system of equations $P_1 = \cdots = P_n = 0$ of degrees $m_1, \ldots, m_n$ will be called nondegenerate if it has exactly $\mu = m_1 \ldots m_n$ distinct roots.

Consider the parallelepiped $\Delta(m)$ in $\mathbb{R}^n$ defined by the inequalities $0 \leq y_1 \leq m_1 - 1, \ldots, 0 \leq y_n \leq m_n - 1$. Let $M(m)$ be the space of polynomials with Newton polyhedron $\Delta(m)$. A polynomial $Q \in M(m)$, if and only if the degree of Q with respect to the variables x_i is less than m_i. The dimension of the space $M(m)$ over the field $\mathbb{C}$ is equal to $m_1 \ldots m_n - n = \mu$.

A nondegenerate system is said to be convenient if every complex valued function f on the set A of roots of the system is the restriction of some polynomial in $M(m)$.

Lemma 2. *The system of equations*

$$\prod_{0 \leq k \leq m_1} (x_1 - k) = \cdots = \prod_{0 \leq k \leq m_n} (x_n - k) = 0$$

is convenient.

Indeed, the set A of roots of this system contains precisely $\mu = m_1 \ldots m_n$ element. In addition, the equations of the system can be rewritten in the form of equalities $x_1^{m_1} = Q_1(x_1), \ldots, x_n^{m_n} = Q_n(x_n)$, in which $Q_1, \ldots, Q_n$ are polynomials of degrees $m_1 - 1, \ldots, m_n - 1$. Using these equations, it is not hard to show that every polynomial $Q(x)$ coincides on the set A with some polynomial in the space $M(m)$. This implies Lemma 1, since every function f on the finite set A is the restriction of some polynomial.

Lemma 3. *The inconvenient systems form a hypersurface in the space of all systems of degree m.*

Indeed, as is well known, the degenerate systems form a hypersurface in the space of all systems of degree m. Take any nondegenerate system and enumerate its roots arbitrarily $a_1, \ldots, a_\mu$. Then enumerate in some way the integer points in the Newton polyhedron $\Delta(m)$. These numerations define bases in the μ-dimensional space $C(A)$ of all complex-valued functions on A and in the μ-dimensional space $M(m)$. Let det denote the determinant of the matrix of the restriction mapping $i \colon M(m) \to C(A)$ with respect to these bases. The number $\det^2$ does not depend on the choice of enumeration; it depends only on the coefficients of the system, this dependence being analytic. A nondegenerate system is convenient if and only if the number $\det^2$ for it is distinct from zero. The function $\det^2$ is not identically equal to zero. Indeed, by Lemma 1, there exist convenient systems of degree m. Lemma 2 is proved.

Let $M(m, R)$ denote the space of polynomials with real coefficients with Newton polyhedron $\Delta(m)$, $M(m, R) = M(m) \cap R(x)$.

Lemma 4. *For a convenient system of equations of degree m with real coefficients, the restriction of polynomials in the space $M(m, R)$ to the set A defines an isomorphism of $M(m, R)$ with the algebra L_τ onto A, where $\tau \colon A \to A$ is the involution of complex conjugation.*

The restrictions of polynomials in $M(m, R)$ clearly lie in the algebra L_τ. In addition, for convenient systems a nonzero polynomial in $M(m)$ corresponds to a nonzero function on A. Lemma 4 follows from the inclusion $M(m, R) \subset M(m)$ and the fact that $\dim M(m, R) = \mu$, and $\dim L_\tau = \mu$.

7. Inequalities for nondegenerate fields.

We conclude the proof of Theorem (see Sec. 2). Let V be a nondegenerate field of degree m. Under a small change in the coefficients of the components $P_1, \dots, P_n$ of V the number ind will not change. It can therefore be assumed, without loss of generality, that $P_1 = \dots = P_n = 0$ is a convenient system of degree m (see Lemma 3) and that the surface $P_0 = 0$ in $\mathbb{C}^n$ does not intersect the set of roots A of the system. By Lemma 4, every function on A in L_τ, where the involution $\tau\colon A \to A$ is complex conjugation, is the restriction of a unique polynomial in $M(m, R)$. We consider the quadratic form $K_\varphi(f)$ on L_τ with $\varphi = 1/j$ (here $j = \det \dfrac{\partial P}{\partial x}$). According to the assertion in Sec. 5, $\sigma K_\varphi =$ ind. It follows from the Euler–Jacobi formula that for all polynomials f of degree less than $\frac{1}{2}\sum_{i>0}(m_i - 1)$, the identity $K_\varphi(f) = 0$ is valid. In our case, the inequality $|\sigma K_\varphi| \le \mu - 2\dim L_0$ takes the form $|\mathrm{ind}| \le \Pi(m)$. Indeed, μ is equal to the number of integer points in the polyhedron $\Delta(m)$, and $\dim L_0$ is equal to the number of integer points in $\Delta(m)$ satisfying the inequality $\sum y_i < 1/2 \sum_{i>0}(m_i - 1)$. The inequality $|\mathrm{ind}| \le \Pi(m)$ has been proved. The congruence ind $\equiv \mu \mod 2$ is almost obvious. Theorem is proved.

