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NONLINEAR INSTABILITY OF DISPERSIVE WAVES

WALTER A. STRAUSS

Dedicated to the Memory of Tosio Kato

Consider a nonlinear evolution equation

$$\frac{du}{dt} = A(u) \tag{1}$$

and an equilibrium solution ϕ; that is, $0 = A(\phi)$. A concept of central importance in science is the concept of stability.

Definition. The equilibrium ϕ is (nonlinearly) **stable** if: $\forall \epsilon > 0, \exists \delta > 0$ such that if $\|u_0 - \phi\|_1 < \delta$, then there exists a unique solution u of (1) with $u(0) = u_0$ defined for $0 \leq t < \infty$ such that

$$\sup_{0 \leq t < \infty} \|u(t) - \phi\|_2 < \epsilon.$$

Remarks. (i) "Unstable" means "not stable".

(ii) The definition may be very sensitive to the norms $\|...\|_1$ and $\|...\|_2$ as well as to the space in which $u(\cdot)$ exists! If X is a Banach space, we define stability in X to mean that X is chosen in all three places.

(iii) The definition must be modified for non-equilibrium solutions such as traveling waves (orbital stability).

(iv) In case an orbit is unstable, a deep question is the following. What happens to it as $t \to +\infty$? Does it blow up? Does it converge to another equilibrium?

Linearization. Consider the linear equation

$$\frac{dv}{dt} = Lv \qquad \text{where } L = A'(\phi). \tag{2}$$

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Definition. The equilibrium ϕ is **linearly stable** if: $\forall\epsilon > 0, \exists\delta > 0$ such that

$$\text{if } \|v_0\|_1 < \delta, \text{ then } \sup_{0\le t<\infty} \|v(t)\|_2 < \epsilon.$$

Again the definition depends on the norms.

The basic theme of this lecture is the question: Does linear instability imply (nonlinear) instability? In what norms? We will mostly consider dispersive waves, which roughly means that most of the spectrum of L is imaginary.

As a baby example, consider the PDE $u_t = xu_x + u^2$ and its equilibrium solution $\phi = 0$. Its linearized equation $v_t = xv_x$ satisfies $\int v^2 dx = ce^{-t}$. Hence it is linearly stable in the L^2 norm. Nevertheless the solutions of the nonlinear PDE blow up (say at $x = 0$). This example shows how carefully the norms have to be chosen!

We introduce the notation $w = u - \phi$ for the difference between a solution and the equilibrium. In terms of w, equation (1) can be written as

$$\frac{dw}{dt} = Lw + N(w) \tag{3}$$

where $L = A'(\phi)$ and $N(w) = O(|w|^2)$ formally. A precise formulation of the main question is as follows.

Assume that we have

(i) two Banach spaces $X \subset Z$,

(ii) a strongly continuous semigroup e^{tL} on Z, and

(iii) a nonlinear operator $N : X \to Z$ that satisfies $\|N(w)\|_Z \le c\|w\|_X^\alpha \|w\|_Z^\beta$ for $\|w\|_X$ small, where $\beta > 1$ and $\alpha \ge 0$.

QUESTION: If spec(e^{tL}) meets the exterior of the closed unit disc, is $w = 0$ (nonlinearly) unstable in X? The next three theorems give an affirmative answer under three conditions.

Theorem 1. *True if there exists some point spectrum $e^{\lambda_0 t}$ "near" the maximal growth of e^{tL}. More precisely,*

$$\Re\lambda_0 > \frac{1}{\beta} \lim_{t\to+\infty} \frac{1}{t} \log \|e^{tL}\|_Z.$$

Theorem 2. *True if there exists a spectral gap outside the unit disk. This means there exists an annulus outside the unit disk that is entirely within the resolvent set of e^L.*

Theorem 3. *True if $X = Z$.*

These theorems are found in [6][4][10], respectively. The following "spectral dangers" occur in these theorems.

1°. $\sigma(e^{tL}) \supset e^{\sigma(tL)}$ but not necessarily $=$.

2°. It is possible that $\|e^{tL}\|$ is greater than the spectral radius of e^{tL}.
3°. e^{tL} could have continuous or residual spectrum.

It should be noted that for many interesting PDEs these theorems do not apply precisely but their basic ideas do. There are also some theorems [10] for equations of Hamiltonian type $\frac{du}{dt} = JE'(u)$ in terms of the spectrum of $E''(\phi)$. We shall now turn to a serious example.

IDEAL PLANE FLOW

The incompressible Euler equation is

$$\partial_t u + (u \cdot \nabla) + \nabla p = 0, \qquad \nabla \cdot u = 0 \tag{4}$$

where $x \in \Omega \subset \mathbb{R}^n, u \in \mathbb{R}^n$. Here Ω is a smooth, bounded, simply connected domain and the boundary condition is $u \cdot \nu = 0$ on $\partial\Omega$. Alternatively, Ω could be the n-torus. Now let $\phi(x)$ be a smooth equilibrium flow (of which there are many possibilities).

Linearization #1. The straightforward linearization is

$$(\partial_t + \phi \cdot \nabla)v + (v \cdot \nabla)\phi + \nabla q = 0, \qquad \nabla \cdot v = 0. \tag{5}$$

It generates a semigroup in $L^2(\Omega)$ with a generator A_1. Its essential spectrum is governed by a system of ODEs, as follows.

$$\begin{cases} \dot{x} = \phi(x) \\ \dot{\xi} = -(\partial_x \phi)^T \xi \\ \dot{b} = -(\partial_x \phi) b + 2(\xi \cdot \partial_x \phi) b\, \xi / |\xi|^2 \end{cases} \tag{6}$$

Indeed, Friedlander and Vishik [3] proved the following theorem.

