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Rational unicuspidal plane curves with $\bar{\kappa}=1$

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1 Introduction

Let C be a curve on $\mathbf{P}^2=\mathbf{P}^2(\mathbf{C})$. A singular point of C is said to be a *cusp* if it is a locally irreducible singular point. We say that C is *cuspidal* if C has only cusps as its singular points. For a cusp P of C, we denote the *multiplicity sequence* of (C, P) by $\bar{m}_P(C)$, or simply $\bar{m}_P$. We use the abbreviation m_k for a subsequence of $\bar{m}_P$ consisting of k consecutive m's. For example, (2_k) means an A_{2k} singularity. We denote by $\bar{\kappa}(\mathbf{P}^2\setminus C)$ the logarithmic Kodaira dimension of the complement $\mathbf{P}^2\setminus C$. Let C' be the strict transform of a rational unicuspidal plane curve C via the minimal embedded resolution of the cusp of C. We define $n(C):=-(C')^2$. By [Y], $\bar{\kappa}(\mathbf{P}^2\setminus C)=-\infty$ if and only if $n(C)<2$. By [Ts1, Proposition 2], there exist no rational cuspidal plane curves with $\bar{\kappa}=0$. Thus $\bar{\kappa}(\mathbf{P}^2\setminus C)\geq 1$ if and only if $n(C)\geq 2$.

Theorem 1. *If C is a rational unicuspidal plane curve with $\bar{\kappa}(\mathbf{P}^2\setminus C)=1$, then there exists a unique pencil Λ on $\mathbf{P}^2$ satisfying the following four conditions.*

(i) *The cusp P of C is a unique base point of Λ.*

(ii) *The pencil Λ has a unique reducible member $C+n(C)B$. Here B is a line or an irreducible conic such that $(CB)_P=(\deg B)(\deg C)-1$.*

(iii) *The pencil Λ has exactly two multiple members $\mu_A A$, $\mu_G G$, where μ_A, μ_G are integers with $\mu_A, \mu_G\geq 2$, $A\setminus\{P\}\cong\mathbf{C}^*$, $G\setminus\{P\}\cong\mathbf{C}$.*

(iv) *The complement of $\{P\}$ to every member other than $\mu_A A$, $\mu_G G$ and $C+n(C)B$ is isomorphic to $\mathbf{C}^*$.*

Let C be a rational unicuspidal plane curve with $\bar{\kappa}(\mathbf{P}^2\setminus C)=1$. We say that C is of *type I* (resp. *type II*) if the curve B in Theorem 1 (ii) is a line (resp. an irreducible conic).

Theorem 2. *Let C be a rational unicuspidal plane curve with $\bar{\kappa}(\mathbf{P}^2 \setminus C) = 1$. Put $n := n(C)$. Let P be the cusp of C.*

(i) *Type I. There exists an integer s with $s \geq 2$ such that $\deg C = (n+1)^2(s-1)+1$, $\bar{m}_P = (n(n+1)(s-1), ((n+1)(s-1))_{2n+1}, (n+1)_{2(s-1)})$, $\mu_A = n+1$ and $\mu_G = (n+1)(s-1)+1$. There exist $a_2, \dots, a_s \in \mathbf{C}$ with $a_s \neq 0$ such that C is projectively equivalent to the curve:*

$$((f^{s-1}y + \sum_{i=2}^{s} a_i f^{s-i} x^{(n+1)i-n})^{\mu_A} - f^{\mu_G})/x^n = 0,$$

where $f = x^n z + y^{n+1}$. Conversely, for arbitrary integers n, s with $n \geq 2$, $s \geq 2$ and $a_2, \dots, a_s \in \mathbf{C}$ with $a_s \neq 0$, the above equation defines a rational unicuspidal plane curve with $\bar{\kappa}(\mathbf{P}^2 \setminus C) = 1$. The multiplicity sequence of the cusp is equal to $\bar{m}_P$.