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LECTURE 3. SYSTEM OF EQUATIONS WITH GENERIC NEWTON POLYHEDRA

Consider the system of equations $P_1 = \dots = P_n = 0$ in $(\mathbb{C}\setminus 0)^n$, where $P_1, \dots, P_n$ are Laurent polynomials with the Newton polyhedra $\Delta_1, \dots, \Delta_n$.

Assume that Newton polyhedra $\Delta_1, \dots, \Delta_n$ are developed (see §1), which means that they are located sufficiently generally with respect to each other.

The geometrical meaning of being developed is especially clear in the two-dimensional case: two polygons on a plane are developed if and only if they do not have parallel sides with identically directed outer normals.

We will present two following results about the system of equations whose Newton polyhedra are developed.

First result: Let Q be a Laurent polynomial. Consider the n-form

$$\omega = (Q/P)\,\frac{dz_1}{z_1} \wedge \dots \wedge \frac{dz_n}{z_n},$$

where $z_1, \dots, z_n$ are independent variables and $P = P_1 \cdot \dots \cdot P_n$. Then the sum of the Grothendieck residues of the form ω over all roots of the system of equations can be explicitly evaluated (see [1]).

Second result: Vieta formula for the product of roots of a polynomial can be generalized to the multidimensional case: one can compute in the group $(\mathbb{C}^*)^n$ the product of all roots of a system of n polynomial equations with sufficiently general Newton polyhedra (see [2]).

1. Combinatorial coefficient.

Let $\Delta_1, \dots, \Delta_n$ be convex polyhedra in $\mathbb{R}^n$ and Δ be their Minkowski sum. Each face of the polyhedron Δ is the sum of faces of Δ_i polyhedra. A face Γ will be called locked if its terms include at least one vertex. A vertex $A \in \Delta$ will be called critical if all faces adjacent to this vertex are locked.

Consider a continuous map $F\colon \Delta \to \mathbb{R}^n$, $F = (f_1, \dots, f_n)$, such that each its components f_i is nonnegative and vanishes on those and only those faces $\Gamma = \Gamma_1 + \dots + \Gamma_n$ whose term Γ_i is a point-vertex of the polyhedron Δ_i.

The restriction $\tilde{F}$ of the map F onto the boundary $\partial\Delta$ of the Δ polyhedron transfers a neighborhood of a critical vertex into a neighborhood of the zero point on the boundary $\partial\mathbb{R}^n_+$ of the positive octant.

The combinatorial coefficient k_A of a critical vertex $A \in \Delta$ is the local degree of the germ of the map $\tilde{F}\colon (\partial\Delta, A) \to (\partial\mathbb{R}^n_+, 0)$. The coefficient k_A is well-defined and depends only on the orientations of the polyhedron Δ and the positive octant $\mathbb{R}^n_+$.

The set of the polyhedra $\Delta_1, \dots, \Delta_n$ is called developed if all faces of the sum polyhedron Δ are locked. Almost all sets of n polyhedra in the space $\mathbb{R}^n$ are developed.

2. Orientations.

The sign of the form ω depends on the order of the independent variables $z_1, \dots, z_n$. This order also determines the orientation of the linear space $\mathbb{R}^n$ that contains me lattice of the monomials z^a and the Newton polyhedron $\Delta = \Delta_+ \dots + \Delta_n$.

The order of the equations $P_1 = \dots = P_n = 0$ (or their Newton polyhedra $\Delta_1, \dots, \Delta_n$) determines the orientation of the space $\mathbb{R}^n_+$, which is involved in the definition of the combinatorial coefficient. The order of the equations also determines the sign of the Grothendieck residue in the roots of the system of equations.

Let us arbitrarily select the orders of the independent variables and the equations. These determine the signs of the form ω, the Grothendieck residue, and the signs of the combinatorial coefficients.

3. Residue of the form in a vertex of a polyhedron.

For each vertex A of the Newton polyhedron $\Delta(P)$ of the Laurent polynomial P, we construct the Laurent series or the function Q/P, where q is an arbitrary Laurent polynomial.