Theorem 4. *The essential spectral radius of e^{tA_1} equals the maximum growth rate of (6).*

In particular, if the solutions of (6) grow exponentially, then e^{tL} has some essential spectrum in the exterior of the closed unit disk. The fluid is stretched along streamlines.

Now we specialize to two dimensions. The following nonlinear stability and instability theorems are known for $n = 2$.

(a) There exist certain flows that are stable in H^1. [1]

(b) There are certain flows that are unstable in H^s for $s > 2$, by making use of the point spectrum of (5). [4]

(c) There are certain flows that are unstable in L^2. [5]

Linearization #2. Consider the vorticity for $n = 2$:

$$\omega = \text{curl } u = \partial_1 u_2 - \partial_2 u_1.$$

From (4) it satisfies

$$(\partial_t + u \cdot \nabla)\omega = 0$$

because $n = 2$. Linearizing this equation, and using the notation $\eta = \delta\omega$, $v = \delta u$, we have

$$(\partial_t + \phi \cdot \nabla)\eta + \nabla(\nabla \times \phi) \cdot v = 0, \qquad \nabla \cdot v = 0. \tag{7}$$

Acting on η, the linearized generator therefore is $A_2 = -\phi \cdot \nabla - \nabla(\nabla \times \phi) \cdot (\mathrm{curl})^{-1}$. Considering A_2 acting on $L^2(\Omega)$ is roughly equivalent to considering A_1 acting on $H^1(\Omega)$, because $n = 2$ and $\nabla \cdot v = 0$.

However, in contrast to e^{tA_1}, the essential spectrum of e^{tA_2} has no growth because the second term in A_2 is a compact operator. That is, if $\lambda \in \mathrm{essspec}(A_2)$, then $\Re\lambda = 0$. So instability in the sense of the second linearization can only occur in the discrete spectrum. The following theorem [2] relates this kind of linear instability to nonlinear instability.

Theorem 5. *If A_2 has point spectrum $\Re\lambda > \sigma$ (with σ given below), then ϕ is unstable in the space $H^1(\Omega)$. (That is, the space $u \in H^1, \omega \in L^2$.)*

This is the space in which Arnold's stability theorem is valid. Here σ is the classical growth rate for the ODE $\dot{x} = \phi(x)$. That is, if $X(t,x)$ denotes the flow for this ODE, then

$$\sigma = \sup_x \lim_{t \to +\infty} \frac{1}{t} \log \left| \frac{\partial X}{\partial x} \right|. \tag{8}$$

In Theorem 5 there is no further restriction on the domain. For shear flows and for simple rotating flows, it is easy to see that $\sigma = 0$.

Recently Zhiwu Lin [9] has proven that many flows in two dimensions are linearly unstable.

COLLISIONLESS PLASMA

The Vlasov-Maxwell equations describe a plasma in the absence of collisions. The unknowns are the density of particles, which we will call $u(t,x,v)$, and the electric and magnetic fields $(E(t,x), B(t,x))$. The momentum $v \in \mathbb{R}^3$ is an independent variable, in addition to the time t and space $x \in \mathbb{R}^3$. The Vlasov equation is

$$(\partial_t + v \cdot \nabla_x)u + (E + v \times B) \cdot \nabla_v u = 0. \tag{9}$$

It is coupled to Maxwell's equations via the charge $\rho = \int u\, dv$ and the current $j = \int vu\, dv$. This system has many kinds of equilibria. Some are stable and some are unstable in the L^1 norm. See [7] [8].

KURAMOTO-SIVASHINSKY EQUATION

This is not a dispersive equation but it will illustrate our general techniques; in fact, we will be able to use a slightly modified version of Theorem 3. The equation is

$$u_t + u_{xxxx} + u_{xx} + uu_x = 0. \tag{10}$$

It is a one-dimensional model in the theory of flame propagation. There are many numerical and some theoretical results showing that some of its solutions engage in very complicated dynamical behavior. It has many traveling wave solutions $u = \phi(x - ct)$ for which the two limits

$$b_\pm = \lim_{t \to \pm\infty} \phi(\xi)$$

exist [12]. These $\phi(\xi)$ have multiple maxima and minima. The following theorem [11] asserts their instability.

Theorem 6. *Any such traveling wave is unstable under $H^1(\mathbb{R})$ perturbations. That is, there exists an $\epsilon_0 > 0$ and solutions $u^\delta(t,x)$ of (10) such that*

$$\|u^\delta(0,\cdot) - \phi(\cdot)\|_{H^1} < \delta \qquad (0 < \delta \le \delta_0)$$

but

$$\sup_{0 \le t \le C|\log\delta|} \|u^\delta(t,\cdot) - \phi(\cdot)\|_{H^1} > \epsilon_0.$$

In fact, we write the linearized generator as

$$L = -\partial^4 - \partial^2 - b_+\partial - (\phi - b_+)\partial - \phi_x.$$

By Fourier transformation the first three terms have unstable essential spectrum while the last two define a relatively compact operator. In this way we prove that e^{tL} has essential spectrum outside the closed unit disk. Then we apply a variant of Theorem 3 to deduce the nonlinear instability.

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