(ii) *Type II and $\deg C = ((4n+1)^2+1)/2$. We have $\bar{m}_P = ((n(4n+1))_4, (4n+1)_{2n}, 3n+1, n_3)$, $\mu_A = 4n+1$ and $\mu_G = 2n+1$. The curve C is projectively equivalent to the curve:*

$$((g^n y + x^{2n+1})^{\mu_A} - (g^{2n}z + 2x^{2n}yg^n + x^{4n+1})^{\mu_G})/g^n = 0,$$

where $g = xz - y^2$. Conversely, for an arbitrary integer n with $n \geq 2$, the above equation defines a rational unicuspidal plane curve with $\bar{\kappa}(\mathbf{P}^2 \setminus C) = 1$. The multiplicity sequence of the cusp is equal to $\bar{m}_P$.

(iii) *Type II and $\deg C \neq ((4n+1)^2+1)/2$. There exists a positive integer s such that, by setting $m := 4n+1$ and $t := 4s-1$, we have $\deg C = (m^2t+1)/2$,*

$$\bar{m}_P = \begin{cases} ((3mn)_4, (3m)_{2n}, (m)_3, 3n+1, n_3) & \text{if } s = 1, \\ ((tmn)_4, (tm)_{2n}, (sm)_3, (s-1)m, m_{2(s-1)}, 3n+1, n_3) & \text{if } s > 1, \end{cases}$$

$\mu_A = m$ and $\mu_G = 2(ms-n)$. There exist $a_1, \dots, a_s \in \mathbf{C}$ with $a_s \neq 0$ such that C is projectively equivalent to the curve:

$$((h^{2s-1}(g^n y + x^{2n+1}) + \sum_{i=1}^{s} a_i h^{2(s-i)} g^{mi-n})^{\mu_A} - h^{\mu_G})/g^n = 0,$$

where $h = g^{2n}z + 2x^{2n}yg^n + x^m$. Conversely, for an arbitrary integer n with $n \geq 2$, a positive integer s and $a_1, \dots, a_s \in \mathbf{C}$ with $a_s \neq 0$, the above equation defines a rational unicuspidal plane curve with $\bar{\kappa}(\mathbf{P}^2 \setminus C) = 1$. The multiplicity sequence of the cusp is equal to $\bar{m}_P$.

A plane curve C is said to be of *type* (d, ν) if the degree of C is d and the maximal multiplicity of C is ν. If C is a rational cuspidal curve of type (d, ν), then the inequality $d < 3\nu$ holds true ([MS]). See also [O].

Corollary 1. *Let C be a rational unicuspidal plane curve of type (d, ν) with $\bar{\kappa}(\mathbf{P}^2 \setminus C) = 1$.*

(i) *Type I. We have $1 < d/\nu \leq 5/3$. The equality holds if and only if C is projectively equivalent to a curve in Theorem 2 (i) with $n = s = 2$.*

(ii) *Type II. We have $2 < d/\nu \leq 41/18$. The equality holds if and only if C is projectively equivalent to the curve in Theorem 2 (ii) with $n = 2$.*

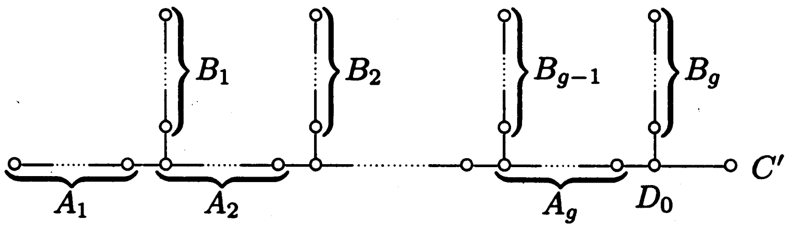
Corollary 2. *Let C be a rational unicuspidal plane curve. Then $\bar{\kappa}(\mathbf{P}^2 \setminus C) = 1$ if and only if the multiplicity sequence of the cusp is one of those in Theorem 2.*

Corollary 3. *Let C be a rational unicuspidal plane curve. Then $\bar{\kappa}(\mathbf{P}^2 \setminus C) = 2$ if and only if $n(C) \geq 2$ and the multiplicity sequence of the cusp is none of those in Theorem 2.*

Remark 4. In [Ts1], Tsunoda claimed to have obtained the defining equations of rational unicuspidal plane curves with $\bar{\kappa} = 1$. Comparing the degrees of his with ours, it seems that the equations he obtained are those of type I, $s = 2$ in Theorem 2 (i).