The monomial z^a corresponding to the vertex A of the polyhedron $\Delta(P)$ is included in P with some nonzero coefficient C_A; therefore, the free term of the Laurent polynomial $\tilde{P} = P/(C_A z^a)$ is equal to unity. Let us specify, the Laurent series for $1/\tilde{P}$ by the formula $1/\tilde{P} = 1 + (1-\tilde{P}) + (1-\tilde{P})^2 + \dots$. Each monomial z^b is included with nonzero coefficients in only a finite number of the terms $(1-\tilde{P})^k$. Therefore, the coefficient of each monomial z^b in this series is well-defined. The formal product of the series obtained and the Laurent polynomial $C_A z^a Q$ will be called the Laurent series of the rational function Q/P at the vertex A of the Newton polyhedron $\Delta(P)$.

The residue $res_A\omega$ of the rational form $\omega = \frac{Q}{P}\frac{dz_1}{z_1}\wedge\cdots\wedge\frac{dz_n}{z_n}$ at the vertex A at the vertex A of the Newton polyhedron $\Delta(P)$ is the free term of the Laurent series of the function Q/P at the vertex A. The residue $\mathrm{res}_A\omega$ can explicitly be written as a polynomial of C_A^{-1} and the coefficients of the Laurent polynomials P and Q.

4. The first main result.

Main Theorem. *If the Newton polyhedra of the equations in the system are developed, then the sum of the Grothendieck residues can be evaluated for a form ω with any Laurent polynomial Q. This sum is equal to $(-1)^n\sum k_A \mathrm{res}_A\omega$, where the summation is over all critical vertices A of the polyhedron Δ.*

Corollary. *The sum $\sum R(a)\mu(a)$ of values of an arbitrary Laurent polynomial R over roots a of a system of equations with developed Newton polyhedra, where the roots are evaluated while taking into account their multiplicities $\mu(a)$ is $(-1)^n\sum k_A \mathrm{res}_A\omega$ where*

$$\omega = R\frac{dP_1}{P_1}\wedge\cdots\wedge\frac{dP_n}{P_n} = \left[Rz_1\ldots z_n \det\left(\frac{dP}{dz}\right)/(P_1\ldots P_n)\right]\frac{dz_1}{z_1}\wedge\cdots\wedge\frac{dz_n}{z_n}.$$

5. Geometric application.

For each vertex A of the polyhedron $\Delta = \Delta_1+\cdots+\Delta_n$, a set of vertices $A_i\in\Delta_i$ such that $A = A_1+\cdots+A_n$ is determined. Put $\det A$ to be equal to the determinant of the matrix formed by the vectors $A_1,\ldots,A_n$.

Theorem. *For the mixed volume V of developed polyhedra $\Delta_1,\ldots,\Delta_n$ with rational vertices,*

$$n!V = (-1)^n\sum k_A \det A.$$

For polyhedra with integer vertices, this theorem is proved by comparing the Bernstein formula for the number of the roots of the system of equations with Corollary for $R = 1$. One can prove the theorem geometrically and eliminate the condition that the vertices are rational [2].

6. Algebraic application.

Corollary makes it possible to construct an explicit theory of elimination for a system of equations in $(\mathbb{C}\setminus 0)^n$ with developed Newton polyhedra. Let us explain, for example, how an equation for the first coordinate z_1 of the roots of the system is obtained. For this purpose it is sufficient to evaluate the sums $\sum R(a)\mu(a)$, where R are polynomials equal to $1, z_1,\ldots,z_1^N$, with $N = n!V-1$, and use the Newton formulas expressing the coefficients in an equation in terms of the sums of powers of its roots.

7. Local version.

Consider a system of analytic equations $p_1 = \cdots = p_n = 0$ in a neighborhood of the point $0\in\mathbb{C}^n$. Let the Newton diagrams of $\Gamma_1,\ldots,\Gamma_n$ of these equations be convenient. The definitions of developed Newton diagrams and the combinatorial coefficients of the vertices of the diagram of the sum $\Gamma = \Gamma_1+\cdots+\Gamma_n$ and the residues $\mathrm{res}_A\omega$ almost literally repeat the definitions given above. The following theorem is valid.