2 Proof of Theorem 1

Let C be a rational unicuspidal curve on $\mathbf{P}^2$ with $\bar{\kappa}(\mathbf{P}^2 \setminus C) = 1$. Let $\sigma : V \to \mathbf{P}^2$ be the composite of the shortest sequence of blowing-ups over P such that the reduced total transform D of C is a normal crossing divisor. Let C' be the strict transform of C. Put $D' := D - C'$. We remark that every irreducible components of D' is a smooth rational curve, whose self-intersection number is less than -1. Let D_0 denote the exceptional curve of the last blowing-up of σ. The dual graph of D has the following shape.



As a convention, A_1 contains the exceptional curve of the first blowing-up. Let A_{i1} denote the leftmost component of A_i in the above figure. In the course of the contraction of D' by σ, $A_g + D_0 + B_g$ is contracted a (-1)-curve E and $A_{g-1} + E + B_{g-1}$ to a (-1)-curve, and so on. Write $\sigma = \sigma_1 \circ \cdots \circ \sigma_g$, where σ_g contracts $A_g + D_0 + B_g$ to a (-1)-curve E, σ_{g-1} contracts $A_{g-1} + E + B_{g-1}$ to a (-1)-curve, and so on. A blowing-up of σ_i is called *sprouting* if it is done at a smooth point of the exceptional curve of the preceding blowing-ups. As a convention, the first blowing-up of σ_1 is not sprouting. Let s_i denote the number of sprouting blowing-ups in σ_i.

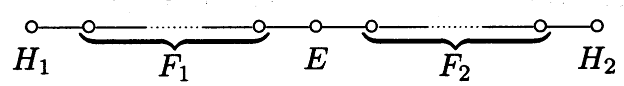
Following [FZ], we consider a strictly minimal model $(\tilde{V}, \tilde{D})$ of (V, D). We successively contract (-1)-curves E such that $E \subset D$ and $(D - E)E \leq 2$, or $E \not\subset D$ and $DE \leq 1$. After a finite number of contractions, we have no (-1)-curves to contract. Let $\pi : V \to \tilde{V}$ denote the composite of the contractions. For a divisor $\Delta \subset V$, write $\tilde{\Delta} = \pi_*(\Delta)$. It is clear that $\tilde{D}$ is a divisor with only simple normal crossings and $\bar{\kappa}(\tilde{V} \setminus \tilde{D}) = 1$. By [Ka, Theorem 2.3] and the fact that $\tilde{V} \setminus \tilde{D}$ is affine, we have the following:

Lemma 5. *There exists a fibration $\tilde{p} : \tilde{V} \to \mathbf{P}^1$ whose general fiber F is $\mathbf{P}^1$ and $\tilde{D}F = 2$.*

It is known that a $\mathbf{P}^1$-fibration over $\mathbf{P}^1$ is obtained from a $\mathbf{P}^1$-bundle $\hat{p} : \Sigma \to \mathbf{P}^1$ by successive blowing-ups $\tilde{\pi} : \tilde{V} \to \Sigma$. Put $p = \tilde{p} \circ \pi$. We have the following commutative diagram.

$$\begin{tikzcd}
V \arrow[r, "\pi"] \arrow[drr, "p"'] & \tilde{V} \arrow[r, "\tilde{\pi}"] \arrow[dr, "\tilde{p}"] & \Sigma \arrow[d, "\hat{p}"] \\
 & & \mathbf{P}^1
\end{tikzcd}$$

Following [FZ], we use the following terminology. The triple $(\tilde{V}, \tilde{D}, \tilde{p})$ is called a $\mathbf{C}^*$*-triple*. A component of $\tilde{D}$ is called *horizontal* if the image of it under $\tilde{p}$ is 1-dimensional. Let $\tilde{H}$ be the sum of the horizontal components of $(\tilde{V}, \tilde{D}, \tilde{p})$. The $\mathbf{C}^*$-triple $(\tilde{V}, \tilde{D}, \tilde{p})$ is called of *twisted type* if $\tilde{H}$ is irreducible; otherwise it is called of *untwisted type*. By [Kiz, Theorem 3], our $\mathbf{C}^*$-triple is of untwisted type. (See also [M2, Theorem 4.7.1, Lemma 4.10.3].) Thus $\tilde{H}$ consists of two irreducible components H_1, H_2. Suppose $\tilde{p}$ has a singular fiber. The dual graph of the sum of the singular fiber and the horizontal components has the following shape (cf. [FZ, Lemma 5.5]).