Theorem. *For a system of analytic equations $p_1 = \cdots = p_n = 0$ with developed Newton diagrams, the point 0 is an isolated solution. The Grothendieck residue at zero of the form $\omega = \dfrac{q}{p_1 \cdots p_n}\dfrac{dz_1}{z_1} \wedge \cdots \wedge \dfrac{dz_n}{z_n}$, where q is an arbitrary analytic function divisible by $z_l \cdot \ldots \cdot z_n$, is equal to $(-1)^{n-1} \sum k_A \operatorname{res}_A \omega)$, where the summation is over all vertices A of the Newton diagram $\Gamma = \Gamma_1 + \cdots + \Gamma_n$ that lie strictly inside the positive octant.*

8. Vieta formula.

According to the classical Vieta formula, the product of the nonzero roots of an equation $a_n x^n + \cdots + a_k x^k = 0$ with $a_n \neq 0$, $a_k \neq 0$ is equal to the number $(-1)^{n-k} a_k a_n^{-1}$. We generalize the Vieta formula to the multidimensional case. More precisely, we compute in the group $(\mathbb{C}^*)^n$ the product of all the roots of the system of equations

$$P_1(x) = \cdots = P_n(x) = 0, \quad x \in (\mathbb{C}^*)^n,$$

whose Newton polyhedra $\Delta_1, \ldots, \Delta_n$ are developed.

We will present two different formulas for this product. In the first formula we use the so called Parshin symbols, in the second formula we use derivatives of the mixed volume with respect to vertices of all polyhedra. Let us start with the one-dimensional case.

The Vieta formula has an interpretation connected with Weil's theorem. First let us recall Weil's theorem (see, for example, [3]). Let X be a complex algebraic curve, and f and g be two nonzero meromorphic functions on X. With each point $a \in X$ is associated the so-called the Weil symbol $[f, g]_a$. Here is its definition. Let u be a local parameter about the point a, $u(a) = 0$, and let $f = c_1 u^{k_1} + \ldots$ and $g = c_2 u^{k_2} + \ldots$ be the leading terms of the expansion of the functions f and g at the point a. The Weil symbol $[f, g]_a$ is the number $(-1)^{k_1 k_2} c_1^{k_2} c_2^{-k_1}$. For all the points on the curve X, except for a finite number of them, the Weil symbol is equal to one. The following holds:

Weil's theorem.

$$\prod_{a \in X} [f, g]_a = 1.$$

Let us apply Weil's theorem in the case that the curve X coincides with the Riemann sphere, the function f is equal to the coordinate function x, and the function g is equal to a polynomial P. We will get

$$\prod x(a) = [x, P]_0^{-1} [x, P]_\infty^{-1},$$

where the product is taken over all nonzero roots a of the polynomial P. This formula coincides with the Vieta formula.

The Vieta formula also has a completely different interpretation.

The Newton polygon of the polynomial $P(x) = a_n x^n + \cdots + a_k x^k$ is a segment $I(n, k)$ on the real line with vertices n and k, where $n > k \geq 0$. The product of the nonzero roots of the polynomial P is equal, up to a sign, to the monomial $a_k a_n^{-1}$ in the coefficients a_n, a_k at the vertices n and k of the Newton polyhedron $I(n, k)$ of the polynomial P. The coefficient a_n enters in this monomial to the power given by the value, taken with opposite sign, of the derivative of the length of the segment $I(n,k)$ by the vertex n, i.e. to the power minus one. The coefficient a_k enters in this monomial to the power also given by the value, taken with opposite sign, of the derivative of the length of the segment $I(n,k)$ by the vertex k, i.e. to the power one.

9. Parshin symbols and the product of roots.

Parshin–Kato theory gives a far-reaching generalization of Weil's theorem. In this theory to $(n+1)$ meromorphic functions on an n-dimensional algebraic manifold X and a flag of submanifolds in the manifold X is associated the so-called Parshin symbol which generalizes the Weil symbol. According to Parshin–Kato theory, the product of Parshin symbols over certain flags of submanifolds also turns out to by equal to one (see [4], [5], [6]).

Let $M(P_1,\dots,P_n)$ be the product in the group $(\mathbb{C}^*)^n$ of the roots of the system of equations (1). The computation of the point $M(P_1,\dots,P_n)$ in the group $(\mathbb{C}^*)^n$ is equivalent to the computation of the value $\chi(M(P_1,\dots,P_n))$ of each character $\chi\colon (\mathbb{C}^*)^n \to \mathbb{C}^*$ at this point.

Let Δ be the Minkowski sum of the polyhedra Δ_i, $\Delta = \Delta_1 + \cdots + \Delta_n$. With each vertex A of the polyhedron Δ is associated an integer — the combinatorial coefficient C_A of the vertex A.

With each vertex A of the polyhedron Δ it is possible to associate a number $[P_1,\dots,P_n,\chi]$, which we will call the Parshin symbol of the functions $P_1,\dots,P_n,\chi$ at the vertex A of the polyhedron Δ (see [2]).