Here E is a (-1)-curve, which is not contained in $\tilde{D}$. The curves F_1, F_2 are connected components of $\tilde{D} - (H_1 + H_2)$. The fiber is contracted by $\tilde{\pi}$ to a fiber of $\hat{p}$. By using [FZ, Theorem 5.8 and 5.11], we have the following lemma. (The case (B2) in [FZ, Theorem 5.8] does not occur.)

Lemma 6. *The* $\mathbf{C}^*$*-triple* $(\tilde{V}, \tilde{D}, \tilde{p})$ *has the following properties.*

(i) *The fibration* $\tilde{p}$ *has exactly one smooth fiber* $\tilde{G}$ *contained in* $\tilde{D}$ *and two singular fibers* $\tilde{F}_A = \tilde{A}_1 + \tilde{E}_A + \tilde{B}_g$, $\tilde{F}_B = \tilde{B}_1 + \tilde{E}_B + \tilde{C}'$, *where* $\tilde{E}_A$ *(resp.* $\tilde{E}_B$*) is the* (-1)*-curve in* $\tilde{F}_A$ *(resp.* $\tilde{F}_B$*).*

(ii) *The curves* $\tilde{D}_0$, $\tilde{A}_{21}$ *are the horizontal components.*

We can verify that π has the following properties.

Lemma 7. *The following assertions hold true.*

(i) π *first contracts a* (-1)*-curve* $E_G \not\subset \tilde{D}$ *and every subsequent blowing-down of* π *is the contraction of a component of* $\tilde{D}$.

(ii) *The curve* E_G *is a component of* $\pi^{-1}(\tilde{G})$. *Every blowing-up of* π *is performed at a point on the total transform of* $\tilde{G}$.

Let E_A, E_B denote the strict transforms of $\tilde{E}_A$, $\tilde{E}_B$ in V, respectively. Write $A = \sigma(E_A)$, $B = \sigma(E_B)$ and $G = \sigma(E_G)$. Let μ_A, μ_B and μ_G denote the coefficients of E_A, E_B and E_G in $p^*(p(E_A))$, $p^*(p(E_B))$ and $p^*(p(E_G))$, respectively. We have $\mu_B = n(C)$ by [F, Proposition 4.8]. Since π does not change $\tilde{F}_B$, it follows that B is smooth and rational with self-intersection number s_1. Thus B is a line ($s_1 = 1$) or an irreducible conic ($s_1 = 4$). Now it is clear that the pencil spanned by $\mu_A A$ and $\mu_G G$ satisfies the whole condition in Theorem 1. The uniqueness of the pencil follows from [I, Theorem 3].

3 Proof of Theorem 2

In order to prove Theorem 2, we determine the weighted dual graph of $D + E_A + E_B + E_G$. By using the properties of σ, π and $\tilde{p}$, we obtain the diagram in Figure 1, where $n = n(C)$ and $*$ (resp. $\bullet$) means a (-1)-curve (resp. (-2)-curve). In Theorem 2, we set $s = s_3$. The curves in Theorem 2 (ii) correspond to those of type II with $g = 2$ and the curves in (iii) to those of type II with

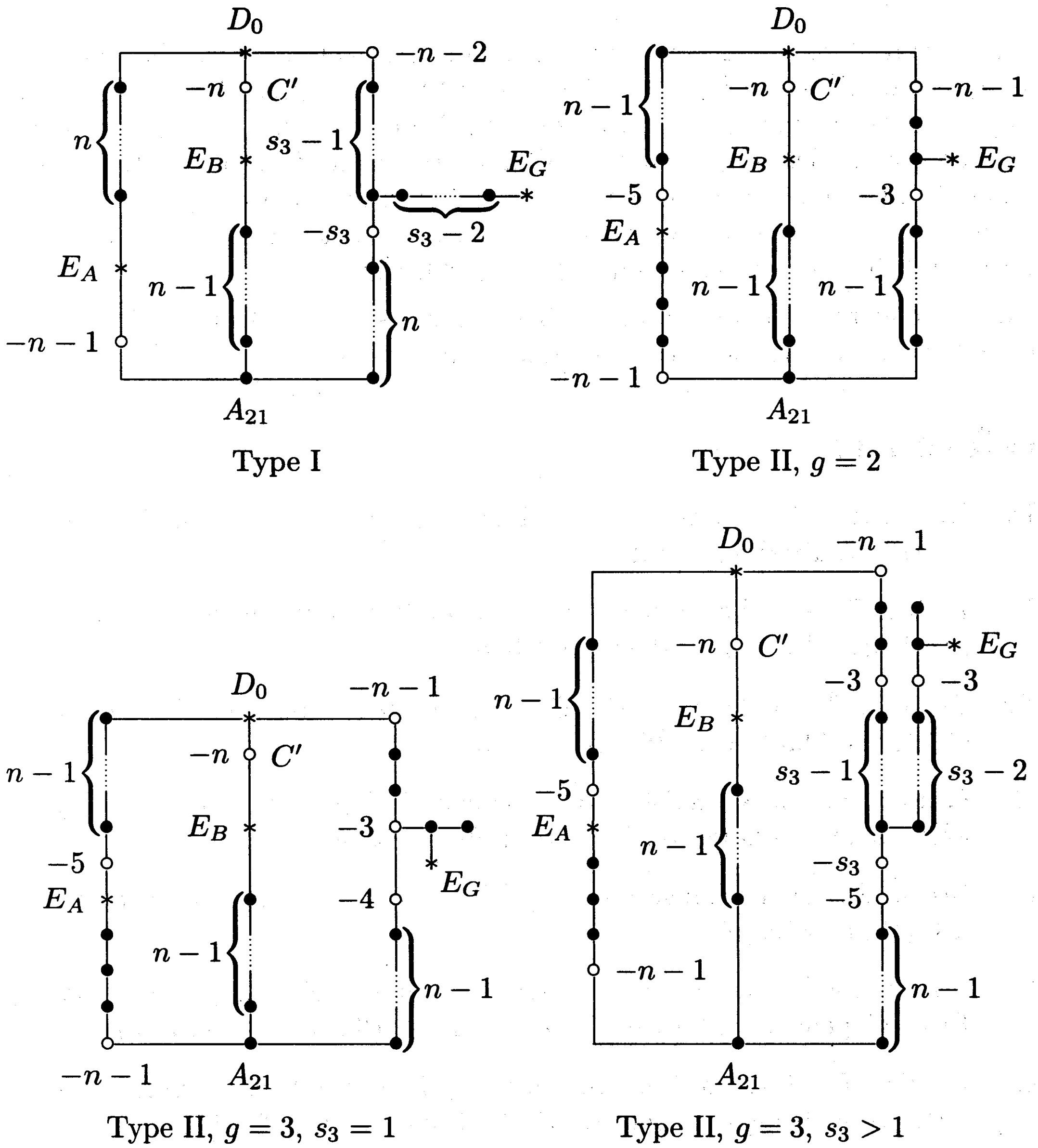


Figure 1: The weighted dual graph of $D+E_A+E_B+E_G$

Remark 8. Our fibration p belongs to the class (D) in the sense of [Kiz]. The last two graphs in [Kiz, Figure 54] coincide with those of type I and type II with $g = 3$, $s_3 > 1$.

The multiplicity sequence of the cusp can be calculated from the weighted dual graph of D' (cf. [BK, p.516, Theorem 12]). The degree of C is calculated from $\bar{m}_P$. We calculate μ_A, μ_G by using [F, Proposition 4.8]. The proof of the assertion for the defining equation of C is based on the following fact. Let f_A, f_B and f_G be the defining polynomials of A, B and G, respectively. Then, since $C + \mu_B B$ is a member of the pencil Λ in Theorem 1, there exists $t \in \mathbf{C}^*$ such that C is defined by the equation $(f_A^{\mu_A} + t f_G^{\mu_G})/f_B^{\mu_B} = 0$.

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