Theorem. *The following equality holds:*

$$\chi(M(P_1,\dots,P_n)) = \prod_{A\in\Delta} [P_1,\dots,P_n,\chi]_A^{(-1)^n C_A}.$$

Here the product is taken over all the vertices A of the polyhedron Δ.

The equality from the theorem is analogous to the interpretation of Vieta formula with the help of Weil's theorem. It could be explained in the framework of Parshin–Kato theory. Our proof, however, is elementary and does not require this theory (see [2]).

10. Volume and the product of roots.

Let Δ be a polyhedron in $\mathbb{R}^n$, let A be one of its vertices, let L be the set of the other vertices of this polyhedron Δ. For every vector h we can consider the point $(A+h)$ and define the polyhedron $\Delta(A+h)$ as the convex hull of the set $L \cup (A+h)$. For example, by definition, the polyhedron $\Delta(A+0)$ coincides with the polyhedron Δ.

Now let $\Delta_1,\dots,\Delta_n$ be a collection of polyhedra in $\mathbb{R}^n$, and let A_i be one of the vertices of the polyhedron Δ_i. Let us consider the mixed volume $V(h)$ of the polyhedra $\Delta_1,\dots,\Delta_{i-1},\Delta_i(A+h),\Delta_{i+1},\dots,\Delta_n$ as a function of the vector h.

Suppose that the function $V(h)$ is differentiable by h at the point 0, and let dV be its differential. We will call this differential the *derivative of the mixed volume by the vertex A of the polyhedron Δ_i* and we will denote this differential by the symbol $d_{A_i}\mathrm{Vol}(\Delta_1,\dots,\Delta_n)$.

One can prove that *the mixed volume of a developed collection of the polyhedra $\Delta_1,\dots,\Delta_n$ is differentiable with respect to every vertex of every polyhedron.*

The Newton polyhedra are located in the space $\mathbb{R}^n$ of characters of the group $(\mathbb{C}^*)^n$; to every integral point $k \in \mathbb{R}^n$ corresponds the character $\chi_k\colon (\mathbb{C}^*)^n \to \mathbb{C}^*$ which maps the point x to the number x^k. The dual space $(\mathbb{R}^n)^*$ to the space $\mathbb{R}^n$ is the space of one-parameter subgroups in the group $(\mathbb{C}^*)^n$; to every integral point $m = m_1, \dots, m_n$ corresponds the one-parameter subgroup $t^m\colon \mathbb{C}^* \to (\mathbb{C}^*)^n$ which assigns to every nonzero number t the point $x = x_1, \dots, x_n$, where $x_1 = t^{m_1}, \dots, x_n = t^{m_n}$.

For a developed collection of integral polyhedra $\Delta_1, \dots, \Delta_n$ every derivative $n! d_{A_i} \operatorname{Vol}(\Delta_1, \dots, \Delta_n)$ is an integral covector on the space $\mathbb{R}^n$. Therefore to the derivative $n! d_{A_i} \operatorname{Vol}(\Delta_1, \dots, \Delta_n)$ corresponds a one-parameter group in the space $(\mathbb{C}^*)^n$. For every nonzero complex number $t \neq 0$ the element $t^{n! d_{A_i} \operatorname{Vol}(\Delta_1, \dots, \Delta_n)}$ in the group $(\mathbb{C}^*)^n$ is defined. In the new version of the formula for the product of roots $M(P_1, \dots, P_n)$ we will use these notations.

Theorem. *For every system of equations $P_1 = \dots = P_n = 0$ with the developed collection of Newton polyhedra $\Delta_1, \dots, \Delta_n$, up to signs the element $M(P_1, \dots, P_n)$ of the group $(\mathbb{C}^*)^n$ is defined by the formula*

$$M(P_1, \dots, P_n) = q \prod_{1 \le i \le n} \prod_{A_i \in \Delta_i} P_i(A_i)^{-n! d_{A_i} \operatorname{Vol}(\Delta_1, \dots, \Delta_n)},$$

where the inner product is conducted over all the vertices A_i of the polyhedra Δ_i; $P_i(A_i)$ is the number equal to the coefficient of Laurent polynomial P_i at the monomial corresponding to the vertex A_i; $n! d_{A_i} \operatorname{Vol}(\Delta_1, \dots, \Delta_n)$ is a one-parameter subgroup in the group $(\mathbb{C}^*)^n$ which corresponds to the derivative of the mixed volume $\operatorname{Vol}(\Delta_1, \dots, \Delta_n)$ by the vertex A_i, q is an element in $(\mathbb{C}^*)^n$ which coordinates are equal to ± 1.

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